TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 6, pp. 2601-2608, December 2011
This paper is available online at http://tjm.math.ntu.edu.tw

DISTORTION THEOREMS OF STARLIKE MAPPINGS IN SEVERAL COMPLEX VARIABLES

Taishun Liu, Jianfei Wang* and Jin Lu

Abstract. In this paper, we first obtain a distortion theorem of Jacobian matrix $J_f(z)$ for biholomorphic starlike mapping f along a unit direction on the unit polydisc. This result extends the classical distortion theorem of starlike functions to higher dimensions. We then give an upper bound estimate of distortion theorem for biholomorphic starlike mappings along a unit direction in a complex Banach space. Finally, we propose two conjectures on starlike mappings in several complex variables.

1. Introduction Introduction and Preliminary Results

In one complex variable, the following classical distortion theorem is well known.

Distortion Theorem A. ([1]). *If f is a biholomorphic function on the unit disk D with $f(0) = 0$ and $f'(0) = 1$, then the following distortion theorem holds*

$$\frac{1-|z|}{(1+|z|)^3} \leqslant |f'(z)| \leqslant \frac{1+|z|}{(1-|z|)^3}, \quad \forall z \in D.$$

In the case of several complex variables, however, there are many counter-examples to show that the distortion theorem fails unless we restrict some subclasses of biholomorphic mappings. For example,

$$f(z) = (z_1, \frac{z_2}{(1-z_1)^k})', \quad k \in \mathbb{N}^+.$$

It is easy to see that $f(0) = 0$ and $J_f(0) = I_2$, where $J_f(0)$ is the Jacobian matrix f at the origin and I_2 is the identity matrix. But $\det J_f(z)$ and $J_f(z)\overline{J_f(z)'}$ for the

Received January 11, 2010, accepted September 13, 2010.
Communicated by Der-Chen Chang.
2010 *Mathematics Subject Classification*: 32A30, 32H02.
Key words and phrases: Starlike mappings, Distortion theorem, Banach space, Growth theorem.
*Corresponding author.

holomorphic mappings f have no finite upper bound and no non-zero lower bound, where $J_f(z)$ is the Jacobian matrix of f at point z. In order to extend the above Distortion Theorem A to high dimensions, H. Cartan suggested the study of convex mappings, starlike mappings and some other subclasses of biholomorphic mappings in several complex variables.

The first affirmative result about the distortion theorem for $\det J_f(z)$ was established by Barnard, FitzGerald and Gong [2], when f is a normalized biholomorphic convex mappings in the unit ball $\mathcal{B}^2$ of $\mathbb{C}^2$. Henceforth, Liu [3] extended it to the unit ball $\mathcal{B}^n$ in $\mathbb{C}^n$. As for the distortion theorem of Jacobian matrix $J_f(z)$, Gong, Wang and Yu [4] in 1993 established the distortion theorem of Jacobian matrix $J_f(z)\overline{J_f(z)'}$ for a normalized biholomorphic convex mapping f in the unit ball $\mathcal{B}^n$ of $\mathbb{C}^n$. In 1999, Gong and Liu [5] generalized the distortion theorem of biholomorphic convex mappings from the unit ball to bounded circular domains in $\mathbb{C}^n$. Later, Liu and Zhang obtained the following general distortion theorem for convex mappings in terms of Carathéodory metric and Kobayashi metric in a complex Banach space.

Distortion Theorem B. ([6]). *Let B be the unit ball in a complex Banach space X. Assume $f : B \to X$ is a biholomorphic convex mapping, $f(0) = 0$ and $Df(0) = I$, then the following distortion theorem holds*

$$(\frac{1-||x||}{1+||x||})F_K^B(x,\xi) \leqslant ||Df(x)\xi|| \leqslant (\frac{1+||x||}{1-||x||})F_C^B(x,\xi),$$

where $F_C^B(x,\xi)$ and $F_K^B(x,\xi)$ are the Carathéodory metric and Kobayashi metric on B, respectively.

In contrast to the distortion theorem of biholomorphic convex mappings, it is natural to consider the distortion theorem for biholomorphic starlike mappings in several complex variables. Until now, although the growth theorems and covering theorems for biholomorphic starlike mappings have been established in several complex variables (see [7-10]), there is no any progress for the corresponding distortion theorem of biholomorphic starlike mappings even on the unit ball or the unit polydisc in $\mathbb{C}^n$. In this paper, we give a distortion theorem $J_f(z)$ for biholomorphic starlike mapping f along a unit direction on the unit polydisc D^n. As an application, this theorem reduces to the classical distortion theorem of starlike function on the unit disk D when $n = 1$. On the other hand, we obtain the upper bound for the biholomorphic starlike mappings f along a unit direction in a complex Banach space by using the growth theorem of starlike mappings. Finally, we propose two conjectures on distortion theorems of starlike mappings.

In all what follows, we define some notation which will recur. Let D be the unit disk in the complex plane $\mathbb{C}$. The symbol X is used to denote complex Banach space with norm $||.||$, and $B = \{z \in X : ||z|| < 1\}$ is the unit ball in X. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, ..., z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w_j}$. Denote by D^n the unit polydisc in $\mathbb{C}^n$, and $(\partial D)^n(0, r)$ the distinguished boundary of the polydisc of radius r with the center 0. The symbol $'$ stands for the transpose of vectors and matrices, and the symbol $\mathbb{N}^*$ represents the set of all positive integers. Let $L(X, X)$ be the space of linear continuous operators from X into X. I means the identity in $L(X, X)$. Let $G \subset \mathbb{C}^n$ be a domain containing 0 and $H(G)$ be the set of all holomorphic mappings from G into $\mathbb{C}^n$. If $f \in H(G)$,we say that f is normalized if $f(0) = 0$ and $J_f(0) = I$.

Let Ω be a domain in X. $f : \Omega \to X$ is said to be holomorphic on Ω if for any $x \in \Omega$, there exists a $Df(x) \in L(X, X)$ such that

$$\lim_{h\to 0} \frac{\|f(x + h) - f(x) - Df(x)(h)\|}{\|h\|} = 0.$$

The linear map $Df(x)$ is called the Fr$\acute{e}$chet derivative of f at point x. In $\mathbb{C}^n$, $Df(x)$ is the Jacobian matrix, always written by $J_f(x)$.

A holomorphic mapping $f : \Omega \to X$ is said to be locally biholomorphic on Ω if its Fr$\acute{e}$chet derivative $Df(x)$ is nonsingular at each $x \in \Omega$. A holomorphic mapping $f : \Omega \to X$ is biholomorphic if the inverse f^{-1} exists, is holomorphic on an open set $V \subset X$ and $f^{-1}(V) = \Omega$.

Let X^* be the dual space of X. For each $x \in X \setminus \{0\}$, we denote by T_z the linear functional on X such that $||T_z|| \leqslant 1$ and $T_z(z) = ||z||$. By the Hahn-Banach theorem, T_z always exists.

DEFINITION 1.1 [11] Let B be the unit ball in a complex Banach space X. A holomorphic map $f : B \to X$ is called starlike if f is a normalized locally biholomorphic mapping and $(1 - t)f(B) \subset f(B)$ for all $t \in [0, 1]$ (thus $f(B)$ is starlike with respect to the origin in the usual sense).

According to the definition 1.1, Suffridge [11] characterized the following property of starlike mappings in a complex Banach space.

Definition 1.2. Let B be the unit ball in a complex Banach space X with norm $||.||$. Suppose $f : B \to X$ is a normalized locally biholomorphic mapping, T_z is a linear functional on X with $||T_z|| \leqslant 1$ and $T_z(z) = ||z||$. We say that f is a normalized starlike mapping if the following conditions holds

$$\Re T_z[(Df(z))^{-1}f(z)] \geqslant 0,\ z \in B.$$

2. Some Lemmas

In order to prove the desired theorems, we need the following lemmas.

Lemma 2.1. ([10]). *If $f : B \to X$ is a normalized biholomorphic starlike mapping, then*

$$\|x\|\frac{1-\|x\|}{1+\|x\|}\leqslant \Re T_x[(Df(x))^{-1}f(x)]\leqslant |T_x[(Df(x))^{-1}f(x)]|\leqslant \|x\|\frac{1+\|x\|}{1-\|x\|}$$

for any $x\in B$.

Lemma 2.2. *If* $f: D^n\to\mathbb{C}^n$ *is a normalized biholomorphic starlike mapping, then*

$$\|z\|\frac{1-\|z\|}{1+\|z\|}\leqslant \|J_f^{-1}(z)f(z)\|\leqslant \|z\|\frac{1+\|z\|}{1-\|z\|} \tag{2.1}$$

for any $z\in D^n$.

Proof. e need only consider $z\in D^n$ and $z\neq 0$.

Take $\xi\in D^n$ such that $|\xi_1|=|\xi_2|=...=|\xi_n|=\|\xi\|$. Then $T_\xi=(0,...,0,\frac{\|\xi\|}{\xi_i},0,...,0)$.

Set $w(z)=J_f^{-1}(z)f(z)$. Then there exists an i, such that

$$\begin{aligned}\|w(z)\|=|w_i(z)|&\leqslant \max_{\xi\in(\partial D)^n(0,\|z\|)}|w_i(\xi)|\\ &=\max_{\xi\in(\partial D)^n(0,\|z\|)}|\frac{\|\xi\|}{\xi_i}w_i(\xi)|\\ &=\max_{\xi\in(\partial D)^n(0,\|z\|)}|T_\xi[w(\xi)]|\\ &=\max_{\xi\in(\partial D)^n(0,\|z\|)}|T_\xi[(J_f(\xi))^{-1}f(\xi)]|.\end{aligned}$$

Lemma 2.1 yields that

$$|T_\xi[J_f^{-1}(\xi)f(\xi)]|\leqslant \|\xi\|\frac{1+\|\xi\|}{1-\|\xi|}\leqslant \|z\|\frac{1+\|z\|}{1-\|z|}.$$

Hence we have

$$||J_f^{-1}(z)f(z)||=\|w(z)\|\leqslant \|z\|\frac{1+\|z\|}{1-\|z\|}.$$

On the other hand, $||T_z||\leqslant 1$ shows that

$$||J_f^{-1}(z)f(z)||\geqslant \|T_z[J_f^{-1}(z)f(z)]||\geqslant \|z\|\frac{1-\|z\|}{1+\|z\|}.$$

from the Lemma 2.1.

Thus we have

$$\|z\|\frac{1-\|z\|}{1+\|z\|}\leqslant \|J_f^{-1}(z)f(z)\|\leqslant \|z\|\frac{1+\|z\|}{1-\|z\|}$$

for any $z \in D^n$.

Lemma 2.3. *If $f : B \to X$ is a normalized biholomorphic starlike mapping, then*

$$\|[Df(x)]^{-1}f(x)\| \geqslant \|x\|\frac{1-\|x\|}{1+\|x\|}$$

for any $x \in B$.

Proof. Suppose T_x is a continuous linear functional on X such that $\|T_x\| = 1$ and $T_x(x) = ||x||$. Then the definition 1.2 shows that

$$\Re T_x[(Df(x))^{-1}f(x)] \geqslant 0.$$

¿From Lemma 2.1, we get

$$||x||\frac{1-||x||}{1+||x||} \leqslant \Re T_x[(Df(x))^{-1}f(x)] \leqslant |T_x[(Df(x))^{-1}f(x)]| \leqslant ||x||\frac{1-||x||}{1+||x||}.$$

Hence

$$||(Df(x))^{-1}f(x)|| \geqslant |T_x[(Df(x))^{-1}f(x)]| \geqslant ||x||\frac{1-||x||}{1+||x||}.$$

Lemma 2.4. ([10]). *Let $f : B \to X$ be a normalized starlike mapping. Then*

(2.2) $$\frac{\|z\|}{(1+\|z\|)^2} \leqslant ||f(z)|| \leqslant \frac{\|z\|}{(1-\|z\|)^2}, z \in B.$$

3. Main Results

In this section, we give the main results of this paper.

Theorem 3.1. *Let $f : D^n \to \mathbb{C}^n$ be a normalized biholomorphic starlike mapping. Then for any $z \in D^n\backslash\{0\}$, there exits a unit vector $\xi(z)$ ($\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}$), such that*

$$\frac{1-\|z\|}{(1+\|z\|)^3} \leqslant \|J_f(z)\xi(z)\| \leqslant \frac{1+\|z\|}{(1-\|z\|)^3},$$

where $\|z\| = \max\limits_{1\leqslant j\leqslant n} |z_j|$.

Proof. Set $\xi(z) = \frac{J_f^{-1}(z)f(z)}{\|J_f^{-1}(z)f(z)\|}$, where $z \in D^n\backslash\{0\}$.

The Lemma 2.2 gives that

$$\|z\|\frac{1-\|z\|}{1+\|z\|} \leqslant \|J_f^{-1}(z)f(z)\| \leqslant \|z\|\frac{1+\|z\|}{1-\|z\|}. \tag{3.1}$$

The Lemma 2.4 then shows that

$$\frac{\|z\|}{(1+\|z\|)^2} \leqslant ||f(z)|| \leqslant \frac{\|z\|}{(1-\|z\|)^2}, z \in D^n. \tag{3.2}$$

So we get from (3.1) and (3.2)

$$\begin{aligned} \frac{1-\|z\|}{(1+\|z\|)^3} &\leqslant \frac{\frac{\|z\|}{(1+\|z\|)^2}}{\|z\|\frac{1+\|z\|}{1-\|z\|}} \leqslant \|J_f(z)\xi(z)\| \\ &= \frac{\|f(z)\|}{\|J_f^{-1}(z)f(z)\|} \leqslant \frac{\frac{\|z\|}{(1-\|z\|)^2}}{\|z\|\frac{1-\|z\|}{1+\|z\|}} \leqslant \frac{1+\|z\|}{(1-\|z\|)^3}, \end{aligned}$$

which leads to the desired result.

Remark. When $n = 1$, the Theorem 3.1 reduces to the classical distortion theorem on normalized starlike mappings on the unit disk D.

Theorem 3.2. *Let $f : B \to X$ be a normalized biholomorphic starlike mapping. Then for any $x \in B\backslash\{0\}$, there exists a unit vector $\xi(x)(\xi(x) = \frac{[Df(x)]^{-1}f(x)}{\|[Df(x)]^{-1}f(x)\|})$, such that*

$$\|Df(x)\xi(x)\| \leqslant \frac{1+\|x\|}{(1-\|x\|)^3}.$$

Proof. Let $\xi(x) = \frac{(Df(x))^{-1}f(x)}{\|(Df(x))^{-1}f(x)\|}$. Then

$$f(x) = Df(x)(Df(x))^{-1}f(x) = \|(Df(x))^{-1}f(x)\|Df(x)\xi(x).$$

The Lemma 2.3 shows that

$$\|[Df(x)]^{-1}f(x)\| \geqslant \|x\|\frac{1-\|x\|}{1+\|x\|}. \tag{3.3}$$

From (2.2) and (3.3), we obtain

$$\|Df(x)\xi(x)\| = \frac{\|f(x)\|}{\|(Df(x))^{-1}f(x)\|} \leqslant \frac{\frac{\|x\|}{(1-\|x\|)^2}}{\|x\|\frac{1-\|x\|}{1+\|x\|}} \leqslant \frac{1+\|x\|}{(1-\|x\|)^3}.$$

4. Two Conjectures

The theorem 3.2 is only about the upper bound of starlike mappings on the unit ball of a complex Banach space. Similar to the theorem 3.1, we will propose the following conjecture 4.1 for the corresponding lower bound estimate.

Conjecture 4.1. Let $f : B \to X$ be a normalized biholomorphic starlike mapping. Then for any $x \in B\backslash\{0\}$, there exits a unit vector $\xi(x)$, such that

$$\|Df(x)\xi(x)\| \geqslant \frac{1-\|x\|}{(1+\|x\|)^3}.$$

Similar to the Distortion Theorem B of convex mappings, the following conjecture 4.2 is also proposed.

Conjecture 4.2. Let B be the unit ball in a complex Banach space X. Assume $f : B \to X$ is a biholomorphic starlike mapping, $f(0) = 0$ and $Df(0) = I$, then the following distortion theorem holds

$$\Big(\frac{1-||x||}{1+||x||}\Big)^2 F_K^B(x,\xi) \leqslant ||Df(x)\xi|| \leqslant \Big(\frac{1+||x||}{1-||x||}\Big)^2 F_C^B(x,\xi),$$

where $F_C^B(x,\xi)$ and $F_K^B(x,\xi)$ are the Carathéodory metric and Kobayashi metric on B, respectively.

Acknowledgments

This work was supported by National Natural Science Foundation of China (No. 10971063, No. 11001246), Natural Science Foundation of Zhejiang province (No. D7080080, No. Y6090694, No. Y6100219, No. Y6090036) and Zhejiang Innovation Project (No. T200924, No. T200905).

References

1. P. L. Duren, *Univalent Function*, Berlin, Springer-Verlag, 1983.
2. R. W. Barnard, C. H. FitzGerald and S. Gong, A distortion theorem for biholomorphic mappings in $\mathbb{C}^2$, *Trans. Amer. Math. Soc.*, **344** (1994), 902-924.
3. T. S. Liu, *The growth theorems and covering theorems for biholomorphic mappings on classical domains*, University of Science and Technology of China, Dissertation, 1989.
4. S. Gong, S. K. Wang and Q. H. Yu, Biholomorphic convex mappings of ball in $\mathbb{C}^2$, *Pacific J. Math.*, **161(2)** (1993), 287-306.

5. S. Gong and T. S. Liu, Distortion theorems for biholomorphic convex mappings on bounded convex circular domains, *Chin. Ann. Math.*, **20B(3)** (1999), 297-304.
6. T. S. Liu and W. J. Zhang, A distortion theorem of biholomorphic convex mappings in a Banach space, *Acta Math. Sinica.*, **46A(6)** (2003), 1041-1046.
7. R. W. Barnard, C. H. FitzGerald and S. Gong, The growth and $\frac{1}{4}$-theorem for starlike mappings in $\mathbb{C}^n$, *Pacific J. Math.*, **150** (1991), 13-22.
8. S. Gong, S. K. Wang and Q. H. Yu, The growth and $\frac{1}{4}$-theorem for starlike mappings on B_p, *Chin. Ann. Math.*, **11B** (1990), 100-104.
9. S. Gong, S. K. Wang and Q. H. Yu, The growth theorems in several complex variables, *Chin. Ann. Math.*, **14B** (1993), 93-105.
10. W. J. Zhang and D. Z. Dong, A growth and $\frac{1}{4}$-theorem for starlike mappings in a Banach space, *Chin. Ann. Math.*, **13A** (1992), 417-423.
11. T. J. Suffridge, Starlikeness, convexity and other geometric properties of holomorphic map in higher dimensions *Lecture Notes in Math.*, **599** (1975), 147-159.

Taishun Liu
Department of Mathematics
Huzhou Teachers College
Huzhou, Zhejiang 313000
P. R. China
E-mail: lts@ustc.edu.cn

Jianfei Wang
Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang 321004
P. R. China
E-mail: wangjf@mail.ustc.edu.cn

Jin Lu
Department of Mathematics
Huzhou Teachers College
Huzhou, Zhejiang 313000
P. R. China
E-mail: lujin@mail.ustc.edu.cn