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GLOBAL STABILITY OF A REACTION-DIFFUSION SYSTEM OF A COMPETITOR-COMPETITOR-MUTUALIST MODEL

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Abstract. In this paper, we study a reaction-diffusion system of a competitor-competitor-mutualist model with Neumann boundary condition. Using iteration method, we investigate the global asymptotic stability of the unique positive constant steady-state solution under some assumptions. We also give some sufficient conditions under which there are no nonconstant positive steady-state solution exist.

1. Introduction

In this paper, we investigate the global asymptotic stability of a competitor-competitor-mutualist model as follows:

$$
(1.1)\qquad \begin{cases} u_t - d_1\Delta u = \alpha u(1-\dfrac{u}{K_1}-\dfrac{\beta v}{1+m\omega}), (t>0, x\in\Omega),\\ v_t - d_2\Delta v = \delta v(1-\dfrac{v}{K_2}) - \eta uv, (t>0, x\in\Omega),\\ \omega_t - d_3\Delta\omega = \gamma\omega(1-\dfrac{\omega}{L_0+lu}), (t>0, x\in\Omega),\\ \dfrac{\partial u}{\partial v} = \dfrac{\partial v}{\partial v} = \dfrac{\partial \omega}{\partial v} = 0, (t>0, x\in\partial\Omega).\\ u(x,0)=u_0(x), v(x,0)=v_0(x), \omega(x,0)=\omega_0(x). \end{cases}
$$

Here $u(t,x), v(t,x), \omega(t,x)$ represent the population of two competitor and mutualist with diffusion constants d_1, d_2 and d_3, respectively, Ω is a bounded domain in R^n, $\frac{\partial\cdot}{\partial v}$ is the outer normal derivatives on $\partial\Omega$. The all parameters in (1.1) are positive constants, for details see [8].

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The model was initiated proposed and studied by Rai, Freedman and Addicott in [4] in the ODE form:

$$\text{(1.2)} \qquad \begin{cases} u_t = \alpha u(1 - \dfrac{u}{K_1} - \beta \dfrac{v}{1+m\omega}), (t>0), \\ v_t = \delta v(1 - \dfrac{v}{K_2}) - \eta uv, (t>0), \\ \omega_t = \gamma\omega(1 - \dfrac{\omega}{L_0 + lu}), (t>0), \\ u(t) = u_0, v(x,0) = v_0, \omega(x,0) = \omega_0. \end{cases}$$

Their model was extended by Zheng [8] to the system (1.1). In [8], under some assumptions the local stability of the unique positive constant steady-state solution is discussed by the method of spectral analysis of linearized operator. The problem (1.2) has been extended to periodic systems by several workers (cf.[1,2,5,7]), where the diffusion coefficients d_i and the various reaction rates α, β, etc. are periodic functions of t. Both the existence and asymptotic behavior of time-periodic solutions were investigated in the above papers.

To investigate the asymptotic behavior of the solution of (1.1), as well as the nonexistence of nonconstant positive steady states of (1.1), we also should consider the corresponding steady-state system

$$\text{(1.3)} \qquad \begin{cases} -d_1\Delta u = \alpha u(1 - \dfrac{u}{K_1} - \dfrac{\beta v}{1+m\omega}), (x \in \Omega), \\ -d_2\Delta v = \delta v(1 - \dfrac{v}{K_2}) - \eta uv, (x\in\Omega), \\ -d_3\Delta\omega = \gamma\omega(1 - \dfrac{\omega}{L_0+lu}), (x\in\Omega), \\ \dfrac{\partial u}{\partial \upsilon} = \dfrac{\partial v}{\partial \upsilon} = \dfrac{\partial \omega}{\partial \upsilon} = 0, (x \in \partial\Omega). \end{cases}$$

The main purpose of this paper is to prove the global asymptotic stability of the unique positive constant steady-state solution of (1.3)(see Theorem 2.1) and the nonexistence of nonconstant positive steady-state solutions under some assumptions (see Theorem 2.2).

2. The Main Result

Before giving our main results, we recall some results in [8](cf.[4]) which we will use later in the proof of our results.

Theorem A. *(1) If $1 + mL_0 = \beta K_2$ and $\eta K_1 < \delta$, then system (1.1) has a unique positive constant steady-state solution $E^* = (u^*, v^*, \omega^*)$, i.e., solution of*

$$\text{(2.1)} \qquad \begin{cases} 1 - \dfrac{u}{K_1} - \beta\dfrac{v}{1+m\omega} = 0, \\ \delta(1 - \dfrac{v}{K_2}) - \eta u = 0, \\ 1 - \dfrac{\omega}{L_0+lu} = 0 \end{cases}$$

and is given by

$$u^* = K_1 - \frac{(\delta - \eta K_1)\beta K_2}{ml\delta}, v^* = K_2 - \frac{\eta}{\delta}u^*, \omega^* = L_0 + lu^*.$$

(2) *If* $1+mL_0 > \beta K_2$ *then system (1.1) has a unique positive constant steady-state solution* $E^* = (u^*, v^*, \omega^*)$ *and is given by the positive value of*

$$u^* = \frac{\tau \pm \sqrt{\tau^2 + 4ml\delta^2 K_1(1 + mL_0 - \beta K_2)}}{2ml\delta}, v^* = K_2 - \frac{\eta}{\delta}u^*, \omega^* = L_0 + lu^*.$$

where

$$\tau = ml\delta K_1 + \beta\eta K_1 K_2 - \delta(1 + mL_0)$$

provided $u^* < \frac{\delta}{\eta}$.

Then our main results are the following Theorems:

Theorem 2.1 *Let* E^* *be the unique equilibrium of* (1.1) *stated in Theorem A. If* $1 + mL_0 \geq \beta K_2$ *and* $\eta K_1 < \delta$*, then* E^* *is globally stable, i.e.,*

$$\lim_{t\to\infty} U(x,t) = E^* \tag{2.2}$$

for any solution $U(x,t) := (u(x,t), v(x,t), w(x,t))$ *of* (1.1) *with* $(u_0(x), v_0(x), w_0(x)) \not\equiv (0,0,0)$*, where* u_0, v_0, w_0 *are nonnegative functions.*

Theorem 2.2 *Let* μ_1 *be the smallest eigenvalue of the operator* $-\Delta$ *(except 0) on* Ω *with the homogenous Neumann boundary condition and denote* $\alpha, \delta, \gamma, \beta, L_0, l, \eta, K_1, K_2, m$ *collectively by* Λ *for notational convenience. Then*

(i) *there exists a positive constant* $d = d(n, \Omega, \Lambda)$*, such that* (1.3) *has no non-constant positive classical solution for* $d_2, d_3 \geq d$ *provided that* $\mu_1 d_1 > \delta$,

(ii) *there exists a positive constant* $d = d(n, \Omega, \Lambda)$*, such that* (1.3) *has no non-constant positive classical solution for* $d_1, d_3 \geq d$ *provided that* $\mu_1 d_2 > \alpha$,

(iii) *there exists a positive constant* $d = d(n, \Omega, \Lambda)$*, such that* (1.3) *has no non-constant positive classical solution for* $d_1, d_2 \geq d$ *provided that* $\mu_1 d_3 > \gamma$.

3. The Proof of the Main Results

Lemma 3.1 (cf. [6]). *Let* f, K *are positive constants,* $T \in [0, \infty)$*, and if* w *satisfies*

$$\begin{cases} w_t - d\Delta w \leq (\geq) fw(K - w), & (x,t) \in \Omega \times (T, \infty), \\ \dfrac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times [T, \infty), \end{cases} \tag{3.1}$$

then

$$\limsup_{t\to\infty}\max_{\bar\Omega} w(.,t) \le K \ \ (\liminf_{t\to\infty}\min_{\bar\Omega} w(x,t) \ge K).$$

Lemma 3.2 *Let $U(x,t) = (u(x,t), v(x,t), w(x,t))$ be a solution of (1.1), and if $U(x,0) = (u_0(x), v_0(x), w_0(x)) \ge (0,0,0),\ \ x \in \bar\Omega$, then*

$$(0,0,0) \le \limsup_{t\to\infty}\max_{\bar\Omega} U(x,t) \le (K_1, K_2, L_0 + lK_1), \ \ \text{for}\ \ t > 0, x \in \bar\Omega.$$

Moreover if for $x \in \bar\Omega$,$U(x,0) \not\equiv (0,0,0)$, then $U(x,t) > (0,0,0)$ for $x \in \bar\Omega, t > 0$.

Proof. The first part easily obtain by Lemma 3.1 and if $x \in \bar\Omega$,$U(x,0) \not\equiv (0,0,0)$, by maximum principle, we can get $U(x,t) > (0,0,0)$.

Lemma 3.3 *For any positive classical solution $(u(x), v(x), w(x))$ of (1.3) the following estimates hold*

$$(u(x), v(x), w(x)) \le (K_1, K_2, L_0 + lK_1), \ \ \text{for}\ \ x \in \bar\Omega.$$

To prove Lemma 3.3, we should use the following Proposition from [3].

Proposition 3.4. *Suppose that $g \in (\bar\Omega \times \mathbb{R})$, then the following results hold:*

(i) *If $w \in C^2(\Omega)\cap C^1(\bar\Omega)$ satisfies $\Delta w + g(x, w(x)) \ge 0$ for $x \in \Omega$, $\frac{\partial w}{\partial \nu} \ge 0$ for $x \in \partial\Omega$ and $w(x_0) = \max_{\bar\Omega} w$, then $g(x_0, w(x_0)) \ge 0$.*

(ii) *If $w \in C^2(\Omega)\cap C^1(\bar\Omega)$ satisfies $\Delta w + g(x, w(x)) \le 0$ for $x \in \Omega$, $\frac{\partial w}{\partial \nu} \le 0$ for $x \in \partial\Omega$ and $w(x_0) = \min_{\bar\Omega} w$, then $g(x_0, w(x_0)) \le 0$.*

Proof of Lemma 3.3 Let $\phi = d_1 u$. From the first equation of (1.3) we have

$$-\Delta\phi = \alpha u(1 - \frac{u}{K_1} - \frac{\beta v}{1 + m\omega}), (x \in \Omega); \frac{\partial\phi}{\partial\nu} = 0, (x \in \partial\Omega).$$

Let $x_0 \in \bar\Omega$ be such that $\phi(x_0) = \max_{\bar\Omega}\phi$. Then by proposition 3.4 and the positiveness of u, v, w, we get $1 - \frac{u(x_0)}{K_1} - \frac{\beta v(x_0)}{1 + m\omega(x_0)} \ge 0$. Hence $u(x_0) \le K_1$ which in turn implies that $\max_{\bar\Omega} u \le K_1$.

Analogously, by setting $V = d_2 v$ and $W = d_3 w$, we can easily get from proposition 3.4 that

$$v(x_0) \le K_2, \ \ w(x_0) \le L_0 + lK_1.$$

These inequalities certainly implies our assertion.

Proof of Theorem 2.1. We first prove if $1 + mL_0 > \beta K_2$ and $\eta K_1 < \delta$ hold, then E^* is globally stable. Since $U(x,t) = (u(x,t), v(x,t), w(x,t))$ is a solution of (1.1), by Lemma 3.2,

$$
\begin{aligned}
&u_t - d_1\Delta u \le \alpha u(1 - \frac{u}{K_1})\\
(3.2)\qquad &v_t - d_2\Delta v \le v(1 - \frac{v}{K_2}),\\
&\omega_t - d_3\Delta\omega \ge \gamma\omega(1 - \frac{\omega}{L_0}).
\end{aligned}
$$

By Lemma 3.1, we have

$$
\begin{aligned}
&\limsup_{t\to\infty} \max_{\bar\Omega} u(.,t) \le K_1 =: \bar u_1,\\
(3.3)\qquad &\limsup_{t\to\infty} \max_{\bar\Omega} v(.,t) \le K_2 =: \bar v_1,\\
&\limsup_{t\to\infty} \max_{\bar\Omega} \omega(.,t) \ge L_0 =: \underline{\omega}_1.
\end{aligned}
$$

For any given $\varepsilon > 0$, there exists T_1^ε sufficient large, such that

$$
u(x,t) < \bar u_1 + \varepsilon, \ \text{ for } \ x \in \bar\Omega, t > T_1^\epsilon,
$$

$$
v(x,t) < \bar v_1 + \varepsilon, \ \text{ for } \ x \in \bar\Omega, t > T_1^\epsilon,
$$

$$
\omega(x,t) > \underline{\omega}_1 + \varepsilon, \ \text{ for } \ x \in \bar\Omega, t > T_1^\epsilon.
$$

By the second and third equations of (1.1), we have for $x \in \Omega, t > T_1^\epsilon$,

$$
\begin{aligned}
v_t - d_2\Delta v &\ge \delta v(1 - \frac{v}{K_2}) - \eta(\bar u_1 + \varepsilon)v,\\
\omega_t - d_3\Delta\omega &\le \gamma\omega(1 - \frac{\omega}{L_0 + l(\bar u_1 + \varepsilon)}),
\end{aligned}
$$

then again by Lemma 3.1, we obtain

$$
\begin{aligned}
\liminf_{t\to\infty} \min_{\bar\Omega} v(x,t) &\ge [1 - \frac{\eta}{\delta}(\bar u_1 + \varepsilon)]K_2,\\
\limsup_{t\to\infty} \max_{\bar\Omega} \omega(.,t) &\le L_0 + l(\bar u_1 + \varepsilon).
\end{aligned}
$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$
\begin{aligned}
\liminf_{t\to\infty} \min_{\bar\Omega} v(x,t) &\ge (1 - \frac{\eta}{\delta}\bar u_1)K_2 =: \underline{v}_1,\\
\limsup_{t\to\infty} \max_{\bar\Omega} \omega(.,t) &\le L_0 + l\bar u_1 =: \bar\omega_1.
\end{aligned}
$$

Hence, there exists T_2^ε sufficient large, such that

$$
\omega(x,t) \le \bar\omega_1 + \varepsilon \ \text{ for } \ x \in \bar\Omega, t \ge T_2^\epsilon.
$$

From the equation for u in (1.1)

$$u_t - d_1\Delta u \geq \alpha u(1 - \frac{u}{K_1} - \frac{\beta \bar{v}_1}{1 + m(\underline{\omega}_1 + \varepsilon)}).$$

Thanks to Lemma 3.1

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(.,t) \geq (1 - \frac{\beta \bar{v}_1}{1 + m(\underline{\omega}_1 + \varepsilon)})K_1.$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(.,t) \geq (1 - \frac{\beta \bar{v}_1}{1 + m\underline{\omega}_1})K_1 =: \underline{u}_1.$$

Hence, for any $0 < \varepsilon < \bar{u}_1$ there exists T_3^ε sufficient large, such that

$$u(x,t) \geq \underline{u}_1 - \varepsilon \text{ for } x \in \bar{\Omega}, t \geq T_3^\epsilon.$$

From the equation for ω and v in (1.1)

$$\omega_t - d_3\Delta\omega \geq \gamma\omega(1 - \frac{\omega}{L_0 + l(\underline{u}_1 - \varepsilon)}),$$
$$v_t - d_2\Delta v \leq \alpha v(1 - \frac{v}{K_2} - \frac{\eta}{\delta}(\underline{u}_1 - \varepsilon)).$$

Thanks to Lemma 3.1

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} \omega(.,t) \geq L_0 + l(\underline{u}_1 - \varepsilon),$$
$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(.,t) \leq (1 - \frac{\eta}{\delta}(\underline{u}_1 - \varepsilon))K_2.$$

By the arbitrariness of $\varepsilon > 0$, it follows that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} \omega(.,t) \geq L_0 + l\underline{u}_1 =: \underline{\omega}_2,$$
$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(.,t) \leq (1 - \frac{\eta}{\delta}\underline{u}_1)K_2 =: \bar{v}_2.$$

Then there exists T_4^ε sufficiently large, for $x \in \bar{\Omega}, t \geq T_4^\epsilon$ such that

$$\omega(x,t) \geq \underline{\omega}_2 - \varepsilon,$$

$$v(x,t) \leq \bar{v}_2 + \varepsilon.$$

From the equation for u in (1.1)

$$u_t - d_1\Delta u \geq \alpha u(1 - \frac{u}{K_1} - \frac{\beta(\bar{v}_2 + \varepsilon)}{1 + m(\underline{\omega}_2 - \varepsilon)}).$$

Thanks to Lemma 3.1 and by the arbitrariness of $\varepsilon > 0$, it follows that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(.,t) \geq (1 - \frac{\beta \bar{v}_2}{1 + m\underline{\omega}_2})K_1 =: \underline{u}_2.$$

Applying the inductive method we can construct sequences $\{\bar{u}_i\}$,$\{\bar{v}_i\}$, $\{\bar{\omega}_i\}$, $\{\underline{u}_i\}$, $\{\underline{v}_i\}$, $\{\underline{\omega}_i\}$ as follows: $\bar{u}_1 = K_1, \bar{v}_1 = K_2, \bar{\omega}_1 = L_0 + l\bar{u}_1, \underline{u}_1 = (1 - \frac{\beta \bar{v}_1}{1+m\underline{\omega}_1})K_1, \underline{v}_1 = (1 - \frac{\eta}{\delta}\bar{u}_1)K_2, \underline{\omega}_1 = L_0$ and for $n > 1$

$$(3.4)\qquad \begin{aligned} &\bar{v}_n = (1 - \frac{\eta}{\delta}\underline{u}_{n-1})K_2, \underline{v}_n = (1 - \frac{\eta}{\delta}\bar{u}_n)K_2, \underline{\omega}_n = L_0 + l\underline{u}_{n-1},\\ &\bar{\omega}_n = L_0 + l\bar{u}_n, \underline{u}_n = (1 - \frac{\beta \bar{v}_n}{1+m\underline{\omega}_n})K_1, \bar{u}_n = (1 - \frac{\beta \underline{v}_{n-1}}{1+m\bar{\omega}_{n-1}})K_1. \end{aligned}$$

The constants $\bar{u}_1, \bar{v}_1, \bar{\omega}_1, \underline{u}_2, \underline{v}_1, \underline{\omega}_1$ constructed above satisfy the relation:

$$(3.5)\qquad \begin{aligned} &\underline{v}_1 \le \bar{v}_1, \underline{u}_1 \le \bar{u}_1,\\ &\underline{u}_1 \le \liminf_{t\to\infty} \min_{\bar{\Omega}} u(.,t) \le \limsup_{t\to\infty} \max_{\bar{\Omega}} u(.,t) \le \bar{u}_1,\\ &\underline{v}_1 \le \liminf_{t\to\infty} \min_{\bar{\Omega}} v(.,t) \le \limsup_{t\to\infty} \max_{\bar{\Omega}} v(.,t) \le \bar{v}_1, \end{aligned}$$

and the sequence $\{\bar{u}_i\}$,$\{\bar{v}_i\}$, $\{\bar{\omega}_i\}$, $\{\underline{u}_i\}$, $\{\underline{v}_i\}$, $\{\underline{\omega}_i\}$ satisfy

$$(3.6)\qquad \begin{aligned} &\underline{u}_i \le \liminf_{t\to\infty} \min_{\bar{\Omega}} u(.,t) \le \limsup_{t\to\infty} \max_{\bar{\Omega}} u(.,t) \le \bar{u}_i,\\ &\underline{v}_i \le \liminf_{t\to\infty} \min_{\bar{\Omega}} v(.,t) \le \limsup_{t\to\infty} \max_{\bar{\Omega}} v(.,t) \le \bar{v}_i,\\ &0 < \underline{u}_1 \le \underline{u}_2 \le \dots \le \underline{u}_{n-1} \le \underline{u}_n \le \bar{u}_n \le \bar{u}_{n-1} \le \dots \le \bar{u}_2 \le \bar{u}_1 = K_1,\\ &0 < \underline{v}_1 \le \underline{v}_2 \le \dots \le \underline{v}_{n-1} \le \underline{v}_n \le \bar{v}_n \le \bar{v}_{n-1} \le \dots \le \bar{v}_2 \le \bar{v}_1 = K_2,\\ &L_0 < \underline{\omega}_1 \le \underline{\omega}_2 \le \dots \le \underline{\omega}_{n-1} \le \underline{\omega}_n \le \bar{\omega}_n \le \bar{\omega}_{n-1} \le \dots \le \bar{\omega}_2 \le \bar{\omega}_1 = L_0 + K_1. \end{aligned}$$

Actually, since $1 + mL_0 > \beta K_2$, we have

$$K_1 = \bar{u}_1 \ge \underline{u}_1 = (1 - \frac{\beta \bar{v}_1}{1+m\underline{\omega}_1})K_1 = (1 - \frac{\beta K_2}{1+mL_0})K_1 > 0.$$

Then noting that $\eta K_1 < \delta$, we obtain

$$\bar{v}_2 = (1 - \frac{\eta}{\delta}\underline{u}_1)K_2 \ge \underline{v}_1 = (1 - \frac{\eta}{\delta}\bar{u}_1)K_2 > 0,$$
$$L_0 < \underline{\omega}_2 = L_0 + l\underline{u}_1 \le \bar{\omega}_1 = L_0 + l\bar{u}_1.$$

Thus

$$\underline{u}_2 = (1 - \frac{\beta \bar{v}_2}{1+m\underline{\omega}_2})K_1 \le \bar{u}_2 = (1 - \frac{\beta \underline{v}_1}{1+m\bar{\omega}_1})K_1 \le \bar{u}_1,$$

and

$$\underline{u}_2 = (1 - \frac{\beta \bar{v}_2}{1+m\underline{\omega}_2})K_1 \ge \underline{u}_1 = (1 - \frac{\beta \bar{v}_1}{1+m\underline{\omega}_1})K_1 > 0.$$

The above conclusions show that

$$0 < \underline{u}_1 \le \underline{u}_2 \le \bar{u}_2 \le \bar{u}_1 = K_1.$$

Use induction principle, it is not hard to prove that

$$0 < \underline{u}_1 \leq \underline{u}_2 \leq ... \leq \underline{u}_{n-1} \leq \underline{u}_n \leq \bar{u}_n \leq \bar{u}_{n-1} \leq ... \leq \bar{u}_2 \leq \bar{u}_1 = K_1.$$

Then using inductive principle, the proof of the remainder of (3.6) is immediately get from (3.5) and

$$\bar{v}_n = (1 - \frac{\eta}{\delta}\underline{u}_{n-1})K_2, \underline{v}_n = (1 - \frac{\eta}{\delta}\bar{u}_n)K_2 \text{ and } \underline{\omega}_n = L_0 + l\underline{u}_{n-1}, \bar{\omega}_n = L_0 + l\bar{u}_n.$$

By monotone bounds principle, we get

$$\lim_{n\to\infty} \bar{u}_n = \bar{u}, \quad \lim_{n\to\infty} \underline{u}_n = \underline{u}.$$

In (3.4), letting $n \to \infty$, we have

$$\bar{v} = (1 - \frac{\eta}{\delta}\underline{u})K_2, \underline{v} = (1 - \frac{\eta}{\delta}\bar{u})K_2, \tag{3.7}$$

$$\underline{\omega} = L_0 + l\underline{u}, \bar{\omega} = L_0 + l\bar{u}, \tag{3.8}$$

$$\underline{u} = (1 - \frac{\beta\bar{v}}{1 + m\underline{\omega}})K_1, \bar{u} = (1 - \frac{\beta\underline{v}}{1 + m\bar{\omega}})K_1. \tag{3.9}$$

It follows from (3.7) (3.8) that

$$\bar{\omega} - \underline{\omega} = l(\bar{u} - \underline{u}), \bar{v} - \underline{v} = \frac{\eta}{\delta}(\bar{u} - \underline{u}). \tag{3.10}$$

Next, we will prove that $\bar{u} = \underline{u}$, and hence $\bar{v} = \underline{v}, \bar{\omega} = \underline{\omega}$ by (3.10). Substituting the first equation of (3.7)and (3.8) to the first equation of (3.9), we get $\underline{u}$ satisfies the following equation:

$$ml\delta u^2 - \tau u - \delta K_1(1 + mL_0 - \beta K_2) = 0, \tag{3.11}$$

where

$$\tau = ml\delta K_1 + \beta\eta K_1 K_2 - \delta(1 + mL_0).$$

Similarly substituting the second equation of (3.7)and (3.8)to the second equation of (3.9), we get $\bar{u}$ also satisfies the equation (3.11). From [8], we known that under the condition

$$1 + mL_0 > \beta K_2 \text{ and } \eta K_1 < \delta,$$

E^* exists uniquely. This implies $\bar{u} = \underline{u} = u^*$. Consequently $\bar{v} = \underline{v} = v^*, \bar{\omega} = \underline{\omega} = \omega^*$. The fact combined with (3.6)implies

$$\lim_{t\to\infty} U(x,t) = E^*$$

uniformly on $\bar{\Omega}$. The proof is complete.

Next, we prove that if $1+mL_0 = \beta K_2$ and $\eta K_1 < \delta$, then E^* is globally stable. By Lemma 3.2, for any nonnegative initial function which is not identical to zero the solution of (1.1) is positive. therefore there exists $t^* > 0, \delta_i > 0, (i = 1, 2, 3)$ such that

$$\delta_i \leq \min\{u(t,x), v(t,x), \omega(t,x); t^* \leq t \leq t^* + \tau, x \in \bar{\Omega}\},$$

then $(\delta_1, \delta_2, \delta_3) \leq (u(t,x), v(t,x), \omega(t,x)) \leq (K_1, K_2, L_0 + lK_1)$ on $[t_1 - \tau, t_1] \times \bar{\Omega}$ where $t_1 = t^* + \tau$. Using $(u(t,x), v(t,x), \omega(t,x))$ as the initial function in the domain $[t_1 - \tau, \infty) \times \Omega$, similar to prove the first part of Theorem 2.1 above, we construct sequences $\{\bar{u}_i\}, \{\bar{v}_i\}, \{\bar{\omega}_i\}, \{\underline{u}_i\}, \{\underline{v}_i\}, \{\underline{\omega}_i\}$ as follows: $\bar{u}_1 = (1 - \frac{\delta_2}{1+m(L_0+K_1)})K_1, \bar{v}_1 = (1 - \frac{\eta}{\delta}\underline{u}_1)K_2, \bar{\omega}_1 = L_0 + l\bar{u}_1, \underline{u}_1 = (1 - \frac{\beta\bar{v}_1}{1+m\underline{\omega}_1})K_1, \underline{v}_1 = (1 - \frac{\eta}{\delta}\bar{u}_1)K_2, \underline{\omega}_1 = L_0$ and for $n > 1$

$$\begin{aligned} &\bar{v}_n = (1 - \frac{\eta}{\delta}\underline{u}_{n-1})K_2, \underline{v}_n = (1 - \frac{\eta}{\delta}\bar{u}_n)K_2, \underline{\omega}_n = L_0 + l\underline{u}_{n-1}, \\ &\bar{\omega}_n = L_0 + l\bar{u}_n, \underline{u}_n = (1 - \frac{\beta\bar{v}_n}{1+m\underline{\omega}_n})K_1, \bar{u}_n = (1 - \frac{\beta\underline{v}_{n-1}}{1+m\bar{\omega}_{n-1}})K_1. \end{aligned} \tag{3.12}$$

Consequently under condition $1 + mL_0 = \beta K_2$ and $\eta K_1 < \delta$ we have $\underline{u}_1 = (1 - \frac{\beta\bar{v}_1}{1+m\underline{\omega}_1})K_1 > 0, \underline{v}_1 = (1 - \frac{\eta}{\delta}\bar{u}_1)K_2 > 0, \underline{\omega}_1 = L_0 > 0$. By inductive principle, we can get the sequence $\{\bar{u}_i\}, \{\bar{v}_i\}, \{\bar{\omega}_i\}, \{\underline{u}_i\}, \{\underline{v}_i\}, \{\underline{\omega}_i\}$ satisfy (3.6) . Then letting $n \to \infty$, the following proof is similar to the proof of part one above, we omit it here. The proof of Theorem 2.1 is complete.

Proof of Theorem 2.2. For any $\varphi \in L^1(\Omega)$, we write $\bar{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi dx$. Let u, v, ω be any positive classical solution of (1.3). Multiplying the corresponding differential equation in (1.3) by $u - \bar{u}, v - \bar{v}, \omega - \bar{\omega}$ respectively, and then integrating over Ω by part, by the $\varepsilon-$ Young's inequality, we have

$$\begin{aligned} &d_1 \int_\Omega |\nabla(u - \bar{u})|^2 dx \\ =& \int_\Omega \{[\alpha u(1 - \frac{u}{K_1} - \frac{\beta v}{1+m\omega})] - [\alpha\bar{u}(1 - \frac{\bar{u}}{K_1} - \frac{\beta\bar{v}}{1+m\bar{\omega}})]\}(u - \bar{u})dx \\ \leq& \int_\Omega (\alpha|u - \bar{u}|^2 + \alpha\beta K_1|u - \bar{u}||v - \bar{v}| + \alpha\beta K_1 K_2|u - \bar{u}||\omega - \bar{\omega}|)dx \\ \leq& (\alpha + \varepsilon)\int_\Omega |u - \bar{u}|^2 dx + C_1(\varepsilon, \Lambda)\int_\Omega |v - \bar{v}|^2 + C_2(\varepsilon, \Lambda)\int_\Omega |\omega - \bar{\omega}|^2 dx, \\ &d_2 \int_\Omega |\nabla(v - \bar{v})|^2 dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \{[\delta v(1 - \frac{v}{K_2} - \frac{\eta u}{\delta})] - [\delta \bar{v}(1 - \frac{\bar{v}}{K_2} - \frac{\eta \bar{u}}{\delta})]\}(v - \bar{v})dx \\
&\leq \int_\Omega (\delta |v - \bar{v}|^2 + \eta K_2 |u - \bar{u}||v - \bar{v}|dx \\
&\leq \varepsilon \int_\Omega |u - \bar{u}|^2 dx + C_3(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx, \\
&\quad d_3 \int_\Omega |\nabla(\omega - \bar{\omega})|^2 dx \\
&= \int_\Omega [\gamma\omega(1 - \frac{\omega}{L_0 + lu}) - \gamma\bar{\omega}(1 - \frac{\bar{\omega}}{L_0 + l\bar{u}})](\omega - \bar{\omega})dx \\
&\leq \int_\Omega (\gamma|\omega - \bar{\omega}|^2 + \gamma(1 + \frac{l}{L_0} K_1)^2) l |u - \bar{u}||\omega - \bar{\omega}|dx \\
&\leq \varepsilon \int_\Omega |u - \bar{u}|^2 dx + C_4(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx + C_5(\varepsilon, \Lambda) \int_\Omega |\omega - \bar{\omega}|^2 dx.
\end{aligned}
$$

Here, we used Lemma 3.3. Consequently, there exists a sufficient small positive constant ε which only depends on Λ such that

$$
\begin{aligned}
&d_1 \int_\Omega |\nabla(u - \bar{u})|^2 dx + d_2 \int_\Omega |\nabla(v - \bar{v})|^2 dx + d_3 \int_\Omega |\nabla(\omega - \bar{\omega})|^2 dx \\
&\leq (\alpha + \varepsilon) \int_\Omega |u - \bar{u}|^2 dx + C(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx + C(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx.
\end{aligned}
$$

It follows from Poincar$\acute{e}$ inequality that

$$
\begin{aligned}
&\mu_1 [\int_\Omega d_1 |(u - \bar{u})|^2 dx + \int_\Omega d_2 |(v - \bar{v})|^2 dx + \int_\Omega d_3 |(\omega - \bar{\omega})|^2] dx \\
&\leq (\alpha + \varepsilon) \int_\Omega |u - \bar{u}|^2 dx + C(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx + C(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx.
\end{aligned}
$$

Since $\mu_1 d_1 > \alpha$ we may chose $\varepsilon > 0$ sufficient small, such that $\mu_1 d_1 > \alpha + \varepsilon$. Consequently, by above inequality, we have

$$
\begin{aligned}
&\mu_1 [\int_\Omega d_2 |(v - \bar{v})|^2 dx + \int_\Omega d_3 |(\omega - \bar{\omega})|^2] dx \\
&\leq C(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx + C(\varepsilon, \Lambda) \int_\Omega |v - \bar{v}|^2 dx.
\end{aligned}
$$

This implies that $v = \bar{v} = \omega = \bar{\omega} =$constant, and in turn $u = \bar{u} =$constant, if $d_2, d_3 > d := \frac{C(\varepsilon,\Lambda)}{\mu_1}$, which asserts our result (i).

The arguments of (ii)(iii) are rather similar to the ones given in the proof of (i), and are thus omitted.

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