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ERGODIC RETRACTIONS FOR SEMIGROUPS IN STRICTLY CONVEX BANACH SPACES

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Abstract. We study the existence of ergodic retractions for semigroups of mappings in strictly convex Banach spaces. We prove, for instance, the following theorem. Let $(X, \|\cdot\|)$ be a strictly convex Banach space and let Γ be a norming set for X. Let C be a bounded and convex subset of X, and suppose C is compact in the Γ-topology. If $\mathcal{S}$ is a right amenable semigroup, $\varphi = \{T_s : s \in \mathcal{S}\}$ is a semigroup on C with a nonempty set $F = F(\varphi)$ of common fixed points, and each T_s is (F-quasi-) nonexpansive, then there exists an (F-quasi-) nonexpansive retraction R from C onto F such that $RT_s = T_sR = R$ for each $s \in \mathcal{S}$, and every Γ-closed, convex and φ-invariant subset of C is also R-invariant.

1. Introduction and Preliminaries

Throughout this paper all Banach spaces are real. Let $(X, \|\cdot\|)$ be a Banach space and let Γ be a nonempty subspace of its dual X^*. If

$$\sup\{x^*(x) : x^* \in \Gamma, \|x^*\| = 1\} = \|x\|$$

for each $x \in X$, then we say that Γ is a norming set for X. In the sequel we will always assume that Γ is indeed a norming set for X.

It is obvious that a norming set generates a Hausdorff linear topology $\sigma(X, \Gamma)$ to which we shall refer as the Γ-topology. Directly from the definition of the Γ-topology we know that it is weaker than the weak topology. Next, it is easy to observe that the norm of X is lower semicontinuous with respect to the Γ-topology, namely,

$$\liminf_{\alpha} \|x_\alpha\| \geq \|x\|$$

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for any net $\{x_\alpha\}$ which converges to x in the Γ-topology.

Throughout this paper we use the following notation. If A is a subset of X, then ${}^{\Gamma}\overline{A}$ will stand for the closure of A in the Γ-topology and $\overline{A}^*$ will denote the weak* closure of A with respect to an appropriate predual. If $x^* \in X^*$ and $x \in X$, then the value of x^* at x will be denoted by $x^*(x)$ or by $\langle x, x^*\rangle$. If X is the dual space of a Banach space Z, then there is a natural embedding n of Z into X^*. To unify our notations we will write $\langle x, z\rangle$ instead of $\langle x, n(z)\rangle$ for each $z \in Z$ and each $x \in X$.

Our paper is motivated by certain recent results obtained by S. Saeidi [18]. We show that his theorems are valid under weaker assumptions. Namely, we replace the weak compactness of the subset C with Γ-compactness of this set and some additional properties of Γ. We refer the reader to [6] and [13] for more information concerning the Γ-topology and its applications.

We now recall several definitions and notations. A Banach space X is said to be strictly convex (or rotund) if $\|(x+y)/2\| < 1$ for every $x, y \in X$ such that $\|x\| = \|y\| = 1$ and $\|x-y\| > 0$ (see [10] and [17]).

Let $T : C \to C$, where $C \subset X$, be a mapping. We denote by $F(T)$ the set of all fixed points of T.

The mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The mapping $T : C \to C$ is said to be F-quasi-nonexpansive if $\|Tx - f\| \le \|x - f\|$ for all $x \in C$ and $f \in F \subset F(T)$. In the case where $F = F(T)$, the mapping T is called quasi-nonexpansive. We say that T is F-quasi-contractive if $\|Tx - f\| < \|x - f\|$ for all $x \in C \setminus F(T)$ and $f \in F \subset F(T)$. If $F = F(T)$ we say that T is quasi-contractive.

In our paper we study semigroups of (quasi-) nonexpansive mappings. Throughout the paper $\mathcal{S}$ is a semigroup and $B(\mathcal{S})$ is the space of all bounded real-valued functions defined on $\mathcal{S}$ with the supremum norm. For $s \in \mathcal{S}$ and $f \in B(\mathcal{S})$, we set

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for $t \in \mathcal{S}$. If Y is a subspace of $B(\mathcal{S})$ such that $1 \in Y$, then $\mu \in Y^*$ is said to be a mean on Y if $\|\mu\| = \mu(1) = 1$. We will sometimes write $\mu_t\,(f(t))$ instead of $\mu(f)$. The following theorem characterizes means [19].

Theorem 1.1. ([8], [11]). *Let Y be a subspace of $B(\mathcal{S})$ containing the constants and $\mu \in Y^*$. Then the following conditions are equivalent:*

(1) *μ is a mean on Y,*

(2) *the inequalities*

$$\inf_{s\in\mathcal{S}} f(s) \le \mu(f) \le \sup_{s\in\mathcal{S}} f(s)$$

hold for each $f \in Y$.

If Y is l_s-invariant (respectively, r_s-invariant), that is, if $l_s(Y) \subset Y$ (respectively, $r_s(Y) \subset Y$), then a mean μ is said to be left (respectively, right) invariant if $\mu(l_s f) = \mu(f)$ (respectively, $\mu(r_s f) = \mu(f)$) for any $s \in \mathcal{S}$. We say that a mean μ is invariant if it is both left and right invariant. Y is said to be (right) amenable provided there is a (right) invariant mean on Y. It is well known that if $\mathcal{S}$ is a commutative semigroup, then $B(\mathcal{S})$ is amenable [8]. A family $\varphi = \{T_s : s \in \mathcal{S}\}$ of self-mappings of C is a semigroup on C if $T_{ts} = T_t T_s$ for any $t, s \in \mathcal{S}$. The common fixed point set of φ will be denoted by $F(\varphi)$.

2. Main Results

We first prove the following lemma.

Lemma 2.1. *Suppose X_1 is the dual space of a Banach space Z and X_2 is a Banach space. Let $X = X_1 \times X_2$ be equipped with the l^2-norm, that is, $\|(x_1, x_2)\| = \sqrt{\|x_1\|_1^2 + \|x_2\|_2^2}$. Let $\Gamma = Z \times {X_2}^*$. Assume that $f : \mathcal{S} \to X$ is a function such that the Γ-closure of $conv\{f(s) : s \in \mathcal{S}\}$ is Γ-compact and let Y be a subspace of $B(\mathcal{S})$ containing all functions of the form $s \mapsto \langle f(s), x^*\rangle$ with $x^* \in \Gamma$. Then for any $\mu \in Y^*$, there exists a unique element $f_\mu \in X$ such that*

$$\langle f_\mu, x^*\rangle = \mu_s \langle f(s), x^*\rangle$$

for all $x^* \in \Gamma$. Moreover, if μ is a mean on Y, then

$$f_\mu \in C_0 = {}^{\Gamma}\overline{conv\,\{f(s) : s \in \mathcal{S}\}}.$$

(*Sometimes we will denote f_μ by $\int f(s) d\mu(s)$.*)

Proof. We define a linear functional $\tilde{f}$ on Γ by setting

$$\tilde{f}(x^*) = \mu_s \langle f(s), x^*\rangle$$

for each $x^* \in \Gamma$. Since

$$\begin{aligned} |\tilde{f}(x^*)| &= |\mu_s \langle f(s), x^*\rangle| \le sup_{s\in\mathcal{S}} |\langle f(s), x^*\rangle| \|\mu\| \\ &\le (sup_{s\in\mathcal{S}} \|f(s)\|) \|x^*\| \|\mu\| \end{aligned}$$

for every $x^* \in \Gamma$, $\tilde{f}$ is continuous, and therefore writing $\tilde{f} = \langle \tilde{f}_1, \tilde{f}_2\rangle$, we have $\tilde{f}_1 \in X_1$ and $\tilde{f}_2 \in {X_2}^{**}$. We need to show that $\tilde{f}_2 \in X_2$. By assumption, the set $C_0 = {}^{\Gamma}\overline{conv\,\{f(s) : s \in \mathcal{S}\}}$ is Γ-compact and convex. Therefore the sets $C_1 = \{\|\mu\| x : x \in C_0\}$, $C_2 = \{ry : 0 \le r \le 1, y \in C_1\}$ and $C_3 = C_2 - C_2$ are also Γ-compact and convex. Moreover, the latter set C_3 is circled. Let n_2 be the natural embedding of X_2 into ${X_2}^{**}$. Setting $n(x) = n(x_1, x_2) = (x_1, n_2(x_2))$ for

$(x_1, x_2) \in X = X_1 \times X_2$, we get the embedding of $X = X_1 \times X_2$ into $X_1 \times X_2{}^{**}$. So, the set $n(C_3) \subset X_1 \times X_2{}^{**}$ is also convex, circled and compact in the Γ-topology on $X_1 \times X_2{}^{**}$. Now it is sufficient to prove that $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \in n(C_3)$. If not, then by the separation theorem, there exists $x^* \in \Gamma$ such that

$$\tilde{f}(x^*) > \sup\{|\langle x^*, c_0\rangle)| : c_0 = (c_1, c_{2,0}^{**}) \in n(C_3) \subset X_1 \times X_2^{**}\}.$$

But we also have

$$\begin{aligned}
&\sup\{|\langle x^*, c_0\rangle| : c_0 = (c_1, c_{2,0}^{**}) \in n(C_3) \subset X_1 \times X_2{}^{**}\} \\
&= \sup\{|\langle c, x^*\rangle| : c = (c_1, c_2) \in C_3 \subset X_1 \times X_2\} \\
&\geq \sup\{|\langle c, x^*\rangle| : c = (c_1, c_2) \in C_2 \subset X_1 \times X_2\} \\
&= \sup\{r|\langle c, x^*\rangle| : 0 \leq r \leq 1, c = (c_1, c_2) \in C_1 \subset X_1 \times X_2\} \\
&\geq \sup\{\|\mu\||\langle f(s), x^*\rangle| : s \in \mathcal{S}\} \\
&= \|\mu\| sup_{s \in \mathcal{S}} |\langle f(s), x^*\rangle| \\
&\geq \left| \int \langle f(s), x^*\rangle d\mu(s) \right| = \tilde{f}(x^*),
\end{aligned}$$

that is, an inequality in the opposite sense. Hence for each $\mu \in Y^*$, there exists a unique $f_\mu \in X$ such that

$$\langle f_\mu, x^*\rangle = \mu_s\langle f(s), x^*\rangle = \int \langle f(s), x^*\rangle d\mu(s)$$

for all $x^* \in \Gamma$.

Now assume that μ is a mean and suppose that $f_\mu \notin C_0 = {}^{\Gamma}\overline{conv\,\{f(s) : s \in \mathcal{S}\}}$. Then by the separation theorem, there exists $x^* \in \Gamma$ such that

$$\langle f_\mu, x^*\rangle < \inf\{\langle x, x^*\rangle : x \in C_0\}.$$

Consequently, by Theorem 1.1, we get the following contradiction:

$$\begin{aligned}
\inf\{\langle f(s), x^*\rangle : s \in \mathcal{S}\} &\leq \mu_s\langle f(s), x^*\rangle = \langle f_\mu, x^*\rangle \\
&< \inf\{\langle x, x^*\rangle : x \in C_0\} \\
&\leq \inf\{\langle f(s), x^*\rangle : s \in \mathcal{S}\}.
\end{aligned}$$

Hence $f_\mu \in C_0 = {}^{\Gamma}\overline{conv\,\{f(s) : s \in \mathcal{S}\}}$. ∎

Taking $X_1 = \{0\}$ and $X_2 = \{0\}$, respectively, we get the following corollaries.

Corollary 2.2. ([12], [14] and [15]). *Suppose X is a Banach space. Let $f : \mathcal{S} \to X$ be a function such that the closure of $conv\{f(s) : s \in \mathcal{S}\}$ is w-compact and let Y be a subspace of $B(\mathcal{S})$ containing all functions $s \mapsto \langle f(s), x^*\rangle$*

with $x^* \in X^*$*. Then for any* $\mu \in Y^*$*, there exists a unique element* $f_\mu \in X$ *such that*

$$\langle f_\mu, x^* \rangle = \mu_s \langle f(s), x^* \rangle$$

for all $x^* \in X^*$*. Moreover, if* μ *is a mean on* Y*, then*

$$f_\mu \in \overline{conv\,\{f(s) : s \in \mathcal{S}}.$$

Corollary 2.3. *Suppose* X *is the dual space of a Banach space* Z*. Let* $f : \mathcal{S} \to X$ *be a function such that the* w^**-closure of* $conv\{f(s) : s \in \mathcal{S}\}$ *is* w^**-compact and let* Y *be a subspace of* $B(\mathcal{S})$ *containing all functions* $s \mapsto \langle f(s), z \rangle$ *with* $z \in Z$*. Then for any* $\mu \in Y^*$*, there exists a unique element* $f_\mu \in X$ *such that*

$$\langle f_\mu, z \rangle = \mu_s \langle f(s), z \rangle$$

for all $z \in Z$*. Moreover, if* μ *is a mean on* Y*, then*

$$f_\mu \in \overline{conv\,\{f(s) : s \in \mathcal{S}\}}^{\,*}.$$

Lemma 2.1 implies that in our applications the Γ-topology can be different from both the weak and the weak* topologies. In our next lemma we need a Γ-topology for which the conclusion of Lemma 2.1 holds. Therefore we introduce the following definition.

Definition 2.1. Let X be a Banach space and let Γ be a norming set. Let Y be a subspace of $B(\mathcal{S})$ containing all functions of the form $s \mapsto \langle f(s), x^* \rangle$ for each $x^* \in \Gamma$ and for each function $f : \mathcal{S} \to X$ such that the Γ-closure of $conv\{f(s) : s \in \mathcal{S}\}$ is Γ-compact. If the following two conditions are satisfied:

(a) for any $\mu \in Y^*$, there exists a unique element $f_\mu \in X$ such that

$$\langle f_\mu, x^* \rangle = \mu_s \langle f(s), x^* \rangle$$

for all $x^* \in \Gamma$;

(b) if μ is a mean on Y, then

$$f_\mu \in C_0 = {}^{\Gamma}\,\overline{conv\,\{f(s) : s \in \mathcal{S}\}},$$

then we say that Γ is mean-admissible or that the Γ-topology is mean-admissible.

Before stating the next lemma, we exhibit a nontrivial example of a mean-admissible Γ-topology on a Banach space X. To apply this example to our lemma, we need, in addition, the strict convexity of X.

Example 2.1. It is known that the Banach space $C([0,1],\mathbb{R})$ does not have a predual space under any renorming (see [1] and [16]), and that $C([0,1],\mathbb{R})$ can be renormed to be a strictly convex Banach space. Also, the Banach space l^1 can be renormed to be a strictly convex Banach space and c_0 with a suitable norm is still a predual of l^1 with the new norm. Now, taking the Cartesian product of these two spaces with the l^2-norm, we get a strictly convex Banach space for which $\Gamma = c_0 \times C([0,1],\mathbb{R})^*$ is a norming set. It is easy to observe that in $l^1 \times C([0,1],\mathbb{R})$ one can find nontrivial, convex and norm-bounded subsets which are compact in the Γ-topology. Clearly, by Lemma 2.1, this Γ-topology is mean-admissible.

Now we can state and prove our second lemma.

Lemma 2.4. *Suppose X is a Banach space and Γ is mean-admissible. Let C be a nonempty Γ-closed convex subset of X and let μ be a mean on X. Let $\mathcal{S}$ be a semigroup and let $\varphi = \{T_s : s \in \mathcal{S}\}$ be an (F-quasi-) nonexpansive semigroup on C such that the Γ-closure of $conv\{T_s x : s \in \mathcal{S}\}$ is Γ-compact for each $x \in C$. Suppose also that Y is a subspace of $B(\mathcal{S})$ containing all functions $s \mapsto \langle T_s x, x^*\rangle$ with $x \in C$ and $x^* \in \Gamma$, and such that $1 \in Y$. If $T_\mu x = \int T_s x d\mu(s)$, then the following conditions are satisfied:*

(i) *T_μ is an (F-quasi-) nonexpansive self-mapping of C.*

(ii) *$T_\mu x = x$ for each $x \in F(\varphi)$.*

(iii) *$T_\mu x \in {}^{\Gamma}\overline{conv\,\{T_s x : s \in \mathcal{S}\}}$ for each $x \in C$.*

(iv) *If Y is r_s-invariant for each $s \in \mathcal{S}$, then $T_\mu T_s = T_\mu$ for any $s \in \mathcal{S}$.*

(v) *If Y is r_s-invariant for each $s \in \mathcal{S}$, $F = F(\varphi)$ and φ is an F-quasi-contractive semigroup on C, then T_μ is also F-quasi-contractive.*

(vi) *If the mappings in φ are affine, then so is T_μ.*

Proof. Let T be nonexpansive. For $\epsilon > 0$ and $x, y \in C$, there exists $x^* \in \Gamma$ such that $\|x^*\| = \|T_\mu x - T_\mu y\|$ and $\|T_\mu x - T_\mu y\|^2 \le \langle T_\mu x - T_\mu y, x^*\rangle + \epsilon$. Thus we have

$$\begin{aligned}
\|T_\mu x - T_\mu y\|^2 &\le \langle T_\mu x - T_\mu y, x^*\rangle + \epsilon \\
&= \mu_s \langle T_s x - T_s y, x^*\rangle + \epsilon \\
&\le \sup_{s\in\mathcal{S}} \|T_s x - T_s y\| \|T_\mu x - T_\mu y\| + \epsilon \\
&\le \|x - y\| \|T_\mu x - T_\mu y\| + \epsilon
\end{aligned}$$

and therefore T_μ is nonexpansive.

If T is F-quasi-nonexpansive and $f \in F$, then for each $\epsilon > 0$ and $x \in C$, there exists $x^* \in \Gamma$ such that $\|x^*\| = \|T_\mu x - f\|$ and $\|T_\mu x - f\|^2 \leq \langle T_\mu x - f, x^* \rangle + \epsilon$. To see that T_μ is F-quasi-nonexpansive, observe that

$$\begin{aligned}\|T_\mu x - f\|^2 &\leq \langle T_\mu x - f, x^* \rangle + \epsilon \\ &= \mu_s \langle T_s x - f, x^* \rangle + \epsilon \\ &\leq \sup_{s \in \mathcal{S}} \|T_s x - f\| \|T_\mu x - f\| + \epsilon \\ &\leq \|x - f\| \|T_\mu x - f\| + \epsilon.\end{aligned}$$

Thus the proof of (i) is complete.

To prove (ii), take $x \in F(\varphi)$ and $x^* \in \Gamma$. Then we have

$$\langle T_\mu x, x^* \rangle = \mu_s \langle T_s x, x^* \rangle = \mu_s \langle x, x^* \rangle = \langle x, x^* \rangle.$$

By the definition of mean admissibility of Γ we get (iii).

To show (iv), it is sufficient to observe that

$$\langle T_\mu(T_s x), x^* \rangle = \mu_{s_1} \langle T_{s_1 s} x, x^* \rangle = \mu_{s_1} \langle T_{s_1} x, x^* \rangle = \langle T_\mu x, x^* \rangle$$

for each $x \in C$ and each $x^* \in \Gamma$.

To prove (v), observe that if $x \notin F$, then there exists $s_0 \in \mathcal{S}$ such that $T_{s_0} x \notin F$ and by (i), (iv) and the F-quasi-contractivity of T_{s_0} this yields

$$\|T_\mu x - f\| = \|T_\mu T_{s_0} x - f\| \leq \|T_{s_0} x - f\| < \|x - f\|$$

for each $f \in F$.

If φ is affine, then for $x_1, x_2 \in C$, $0 \leq \alpha \leq 1$ and $x^* \in \Gamma$, we obtain

$$\begin{aligned}\langle T_\mu(\alpha x_1 + (1-\alpha) x_2), x^* \rangle &= \mu_s \langle T_s(\alpha x_1 + (1-\alpha) x_2), x^* \rangle \\ &= \alpha \mu_s \langle T_s x_1, x^* \rangle + (1-\alpha) \mu_s \langle T_s x_2, x^* \rangle \\ &= \alpha \langle T_\mu x_1, x^* \rangle + (1-\alpha) \langle T_\mu x_2, x^* \rangle \\ &= \langle \alpha T_\mu x_1 + (1-\alpha) T_\mu x_2, x^* \rangle\end{aligned}$$

and this completes the proof of the lemma. ∎

We are now able to establish a nonlinear ergodic theorem using Bruck's method ([2-4] and [5]) and some ideas from [18].

Theorem 2.5. *Suppose X is a Banach space and Γ is mean-admissible. Let C be a nonempty locally Γ-compact and convex subset of X, and let μ be a right invariant mean on X. Let $\mathcal{S}$ be a semigroup and let $\varphi = \{T_s : s \in \mathcal{S}\}$ be a semigroup on C such that $F(\varphi) \neq \emptyset$. Suppose also that $Y \subset B(\mathcal{S})$ is an r_s-invariant subspace for any $s \in \mathcal{S}$ containing all functions $s \mapsto \langle T_s x, x^* \rangle$ with $x \in C$ and $x^* \in \Gamma$, and such that $1 \in Y$.*

(a) *If $F = F(\varphi)$ and every mapping T_s is F-quasi-contractive, then there is an F-quasi-contractive retraction $R : C \to F(\varphi)$ such that $RT_s = T_sR = R$ for any $s \in \mathcal{S}$, and each Γ-closed and convex φ-invariant subset of C is also R-invariant.*

(b) *If X is strictly convex and every mapping T_s is (F-quasi-) nonexpansive, then there is an (F-quasi-) nonexpansive retraction $R : C \to F(\varphi)$ such that $RT_s = T_sR = R$ for any $s \in \mathcal{S}$, and each Γ-closed and convex φ-invariant subset of C is also R-invariant.*

(c) *If every mapping T_s is continuous and affine, and either condition (a) or (b) is satisfied, then R is also affine.*

Proof. Without loss of generality we may assume that C is Γ-compact. Suppose that $\varphi = \{T_s : s \in \mathcal{S}\}$ is an F-quasi-nonexpansive semigroup on C. Let the family $\mathcal{N}$ consist of all $T \in C^C$ such that

$$\|Tx - f\| \leq \|x - f\| \quad \text{for any} \quad x \in C,\ f \in F,$$

$$TT_s = T \quad \text{for any} \quad s \in \mathcal{S},$$

and such that every Γ-closed and convex φ-invariant subset of C is also T-invariant. Note that by Lemma 2.4 we get $T_\mu \in \mathcal{N}$. Next, observe that by Tychonoff's theorem, C^C with the product topology generated by the Γ-topology on C is compact [9]. Since the norm is lower semicontinuous with respect to the Γ-topology, we see that $\mathcal{N}$ is also compact in the Γ-topology. Let us preorder $\mathcal{N}$ in the following way:

$$U \preceq V \quad \text{if and only if} \quad \|U(x) - f\| \leq \|V(x) - f\|$$

for any $x \in C$ and $f \in F$. Using once more the lower semicontinuity of the norm with respect to the Γ-topology and Zorn's lemma, we conclude that $\mathcal{N}$ contains a minimal element R with respect to the above preordering. This means that if $T \in \mathcal{N}$ and

$$\|Tx - f\| \leq \|Rx - f\|$$

for each $x \in C$ and $f \in F$, then $\|Tx - f\| = \|Rx - f\|$. Since T_sR belongs to $\mathcal{N}$, $\|T_sRx - f\| \leq \|Rx - f\|$ for any $x \in C$, and T_sR is F-quasi-contractive for each $s \in \mathcal{S}$, we see that $T_sRx = Rx$ for every $x \in C$ and $s \in \mathcal{S}$, that is, $Rx \in F$. We also have $T_sR = RT_s = R$. Thus R is indeed an F-quasi-contractive retraction.

When X is strictly convex, we repeat our arguments and in the end instead of the F-quasi-contractivity of T_s we apply the inequalities $\|T_sRx - f\| \leq \|Rx - f\|$ and $\|\frac{1}{2}(T_sRx + Rx) - f\| \leq \|Rx - f\|$ to get $T_sRx = Rx$ for every $x \in C$ and $s \in \mathcal{S}$ by appealing to the strict convexity of X. Hence R is an F-quasi-nonexpansive retraction, as reuqired.

For affine mappings we proceed as above. We only replace the family $\mathcal{N}$ by the family $\widetilde{\mathcal{N}}$ consisting of all affine $T \in C^C$ such that

$$\|Tx - f\| \leq \|x - f\| \quad \text{for any} \quad x \in C,\ f \in F,$$

$$TT_s = T \quad \text{for any} \quad s \in \mathcal{S},$$

and such that every Γ-closed, convex and φ-invariant subset of C is also T-invariant. Note that by Lemma 2.4 we get $T_\mu \in \widetilde{\mathcal{N}}$. Hence there exists an F-quasi-nonexpansive affine retraction $R : C \to F(\varphi)$ such that $T_s R = RT_s = R$ for each $s \in \mathcal{S}$, and each Γ-closed, convex and φ-invariant subset of C is also R-invariant.

If φ is a nonexpansive semigroup on C, then the proof is analogous. Thus the proof of our theorem is complete. ■

Finally, we mention that a recent general result on intersections of nonexpansive retracts in strictly convex Banach spaces can be found in [7].

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