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GENERIC WELL-POSEDNESS FOR PERTURBED OPTIMIZATION PROBLEMS IN BANACH SPACES

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Abstract. Let X be a Banach space and Z a relatively weakly compact subset of X. Let $J : Z \to \mathbb{R}$ be a upper semicontinuous function bounded from above and $p \geq 1$. This paper is concerned with the perturbed optimization problem of finding $z_0 \in Z$ such that $\|x - z_0\|^p + J(z_0) = \sup_{z\in Z}\{\|x - z\|^p + J(z)\}$, which is denoted by $\max_J(x, Z)$. We prove in the present paper that if X is Kadec w.r.t. Z, then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is generalized well-posed is a dense G_δ-subset of X. If X is additionally J-strictly convex w.r.t. Z and $p > 1$, we prove that the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is well-posed is a dense G_δ-subset of X.

1. INTRODUCTION

Let X be a real Banach space endowed with the norm $\|\cdot\|$. Let Z be a nonempty closed subset of X, $J : Z \to \mathbb{R}$ a function defined on Z and let $p \geq 1$. The perturbed optimization problem considered here is of finding an element $z_0 \in Z$ such that

$$\|x - z_0\|^p + J(z_0) = \sup_{z\in Z}\{\|x - z\|^p + J(z)\} \tag{1.1}$$

which is denoted by $\max_J(x, Z)$. Any point z_0 satisfying (1.1) (if exists) is called a solution of the problem $\max_J(x, Z)$. In particular, if $J \equiv 0$, then the perturbed optimization problem $\max_J(x, Z)$ reduces to the well-known furthest point problem.

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The perturbed optimization problem $\max_J(x, Z)$ was presented and investigated by Baranger in [3, 4] for the case when $p = 1$, and by Bidaut in [5] for the case when $p \geq 1$. The existence results have been applied to optimal control problems governed by partial differential equations, see for example, [2, 3, 5, 6, 7, 8, 12, 21].

Let Z be a bounded closed subset of X and let J be a upper semicontinuous, bounded from above on Z. In the case when $p = 1$, Baranger proved in [4] that if X is a reflexive and locally uniformly convex Banach space then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ has a solution is a dense G_δ-subset of X. This result extends Edelstein's [14] and Asplund's [1] results on farthest points. In the recent paper [24], we extended this result to the setting of nonreflexive Banach spaces, and established porosity results. Consider the problem in an arbitrary Banach space, Cobzas proved in [8] that if Z is a weakly compact subset of X, then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ has a solution is a dense G_δ-subset of X, which extends Lau's result in [17].

In the case when $p > 1$, this kind of perturbed optimization problems was studied by Bidaut in [5]. Recall that a sequence $\{z_n\} \subseteq Z$ is a maximizing sequence of the problem $\max_J(x, Z)$ if

$$\lim_{n\to\infty} (\|x - z_n\|^p + J(z_n)) = \sup_{z\in Z}(\|x - z\|^p + J(z)),$$

and that the problem $\max_J(x, Z)$ is well-posed if $\max_J(x, Z)$ has a unique solution and any maximizing sequence of the problem $\max_J(x, Z)$ converges to the solution. Bidaut proved that if X is a reflexive, strictly convex and Kadec Banach space, then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is well-posed is a dense G_δ-subset of X. The approach used there depends closely on the reflexivity property of the underlying space X. The corresponding perturbed minimization problems have be studied extensively, and the reader is referred to [2, 5, 8, 9, 18, 19, 23, 24] and the references there.

The purpose of the present paper is to extend the results due to Bidaut in [5] to the general setting of nonreflexive Banach spaces. More precisely, assume that Z is a relatively weakly compact subset of X and X is Kadec w.r.t.Z. Then we show in the present paper that the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is generalized well-posed is a dense G_δ-subset of X. If X is additionally J-strictly convex w.r.t.Z and if $p > 1$, then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is well-posed is a dense G_δ-subset of X. It should be noted that, as it will be seen, such an extension is nontrivial. A similar work was done for the case of minimization problems in the recent paper [25], where the main technique is the use of the Hölder inequality. However, this technique used there doesn't work here because $J(z)$ is negative for most points $z \in Z$, which makes the maximization problem more complicated.

2. Preliminaries

We begin with some standard notations. Let X be a Banach space with the dual X^*. We use $\langle \cdot, \cdot \rangle$ to denote the inner product connecting X^* and X. The closed (resp. open) ball in X at center x with radius r is denoted by $\mathbf{B}_X(x, r)$ (resp. $\mathbf{U}(x, r)$) while the corresponding sphere by $\mathbf{S}_X(x, r)$. In particular, we write $\mathbf{B}_X = \mathbf{B}_X(0, 1)$ and $\mathbf{S}_X = \mathbf{S}_X(0, 1)$. Sometimes, the subscripts are omitted if no confusion caused. For a subset A of X, the linear hull and the closure of A are respectively denoted by $\operatorname{span} A$ and $\overline{A}$. For $x \in X$, the distance from x to A is denoted by $d(x, A)$ and defined by $d(x, A) := \inf_{a \in A} \|x - a\|$.

Let Z be a subset of X and J be a real-valued function on Z. We introduce the following definition, where items **(i)** and **(ii)** are well-known in [11, 22], while items **(iii)-(v)** are extensions of **(i)** and **(ii)**, which were first introduced in [25].

Definition 2.1. X is said to be

(i) strictly convex if, for any $x_1, x_2 \in \mathbf{S}$, the condition $\|x_1 + x_2\| = 2$ implies that $x_1 = x_2$;

(ii) (sequentially) Kadec if, for any sequence $\{x_n\} \subseteq \mathbf{S}, x \in \mathbf{S}$, the condition $x_n \to x$ weakly implies that $\lim_{n\to\infty} \|x_n - x\| = 0$.

(iii) J-strictly convex with respect to (w.r.t) Z, if, for any $z_1,\ z_2 \in Z$ such that $\|x - z_1\| = \|x - z_2\|$ for some $x \in X$, the conditions that $\|x - z_1 + x - z_2\| = \|x - z_1\| + \|x - z_2\|$ and $J(z_1) = J(z_2)$ imply that $z_1 = z_2$;

(iv) J-strictly convex, if X is J-strictly convex w.r.t X;

(v) (sequentially) Kadec with respect to (w.r.t) Z, if, for any sequence $\{z_n\} \subseteq Z$ and $z_0 \in Z$ such that there exists a point $x \in X$ satisfying $\lim_{n\to+\infty} \|x - z_n\| = \|x - z_0\|$, the condition $z_n \to z_0$ weakly implies that $\lim_{n\to\infty} \|z_n - z_0\| = 0$.

In the case when $Z = X$, the Kadec property w.r.t Z reduces to the Kadec property, while in the case when $J \equiv 0$, the J-strict convexity w.r.t Z reduces to the strict convexity w.r.t Z. Moreover, the following implications are clear for any subset Z of X and real-valued function J on Z:

$$\begin{aligned} &\text{the strict convexity} \Longrightarrow \text{the } J\text{-strict convexity} \\ &\qquad \Longrightarrow \text{the } J\text{-strict convexity w.r.t.} Z \end{aligned} \tag{2.1}$$

and

$$\text{the Kadec property} \Longrightarrow \text{the Kadec property w.r.t. } Z. \tag{2.2}$$

It should be noted that each converse of implications (2.1) and (2.2) doesn't hold, in general, see [25, Examples 2.1 and 2.2].

The following two propositions are known (see [26] for the fist one and [13] for the second one) and play an important role for our study. Recall that a real-valued function f on an open subset $D \subseteq X$ is Fréchet differentiable at $x \in D$ if there exists $x^* \in X^*$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\|y - x\|} = 0.$$

The element x^* is called the Fréchet differential at x and is denoted by $\mathrm{D}f(x)$.

Proposition 2.1. *Let f be a locally Lipschitz continuous function on an open subset D of X. Suppose that X is a reflexive Banach space. Then f is Fr échet differentiable on a dense subset of D.*

Proposition 2.2. *Let A be a weakly compact subset of a Banach space X and let $Y = \overline{\operatorname{span} A}$. Then there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T : R \to Y$ such that $T(\mathbf{B}_R) \supseteq A$.*

3. Generic Existence and Well-posedness Results

Let $p \geq 1$. For the remainder of the present paper, we always assume that Z is a nonempty bounded closed subset of X, $J : Z \to \mathbb{R}$ is a upper semicontinuous function bounded from above. Furthermore, without loss of generality, we also assume that

$$\sigma := \sup_{z \in Z} J(z) > 0. \tag{3.1}$$

Hence,

$$\sup_{z \in Z} (\|x - z\|^p + J(z)) \geq \sigma > 0 \quad \text{for each } x \in X. \tag{3.2}$$

Define functions $\xi : X \times Z \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$ respectively by

$$(x, z) = \begin{cases} \{\|x - z\|^p + J(z)\}^{\frac{1}{p}} & \text{if } \|x - z\|^p + J(z) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$
$$\text{for each } (x, z) \in X \times Z.$$

and

$$\psi(x) = \sup_{z \in Z} \xi(x, z) \quad \text{for each } x \in X. \tag{3.4}$$

Then, $z_0 \in Z$ is a solution to the problem $\max_J(x, Z)$ if and only if z_0 satisfies that

$$\xi(x, z_0) = \sup_{z \in Z} \xi(x, z) = \psi(x). \tag{3.5}$$

The set of all solutions to the problem $\max_J(x, Z)$ is denoted by $F_{Z,J}(x)$, that is,

$$F_{Z,J}(x) := \{z_0 \in Z : \ \xi(x, z_0) = \psi(x)\} = \{z_0 \in Z : \ \{\|x - z_0\|^p + J(z_0)\}^{\frac{1}{p}} = \psi(x)\}.$$

Again define the function $b : X \mapsto \mathbb{R}$ by

$$b(x) = \lim_{\delta \to 0^+} \inf_{z \in Z^J(x,\delta)} \|x - z\| \quad \text{for each } x \in X, \tag{3.6}$$

where, for each $x \in X$ and each $\delta > 0$,

$$Z^J(x, \delta) = \{z \in Z : \xi(x, z) > \psi(x) - \delta\}. \tag{3.7}$$

Obviously, the function b is Lipschitz continuous.

Lemma 3.1. *Let $\lambda > 0$ and $x \in X$. There exists $L > 0$ such that*

$$|\xi(y, z) - \xi(x, z)| \le L\|y - x\|^{\frac{1}{p}} \quad \textit{for any } y \in \mathbf{B}(x, \lambda) \textit{ and } z \in Z. \tag{3.8}$$

Proof. Let $s \ge 0$ and $t \ge 0$. We first note the following elementary inequalities:

$$|s^{\frac{1}{p}} - t^{\frac{1}{p}}| \le |s - t|^{\frac{1}{p}} \quad \text{and} \quad |s^p - t^p| \le p \max\{s, t\} \, |s - t|. \tag{3.9}$$

Let x, $y \in X$ and $z \in Z$. We claim that

$$|\xi(y, z) - \xi(x, z)| \le |\|x - z\|^p - \|y - z\|^p|^{\frac{1}{p}}. \tag{3.10}$$

To verify this claim, without loss of generality, assume $\|x - z\|^p + J(z) > 0$. Thus, if $\|y - z\|^p + J(z) > 0$, then (3.10) follows directly from the first inequality of (3.9) (with $\|y - z\|^p + J(z)$ and $\|x - z\|^p + J(z)$ in place of s and t respectively). Now assume $\|y - z\|^p + J(z) \le 0$, then $J(z) < -\|y - z\|^p$ and

$$0 < \|x - z\|^p + J(z) < \|x - z\|^p - \|y - z\|^p.$$

Hence

$$|\xi(y, z) - \xi(x, z)| = |\|x - z\|^p + J(z)|^{\frac{1}{p}} \le |(\|x - z\|^p - \|y - z\|^p)|^{\frac{1}{p}}. \tag{3.11}$$

Hence the claim (3.10) holds. Since Z is bounded, it follows that $\Delta := \sup_{z\in Z}\|x-z\| < +\infty$. Thus applying the second inequality of (3.9) (to $\|x-z\|$ and $\|y-z\|$ in place of s and t respectively), we deduce from (3.10) that

$$|\xi(x,z)-\xi(y,z)| \le |\|x-z\|^p - \|y-z\|^p|^{\frac{1}{p}} \le (p\,(\Delta+\lambda))^{\frac{1}{p}}\|x-y\|^{\frac{1}{p}}. \tag{3.12}$$

This means that (3.8) holds with $L := (p\,(\Delta+\lambda))^{\frac{1}{p}}$ and the proof is complete. ■

The following lemma shows that the function ψ is locally Lipschitz on X.

Lemma 3.2. *Let $x \in X$. There are $\lambda > 0$ and $L > 0$ such that*

$$|\psi(y)-\psi(x)| \le L\|y-x\| \quad \textit{for each } y \in \mathbf{B}(x,\lambda). \tag{3.13}$$

Proof. It suffices to verify that there exist $\lambda > 0$ and $L > 0$ such that

$$\psi(x)-\psi(y) \le L\|x-y\| \quad \text{for each } y \in \mathbf{B}(x,\lambda). \tag{3.14}$$

Let $\sigma > 0$ be given by (3.1). Then, by (3.2), $\psi(x) \ge \sigma$ holds for each $x \in X$. Let $y \in X$ and $r > 0$. Set

$$\Gamma(y,r) = \{z \in Z : \xi(y,z) > r\}.$$

Since

$$\xi(x,z) \le \frac{3}{4}\sigma < \psi(x) \quad \text{for each } z \in Z\setminus\Gamma(x,\frac{3}{4}\sigma),$$

it follows that

$$\Gamma\left(x,\frac{3}{4}\sigma\right) \ne \emptyset \quad \text{and} \quad \sup_{z\in\Gamma(x,\frac{3}{4}\sigma)} \xi(x,z) = \psi(x). \tag{3.15}$$

By Lemma 3.1, there exist $\lambda_1 > 0$ and $L_1 > 0$ such that (3.8) holds. Let $\lambda = \left(\frac{\sigma}{4L_1}\right)^{\frac{1}{p}}$. Then for each $y \in \mathbf{B}(x,\lambda)$ and $z \in Z(x,\frac{3}{4}\sigma)$, we have

$$\xi(y,z) > \xi(x,z) - L_1\|x-y\|^p > \frac{3}{4}\sigma - \frac{1}{4}\sigma = \frac{1}{2}\sigma. \tag{3.16}$$

That is, $\Gamma(x,\frac{3}{4}\sigma) \subseteq \Gamma(y,\frac{1}{2}\sigma)$ for each $y \in \mathbf{B}(x,\lambda)$. Write $\Delta := \sup_{z\in Z}\|x-z\| < \infty$. Let $z \in \Gamma(x,\frac{3}{4}\sigma)$ and $y \in \mathbf{B}(x,\lambda)$. Then, by the Mean-Valued Theorem, there exists θ satisfying

$$\min\{\|x-z\|,\|y-z\|\} \le \theta \le \max\{\|x-z\|,\|y-z\|\} \le \Delta+\lambda \tag{3.17}$$

such that

$$\begin{aligned}\xi(x,z)-\psi(y) &\leq \{\|x-z\|^p+J(z)\}^{\frac{1}{p}}-\{\|y-z\|^p+J(z)\}^{\frac{1}{p}}\\ &= (\theta^p+J(z))^{\frac{1-p}{p}}\theta^{p-1}(\|x-z\|-\|y-z\|).\end{aligned} \tag{3.18}$$

By (3.17) and the fact that $z\in\Gamma(x,\frac{3}{4}\sigma)\subseteq\Gamma(y,\frac{1}{2}\sigma)$, one gets that

$$(\theta^p+J(z))^{\frac{1}{p}}\geq\min\{\xi(x,z),\xi(y,z)\}\geq\frac{1}{2}\sigma.$$

This together with (3.18) and (3.17) implies that

$$\begin{aligned}\xi(x,z)-\psi(y) &\leq (\theta^p+J(z))^{\frac{1-p}{p}}\theta^{p-1}(\|x-z\|-\|y-z\|)\\ &\leq\left(\frac{1}{2}\sigma\right)^{1-p}(\Delta+\lambda)^{p-1}\|x-y\|.\end{aligned}$$

Hence (see (3.15))

$$\psi(x)-\psi(y)=\sup_{z\in Z(x,\frac{3}{4}\sigma)}(\xi(x,z)-\psi(y))\leq\left(\frac{1}{2}\sigma\right)^{1-p}(\Delta+\lambda)^{p-1}\|x-y\|$$

and (3.14) is seen to hold with $L=\left(\frac{1}{2}\sigma\right)^{1-p}(\Delta+\lambda)^{p-1}$. ∎

Lemma 3.3. *Let Y be a subspace of X containing Z. Let $x\in Y$ and $y^*\in Y^*$. Suppose that*

$$\lim_{t\to 0^-}\left(\frac{\psi(x+th)-\psi(x)}{t}-\langle y^*,h\rangle\right)=0 \tag{3.19}$$

holds for each $h\in Y$, and holds uniformly for all $h\in Z-x$. Then

$$\|y^*\|=\psi^{1-p}(x)b^{p-1}(x). \tag{3.20}$$

Furthermore, if $\{z_n\}\subseteq Z$ is a maximizing sequence of the problem $\max_J(x,Z)$, then

$$\lim_{n\to+\infty}\|x-z_n\|=b(x)\quad\textit{and}\quad\lim_{n\to\infty}\langle y^*,x-z_n\rangle=\psi^{1-p}(x)b^p(x). \tag{3.21}$$

Proof. Let $\{z_n\}\subseteq Z$ be a maximizing sequence of the problem $\max_J(x,Z)$, without loss of generality, assume that

$$c(x):=\lim_{n\to+\infty}\|x-z_n\| \tag{3.22}$$

exists. Then

$$\lim_{n\to\infty} J(z_n) = \psi^p(x) - c^p(x). \tag{3.23}$$

Below we first show that

$$\|y^*\| \le \psi^{1-p}(x)c^{p-1}(x). \tag{3.24}$$

By the assumption (3.19), it suffices to verify that

$$\lim_{t\to 0^-} \frac{\psi(x+th)-\psi(x)}{t} \le \psi^{1-p}(x)c^{p-1}(x)\|h\| \quad \text{for each } 0 \ne h \in Y. \tag{3.25}$$

Suppose on the contrary that (3.25) doesn't hold. Then, there exist $\epsilon > 0$ and $h \in Y$ with $\|h\| = 1$ such that

$$\lim_{t\to 0^-} \frac{\psi(x+th)-\psi(x)}{t} \ge \psi^{1-p}(x)c^{p-1}(x) + \epsilon. \tag{3.26}$$

This implies that there exists some $t_0 < 0$ such that

$$\psi(x+th) - \psi(x) < t(\psi^{1-p}(x)c^{p-1}(x) + \epsilon) \le t\epsilon \quad \text{for each } t \in [t_0, 0). \tag{3.27}$$

Fix $t \in [t_0, 0)$. There exists $N_{t,\epsilon} > 0$ such that

$$\xi(x, z_n) > \psi(x) + \frac{\epsilon}{2}t \quad \text{for each } n \ge N_{t,\epsilon} \tag{3.28}$$

(see (3.4)). By (3.27) and (3.28), one has that

$$\xi(x+th, z_n) - \xi(x, z_n) \le \psi(x+th) - \psi(x) - \frac{\epsilon}{2}t < \frac{\epsilon}{2}t \quad \text{for each } n \ge N_{t,\epsilon}. \tag{3.29}$$

Fix $n > N_{t,\epsilon}$ and write $s_n = \|x+th-z_n\| - \|x-z_n\|$. Then

$$0 > s_n \ge t\|h\|. \tag{3.30}$$

By the Mean-Value Theorem, we have that

$$\begin{aligned} &\frac{\xi(x+th, z_n) - \xi(x, z_n)}{s_n} \\ &= [(\|x-z_n\| + \theta s_n)^p + J(z_n)]^{\frac{1-p}{p}} (\|x-z_n\| + \theta s_n)^{p-1} \\ &\le [(\|x-z_n\| + t\|h\|)^p + J(z_n)]^{\frac{1-p}{p}} \|x-z_n\|^{p-1}, \end{aligned} \tag{3.31}$$

where $\theta \in (0,1)$ and the inequality holds because of (3.30) (noting $1-p \leq 0$). Hence,

$$\begin{aligned} &\frac{\xi(x+th,z_n)-\xi(x,z_n)}{t} \\ &= \left(\frac{\xi(x+th,z_n)-\xi(x,z_n)}{s_n}\right)\frac{s_n}{t} \\ &\leq [(\|x-z_n\|+t\|h\|)^p + J(z_n)]^{\frac{1-p}{p}} \|x-z_n\|^{p-1}\,\|h\| \end{aligned} \tag{3.32}$$

and

$$\limsup_{n\to+\infty} \frac{\xi(x+th,z_n)-\xi(x,z_n)}{t} \leq [(c(x)+t\|h\|)^p+\psi^p(x)-c^p(x)]^{\frac{1-p}{p}} c^{p-1}(x)\,\|h\|$$

thanks to (3.22) and (3.23). Consequently,

$$\limsup_{t\to 0^-}\limsup_{n\to+\infty} \frac{\xi(x+th,z_n)-\xi(x,z_n)}{t} \leq \psi^{1-p}(x)c^{p-1}(x)\|h\|. \tag{3.33}$$

By (3.29), we have that

$$\frac{\psi(x+th)-\psi(x)}{t} \leq \frac{\xi(x+th,z_n)-\xi(x,z_n)}{t} + \frac{\epsilon}{2}.$$

Combining this with (3.33), we get that

$$\lim_{t\to 0^-} \frac{\psi(x+th)-\psi(x)}{t} \leq \psi^{1-p}(x)c^{p-1}(x)\|h\| + \frac{\epsilon}{2}.$$

This together with assumption (3.19) implies that

$$\langle y^*, h\rangle = \lim_{t\to 0^-} \frac{\psi(x+th)-\psi(x)}{t} \leq \psi^{1-p}(x)c^{p-1}(x)\|h\|, \tag{3.34}$$

which contradicts (3.26).

Next we shall prove that

$$\liminf_{n\to\infty} \langle y^*, x-z_n\rangle \geq \psi^{1-p}(x)c^p(x) \tag{3.35}$$

For this purpose, take $t_n \in (-1,0)$ such that

$$t_n \to 0 \quad \text{and} \quad t_n^2 > \psi^p(x) - \xi^p(x,z_n). \tag{3.36}$$

Write $\Phi_n = \max\{\psi(x), \psi(x+t_n(z_n-x)\}$. Then

$$\lim_{n\to\infty} \Phi_n = \psi(x). \tag{3.37}$$

By the Mean-value Theorem, one can conclude that

$$\begin{aligned}
&\frac{\psi(x+t_n(z_n-x))-\psi(x)}{\psi^p(x+t_n(z_n-x))-\psi^p(x)}\\
&=\frac{1}{p}[\psi(x)+\theta_n(\psi(x+t_n(z_n-x))-\psi(x))]^{1-p}\le\frac{\Phi_n^{1-p}}{p},
\end{aligned}\tag{3.38}$$

where $\theta_n\in(0,1)$. Since

$$\begin{aligned}
\psi^p(x+t_n(z_n-x))-\psi^p(x) &\ge (\|x+t_n(z_n-x)-z_n\|^p+J(z_n))-\psi^p(x)\\
&= ((1-t_n)^p-1)\|(x-z_n)\|^p-[\psi^p(x)-\xi^p(x,z_n)],
\end{aligned}$$

it follows from (3.36) that

$$\frac{\psi^p(x+t_n(z_n-x))-\psi^p(x)}{t_n} < \frac{((1-t_n)^p-1)\|(x-z_n)\|^p}{t_n}-t_n.$$

Combining this together with (3.38), we get that

$$\begin{aligned}
&\frac{\psi(x+t_n(z_n-x))-\psi(x)}{t_n}\\
&=\frac{\psi(x+t_n(z_n-x))-\psi(x)}{\psi^p(x+t_n(z_n-x))-\psi^p(x)}\cdot\frac{\psi^p(x+t_n(z_n-x))-\psi^p(x)}{t_n}\\
&\le\frac{\Phi_n^{1-p}}{p}\cdot\left(\frac{((1-t_n)^p-1)\|(x-z_n)\|^p}{t_n}-t_n\right).
\end{aligned}\tag{3.39}$$

Passing to the limits and by the given assumption, we have that

$$\liminf_{n\to\infty}\left(\langle y^*,x-z_n\rangle+\frac{\Phi_n^{1-p}}{p}\cdot\left(\frac{((1-t_n)^p-1)\|(x-z_n)\|^p}{t_n}-t_n\right)\right)\ge 0.\tag{3.40}$$

From (3.22) and (3.37), one sees that (3.35) holds. Consequently,

$$\|y^*\|\ge\psi^{1-p}(x)c^{p-1}(x),$$

and, together with (3.34),

$$\|y^*\|=\psi^{1-p}(x)c^{p-1}(x).\tag{3.41}$$

Thus we have proved that, for any maximizing sequence $\{z_n\}\subseteq Z$ of the problem $\max_J(x,Z)$,

$$\lim_{n\to\infty}\|x-z_n\|=\psi(x)\|y^*\|^{\frac{1}{p-1}}.\tag{3.42}$$

In particular, let $\{z_n\} \subseteq Z$ be such that $\lim_{n\to\infty} \|x-z_n\| = b(x)$ and $z_n \in Z^J(x, \frac{1}{n})$ for each n (by the definition of $b(x)$, such a sequence $\{z_n\} \subseteq Z$ exists). Then $\{z_n\} \subseteq Z$ is a maximizing sequence $\{z_n\} \subseteq Z$ of the problem $\max_J(x, Z)$, and $b(x) = \psi(x)\|y^*\|^{\frac{1}{p-1}}$ by (3.42). Thus (3.20) is seen to hold. To show (3.21), we note by (3.20) that

$$\limsup_{n\to\infty}\langle y^*, x - z_n\rangle \le \lim_{n\to\infty} \|y^*\|\|x - z_n\| = \|y^*\|b(x) = \psi^{1-p}(x)b^p(x).$$

Hence (3.21) holds by (3.35). Thus the proof is complete. ∎

Define the real-valued function a on X by

$$a(x) = \psi^{1-p}(x)b^{p-1}(x) \quad \text{for each } x \in X.$$

Then a is continuous on X. Set, for each $n \in \mathbb{N}$,

$$H_n^\psi(Z) = \left\{ x \in X : \begin{array}{c} \text{there are } \delta > 0 \text{ and } x^* \in X^* \text{ such that } |\|x^*\| - a(x)| < 2^{-n} \\ \text{and } \inf_{z\in Z^J(x,\delta)}\{\langle\, x^*, x-z\rangle + \psi^{1-p}(x)J(z)\} > (1-2^{-n})\psi(x) \end{array} \right\}.$$

Also set

$$H^\psi(Z) = \bigcap_{n=1}^{\infty} H_n^\psi(Z). \tag{3.43}$$

Let $\Lambda^\psi(Z)$ denote the set of all point $x \in X$ for which there exists $x^* \in X^*$ with $\|x^*\| = a(x)$ such that, for each $\epsilon \in (0, 1)$, there is $\delta > 0$ such that

$$\inf_{z\in Z^J(x,\delta)}\{\langle\, x^*, x - z\rangle + \psi^{1-p}(x)J(z)\} > (1 - \epsilon)\psi(x). \tag{3.44}$$

Obviously,

$$\Lambda^\psi(Z) \subseteq H^\psi(Z). \tag{3.45}$$

Lemma 3.4. *Suppose that Z is a relatively weakly compact closed subset of X. Then $H^\psi(Z)$ is a dense G_δ-subset of X.*

Proof. To show that $H^\psi(Z)$ is a G_δ-subset of X, we only need to prove that $H_n^\psi(Z)$ is open for each n. For this end, let $n \in \mathbb{N}$ and $x \in H_n^\psi(Z)$. Then there exist $x^* \in X^*$ and $\delta > 0$ such that

$$\alpha := 2^{-n} - |\|x^*\| - a(x)| > 0 \tag{3.46}$$

and

(3.47) $\beta := \inf\{\langle\, x^*, x-z\rangle + \psi^{1-p}(x)J(z) : z \in Z^J(x,\delta)\} - (1-2^{-n})\psi(x) > 0.$

without loss of generality, assume that $\delta > 0$ is such that $\xi(x,z) > 0$ for each $z \in Z^J(x,\delta)$. Thus

(3.48) $M = M(x,\delta) := \sup_{z\in Z^J(x,\delta)} |J(z)| \le \sup_{z\in Z^J(x,\delta)} \{|\xi(x,z)|^p + \|x-z\|^p\} < \infty$

as Z is bounded. Since the functions $\psi^{1-p}(\cdot)$ and $a(\cdot)$ are continuous on X, it follows that there exists $\lambda_0 > 0$ such that

(3.49) $|a(y)-a(x)| < \dfrac{\alpha}{2}$ and $|\psi^{1-p}(y)-\psi^{1-p}(x)| < \dfrac{\beta}{2M}$ for each $y \in \mathbf{U}(x,\lambda_0)$.

By Lemmas 3.1 and 3.2, there exist $0 < \lambda \le \lambda_1$ and $L > 0$ such that (3.8) and (3.13) hold. Without loss of generality, assume that $\lambda \le 1$ and $L \ge 1$. Thus (3.13) implies that

(3.50) $$|\psi(y) - \psi(x)| \le L\|y-x\|^{\frac{1}{p}} \quad \text{for each } y \in \mathbf{B}(x,\lambda)$$

(as $\|x-y\| \le \lambda < 1$ and $\frac{1}{p} \le 1$). Let $\bar\lambda > 0$ be such that

$$\bar\lambda^{\frac{1}{p}} < \min\left\{\lambda, \frac{\delta}{2L}, \frac{\beta}{2(a(x)+2L)}\right\}.$$

Then $\mathbf{U}(x,\bar\lambda) \subset \mathbf{U}(x,\lambda)$ and

(3.51) $$\frac{\beta}{2} - (a(x)+1+L)\bar\lambda^{\frac{1}{p}} \ge \frac{\beta}{2} - (a(x)+2L)\bar\lambda^{\frac{1}{p}} > 0.$$

Below we will show that $\mathbf{U}(x,\bar\lambda) \subset H_n^\psi(Z)$. Granting this, the openness of $H_n^\psi(Z)$ is proved. Let $y \in \mathbf{U}(x,\bar\lambda)$. Set $\delta^* := \delta - 2L\bar\lambda^{\frac{1}{p}} > 0$ and let $z \in Z^J(y,\delta^*)$. Then, by (3.7), $\xi(y,z) > \psi(y) - \delta^*$. Thus, using (3.8) and (3.50), one has that

$$\begin{aligned}\xi(x,z) &\ge \xi(y,z) - L\|y-x\|^{\frac{1}{p}} \\ &> \psi(y) - \delta^* - L\bar\lambda^{\frac{1}{p}} \\ &\ge \psi(x) - \delta^* - 2L\bar\lambda^{\frac{1}{p}} \\ &= \psi(x) - \delta;\end{aligned}$$

hence $z \in Z^J(x,\delta)$. Consequently,

(3.52) $$\langle\, x^*, x-z\rangle + \psi^{1-p}(x)J(z) \ge \beta + (1-2^{-n})\psi(x)$$

thanks to (3.47). Note that

$$\langle\, x^*, y-x\rangle \geq -\|x^*\|\|x-y\| \geq -(a(x)+2^{-n})\|x-y\| !ge -(a(x)+1)\|x-y\|^{\frac{1}{p}}. \tag{3.53}$$

It follows from Lemma 3.1 that

$$\begin{aligned}
&\langle\, x^*, y-z\rangle + \psi^{1-p}(y)J(z)\\
=\ &\langle\, x^*, x-z\rangle + \psi^{1-p}(x)J(z)\\
&+\langle\, x^*, y-x\rangle + (\psi^{1-p}(y) - \psi^{1-p}(x))J(z)\\
\geq\ &\frac{\beta}{2} + (1-2^{-n})\psi(x) - (a(x)+1)\|x-y\|^{\frac{1}{p}}\\
\geq\ &\frac{\beta}{2} + (1-2^{-n})\psi(y) - (a(x)+1+(1-2^{-n})L)\|x-y\|^{\frac{1}{p}}\\
\geq\ &(1-2^{-n})\psi(y) + \frac{\beta}{2} - (a(x)+1+L)\bar{\lambda}^{\frac{1}{p}},
\end{aligned}$$

where the first inequality holds because of (3.49), (3.52) and (3.53), while the second one because of (3.50). By (3.51),

$$\inf\{\langle\, x^*, y-z\rangle + \psi^{1-p}(y)J(z) : z \in Z^J(y,\delta^*)\} > (1-2^{-n})\psi(y) \tag{3.54}$$

since $z \in Z^J(y,\delta^*)$ is arbitrary. On the other hand, by (3.46) and (3.49),

$$|\|x^*\| - a(y)| \leq |\|x^*\| - a(x)| + |a(x) - a(y)| \leq 2^{-n} - \alpha + \frac{\alpha}{2} < 2^{-n}.$$

This together with (3.54) implies that $y \in H_n^{\psi}(Z)$ and so $\mathbf{U}(x,\bar{\lambda}) \subset H_n^{\psi}(Z)$.

To prove the density of $H^{\psi}(Z)$ in X, it suffices to prove that $\Lambda^{\psi}(Z)$ is dense in X since $\Lambda^{\psi}(Z) \subset H^{\psi}(Z)$. To this end, take $x_0 \in X$ and $\delta > 0$ such that $M(x_0,\delta)$ defined by (3.48) is finite. Let K denote the weak closure of the set $(\mathbf{B}(0,N) \cap Z) \cup \{x_0\}$, where $N = \|x_0\| + (\psi^p(x_0) + M + L_1)^{1/p} + 1$. Then K is weakly compact in $Y := \overline{\text{span}K}$. By Proposition 2.2, there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T : R \to Y$ such that $T(\mathbf{B}_R) \supseteq K$. Define a function $f_Z : R \to (-\infty, +\infty)$ by

$$f_Z(u) = \psi(x_0 + Tu) \quad \text{for each } u \in R. \tag{3.55}$$

Then f_Z is locally Lipschitz continuous on R by Lemma 3.2. Thus Proposition 2.1 is applicable to concluding that f_Z is Fréchet differentiable on a dense subset of R. Let $1/3 > \epsilon > 0$. It follows that there exists a point of differentiability $v \in R$ with $y = Tv \in \mathbf{U}(0,\epsilon)$. Let $v^* = \mathrm{D}f_Z(v)$. Then

$$\lim_{h\to 0} \frac{\psi(x_0 + T(v+h)) - \psi(x_0 + Tv) - \langle v^*, h\rangle}{\|h\|} = 0, \tag{3.56}$$

and hence

$$\lim_{h\to 0}\frac{\psi(x_0+y+Th)-\psi(x_0+y)-\langle v^*,h\rangle}{\|h\|}=0. \tag{3.57}$$

For each $u \in R$, substituting tu for h in the above expression as $t \to 0$ and using Lemma 3.2, we have there exists $L > 0$ such that

$$\langle v^*, u\rangle \le L\|Tu\| \quad \text{for each } u \in R. \tag{3.58}$$

Define a linear functional y^* on TR by

$$\langle y^*, Tu\rangle = \langle v^*, u\rangle \quad \text{for each } u \in R.$$

Then, $y^* \in (TR)^*$ by (3.58) and so $y^* \in Y^*$ because T has dense range. Clearly, $v^* = T^*y^*$ by definition. Set $x = y+x_0$. Then $\|x-x_0\| < \epsilon$ and $x \in K+Tv \subset TR$. Moreover, by (3.57), we have that

$$\lim_{TR\ni h\to 0}\frac{\psi(x+h)-\psi(x)-\langle y^*,h\rangle}{\|h\|}=0. \tag{3.59}$$

To complete the proof, it suffices to show that $x \in \Lambda^{\psi}(Z)$, that is, there exists $x^* \in X^*$ with $\|x^*\| = a(x)$ such that, for each $\epsilon > 0$, there is $1 > \delta > 0$ such that

$$\langle x^*, x-z\rangle + \psi^{1-p}(x)J(z) > (1-\epsilon)\psi(x) \quad \text{for each } z \in Z^J(x,\delta). \tag{3.60}$$

To do this, note by the Hahn-Banach theorem that, y^* can be extended to an element $x^* \in X^*$ such that $\|x^*\| = \|y^*\|$. Below we shall show that x^* is as desired. Since $TR \supseteq K$, it follows (3.59) that (3.19) holds for each $h \in Y$ and holds uniformly for all $h \in Z - x$. Thus, Lemma 3.3 is applicable and hence $\|x^*\| = \|y^*\| = a(x)$. Suppose on the contrary that there exist $\varepsilon_0 > 0$ and a sequence $\{z_n\}$ in Z such that

$$\lim_{n\to\infty}(\|x-z_n\|^p + J(z_n))^{\frac{1}{p}} = \psi(x) \tag{3.61}$$

but

$$\langle x^*, x-z_n\rangle + \psi^{1-p}(x)J(z_n) \le (1-\epsilon_0)\psi(x) \quad \text{for each } n \in \mathbb{N}. \tag{3.62}$$

Then, by (3.21) and (3.61), one concludes that

$$\lim_{n\to\infty}\|x-z_n\| = b(x) \quad \text{and} \quad \lim_{n\to\infty}J(z_n) = \psi^p(x) - b^p(x).$$

Hence

$$\lim_{n\to\infty}(\langle x^*, x-z_n\rangle + \psi^{1-p}(x)J(z_n)) = \psi^{1-p}(x)b^p(x) + \psi^{1-p}(x)(\psi^p(x)-b^p(x)) = \psi(x),$$

which contradicts (3.62) and the proof is complete. ■

For the main theorem of the present paper we introduce the notion of generalized well-posedness, see for example [15, 16, 20, 27].

Definition 3.2. Let $x \in X$. The problem $\max_J(x, Z)$ is said to be generalized well-posed if any maximizing sequence $\{z_n\}$ of the problem $\max_J(x, Z)$ has a convergent subsequence.

It is clear that the well-posedness implies the generalized well-posedness for the problem $\max_J(x, Z)$ and the converse is true if $F_{Z,J}(x)$ is a singleton.

Now we are ready to prove the main theorem.

Theorem 3.1. *Let Z be a relatively weakly compact subset of X. Suppose that X is Kadec w.r.t. Z. Then the following assertions hold.*

(i) *The set of all $x \in X$ such that the problem $\max_J(x, Z)$ is generalized well-posed is a dense G_δ-subset of X.*

(ii) *If X is J-strictly convex w.r.t. Z and $p > 1$, then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is well-posed is a dense G_δ-subset of X.*

Proof. (i). By Lemma 3.4, it suffices to verify that, for each $x \in H^\psi(Z)$, any maximizing sequence of the problem $\max_J(x, Z)$ has a convergent subsequence. For this purpose, let $x \in H^\psi(Z)$. In view of definition, there exist a positive sequence $\{\delta_n\}$ and a sequence $\{x_n^*\} \subseteq X^*$ with $|\|x_n^*\| - a(x)| < 2^{-n}$ such that

$$\begin{aligned} &\inf\{\langle\, x_n^*, x - z\rangle + \psi^{1-p}(x)J(z) : z \in Z^J(x, \delta_n)\} \\ &> (1 - 2^{-n})\psi(x) \quad \text{for each } n \in \mathbb{N}. \end{aligned} \tag{3.63}$$

Without loss of generality, assume that $\delta_n \le \delta_m$ if $m < n$. Let $\{z_n\}$ be any maximizing sequence of the problem $\max_J(x, Z)$, i.e.,

$$\lim_{n\to\infty} (\|x - z_n\|^p + J(z_n))^{\frac{1}{p}} = \psi(x). \tag{3.64}$$

Note that $\{z_n\}$ is bounded and Z is relatively weakly compact. Without loss of generality, we may assume that $\{\|x - z_n\|\}$ and $\{J(z_n)\}$ are convergent, and that $\{z_n\}$ converges to z_0 weakly for some $z_0 \in X$. Then we have that

$$\|x - z_0\| \le \lim_{n\to\infty} \|x - z_n\| \quad \text{and} \quad b(x) \le \lim_{n\to\infty} \|x - z_n\|. \tag{3.65}$$

Furthermore, we assume that $z_n \in Z^J(x, \delta_m)$ for all $n > m$. Thus,

$$\langle\, x_m^*, x - z_n\rangle + \psi^{1-p}(x)J(z_n) > (1 - 2^{-m})\psi(x) \quad \text{for all } n > m \tag{3.66}$$

and so, for each m,

$$\begin{aligned} &\|x_m^*\|\|x - z_0\| + \psi^{1-p}(x) \lim_{n\to\infty} J(z_n) \\ &\ge \langle\, x_m^*, x - z_0\rangle + \psi^{1-p}(x) \lim_{n\to\infty} J(z_n) \ge (1 - 2^{-m})\psi(x). \end{aligned} \tag{3.67}$$

Because $\lim_{m\to\infty}\|x_m^*\| = \psi^{1-p}(x)b^{p-1}(x)$, letting $m \to \infty$, we get that

$$\psi^{1-p}(x)b^{p-1}(x)\|x - z_0\| + \psi^{1-p}(x)\lim_{n\to\infty} J(z_n) \geq \psi(x),$$

that is

$$b^{p-1}(x)\|x - z_0\| + \lim_{n\to\infty} J(z_n) \geq \psi^p(x).$$

This together with (3.64) implies that

$$b^{p-1}(x)\|x - z_0\| \geq \lim_{n\to\infty}\|x - z_n\|^p$$

Combining this and (3.65), one has that

$$\lim_{n\to\infty}\|x - z_n\| = \|x - z_0\|. \tag{3.68}$$

Since X is Kadec w.r.t. Z and $z_n \to z_0$ weakly, it follows that $\lim_{n\to\infty}\|z_0 - z_n\| = 0$ and hence $z_0 \in Z$, which completes the proof of (i).

(ii). By the proof for assertion (i), one sees that the problem $\max_J(x, Z)$ is generalized well-posed for each $x \in H^\psi(Z)$. Thus we only need to prove that $F_{Z,J}(x)$ is a singleton for each $x \in H^\psi(Z)$. Let $x \in H^\psi(Z)$ and suppose $z_1, z_2 \in F_{Z,J}(x)$. Then, by the definition of $H^\psi(Z)$, for each $n \in \mathbb{N}$, there exists $x_n^* \in X^*$ such that $|\|x_n^*\| - a(x)| < 2^{-n}$ and

$$\langle\, x_n^*, x - z_i\rangle + \psi^{1-p}(x)J(z_i) > (1 - 2^{-n})\psi(x) \quad \text{for each } i = 1, 2.$$

Without loss of generality, we may assume that $\{x_n^*\}$ converges weakly* to some $x^* \in X^*$. Then $\|x^*\| = a(x)$ and

$$\langle\, x^*, x - z_i\rangle + \psi^{1-p}(x)J(z_i) = \psi(x) \quad \text{for each } i = 1, 2. \tag{3.69}$$

Since

$$\|x^*\| = \psi^{1-p}(x)b^{p-1}(x) \leq \psi^{1-p}(x)\|x - z_i\|^{p-1} \quad \text{for each } i = 1, 2,$$

it follows that

$$\begin{aligned}
2\psi(x) &= \langle x^*, x - z_1 + x - z_2\rangle + \psi^{1-p}(x)J(z_1) + \psi^{1-p}(x)J(z_2) \\
&\leq \|x^*\|\|x - z_1 + x - z_2\| + \psi^{1-p}(x)J(z_1) + \psi^{1-p}(x)J(z_2) \\
&\leq \|x^*\|(\|x - z_1\| + \|x - z_2\|) + \psi^{1-p}(x)J(z_1) + \psi^{1-p}(x)J(z_2) \\
&= \psi^{1-p}(x)[b^{p-1}(x)\|x - z_1\| + b^{p-1}(x)\|x - z_2\| + J(z_1) + J(z_2)] \\
&\leq \psi^{1-p}(x)[\|x - z_1\|^p + \|x - z_2\|^p + J(z_1) + J(z_2)] \\
&= 2\psi(x).
\end{aligned}$$

This means that

$$\|x - z_1 + x - z_2\| = \|x - z_1\| + \|x - z_2\| \tag{3.70}$$

and $\|x - z_1\| = \|x - z_2\| = b(x)$. Consequently,

$$J(z_1) = \psi^p(x) - \|x - z_1\|^p = \psi^p(x) - \|x - z_2\|^p = J(z_2).$$

Thus the assumed J-strict convexity of X implies that $x - z_1 = x - z_2$ and so $z_1 = z_2$. This completes the proof. ■

By (2.1) and (2.2), the following corollary is a direct consequence of Theorem 3.1.

Corollary 3.1. *Let Z be a relatively weakly compact subset of X. Suppose that X is Kadec. Then the following assertions hold.*

(i) *The set of all $x \in X$ such that the problem $\max_J(x, Z)$ is generalized well-posed is a dense G_δ-subset of X.*

(ii) *If X is strictly convex and $p > 1$. Then the set of all $x \in X$ such that the problem $\max_J(x, Z)$ is well-posed is a dense G_δ-subset of X.*

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