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WEIGHTED HARDY SPACES ASSOCIATED TO PARA-ACCRETIVE FUNCTIONS

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. In this article, after establishing weighted Plancherel-Pôlya-type inequalities, we introduce a new class of weighted Hardy spaces $H^p_{b,w}$ by using g-function, where w is a Muckenhoupt's weight and b is a para-accretive function. Then we show the atomic decomposition and molecular characterization of $H^p_{b,w}$. As applications, we prove the boundedness of Calderón-Zygmund operators between $H^p_{b,w}$ and classical weighted Hardy spaces H^p_w.

1. Introduction

It is well-known that Calderón-Zygmund operators T are bounded on H^p for $n/(n+\varepsilon) < p \le 1$ provided $T^*1 = 0$. In general, however, such operators are not bounded on H^p even if T satisfies $Tb = T^*b = 0$ for a para-accretive function b. Meyer observed that if b is bounded function and $1 \le \operatorname{Re} b(x)$, the space H^1_b and its dual BMO_b can be simply defined by coping the classical H^1 and BMO, respectively. These spaces have the advantage of a cancellation adapted to the complex measure $b(x)dx$ and are closely related to the Tb theorem. For more details about the space H^1_b, we refer the reader to [14]. However, the method for defining space H^1_b cannot be extended to H^p_b for $p < 1$ because, in general, bf does not make sense when f belongs to classical Hardy spaces H^p for $p < 1$. Recently,

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by establishing a discrete Calderón-type reproducing formula and Plancherel-Pôlya-type inequalities associated to a para-accretive function b, a new Hardy space H_b^p was introduced by Han, Lee, and Lin [9] who also proved that a Calderón-Zygmund operator T is bounded from H^p to H_b^p provided $T^*b = 0$. On the other hand, a remarkable direction of extending classical function or distribution spaces is to study the weighted case, where the weight is in Muckenhoupt's A_p classes. Weighted Hardy spaces H_w^p have been extensively studied by Garc´a-Cuerva [6] and Strom̈berg and Torchinsky [15].

The main purpose of this article is to develop the theory of the weighted Hardy spaces $H_{b,w}^p$, where b is a para-accretive function and w is a Muckenhoupt's weight. We define $H_{b,w}^p$ by g-function, and get its S-function characterization. Also, we show the atomic decomposition and molecular characterization of $H_{b,w}^p$. These new weighted Hardy spaces are related to the Calderón-Zygmund operator theory, as T is bounded from H_w^p to $H_{b,w}^p$ provided the Calderón-Zygmund operator T satisfies $T^*b = 0$. If we denote M_b the multiplication operator by b, i.e. $M_b f = bf$, then TM_b is bounded from $H_{b,w}^p$ to H_w^p provided $T^*1 = 0$, and TM_b is bounded on $H_{b,w}^p$ provided $T^*b = 0$. The main tool used in this article is the discrete Calderón-type reproducing formula developed in [9].

This article is organized as follows. In the next section, recalling some well known results, we establish the weighted Plancherel-Pôlya-type inequalities and define the weighted Hardy spaces $H_{b,w}^p$. The atomic decomposition and molecular characterizations for $H_{b,w}^p$ are given in Section 3. In the last section, we establish the $H_{b,w}^p - L_w^p$, $H_w^p - H_{b,w}^p$, $H_{b,w}^p - H_w^p$, and $H_{b,w}^p - H_{b,w}^p$ boundedness of Calderón-Zygmund operators.

Throughout the article C denotes a positive constant not necessarily the same at each occurrence. We also use $a \approx b$ to denote the equivalence of a and b; that is, there exist two positive constants C_1, C_2 independent of a, b such that $C_1 a \leq b \leq C_2 a$. For a measurable set $E \subseteq \mathbb{R}^n$, $|E|$ will denote the Lebesgue measure of E, and $w(E) = \int_E w(x)dx$. All cubes mentioned in this article mean cubes with their sides parallel to the axes. Given a cube Q, λQ will denote the cube with the same center as Q and with sides parallel to those of Q and λ times as long.

2. Weighted Plancherel-pôlya-type Inequalities and the Definition of $H_{b,w}^p$

We begin by recalling some basic results about Calderón-Zygmund operator theory. As usual, we denote by $\mathscr{D}$ the collection of C^∞ functions on $\mathbb{R}^n$ with compact support.

Definition 2.1. ([14]). A singular integral operator T is a continuous linear operator from $\mathscr{D}$ into its dual associated to a kernel $K(x, y)$, a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\}$, satisfying the following conditions: there exist a constant $C > 0$ and $0 < \varepsilon \leq 1$, such that

$$|K(x,y)| \leq C|x-y|^{-n} \qquad \text{for all } x \neq y, \tag{2.1}$$

$$|K(x,y) - K(x',y)| \leq C|x-x'|^{\varepsilon}|x-y|^{-n-\varepsilon} \tag{2.2}$$

for all x, x', and y in $\mathbb{R}^n$ with $|x - x'| \leq |x-y|/2$, and

$$|K(x,y) - K(x,y')| \leq C|y-y'|^{\varepsilon}|x-y|^{-n-\varepsilon} \tag{2.3}$$

for all y, y', and x in $\mathbb{R}^n$ with $|y-y'| \leq |x-y|/2$. Moreover, the operator T can be represented by

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)g(x)dydx$$

for all $f, g \in \mathscr{D}$ with $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$. We say that a singular integral operator is a *Calderón-Zygmund operator* if it can be extended to be a bounded operator on $L^2(\mathbb{R}^n)$.

Definition 2.2. ([3]). A bounded complex-valued function b defined on $\mathbb{R}^n$ is said to be *para-accretive* if there exist constants $C, \gamma > 0$ such that, for all cubes $Q \subseteq \mathbb{R}^n$, there is a sub-cube Q' with $\gamma|Q| \leq |Q'|$ satisfying

$$\frac{1}{|Q|}\left|\int_{Q'} b(x)dx\right| \geq C.$$

If T is a Calderón-Zygmund operator, then T^* is a Calderón-Zygmund operator as well. Thus Tb can be well defined by

$$\langle Tb, f\rangle = \langle b, T^* f\rangle \qquad \text{for all } f \in H^1,$$

since T and T^* are bounded from H^1 into L^1, and therefore $Tb = 0$ means $\int T^* f(x)b(x)dx = 0$ for all $f \in H^1$. Similarly, $T^*b = 0$ means $\int Tf(x)b(x)dx = 0$ for all $f \in H^1$. Suppose that T is an L^2 bounded operator with kernel $K(x,y)$ satisfying (2.1). If $K(x,y)$ satisfies (2.3), then T is bounded from H^1 to L^1. If $b^{-1}(x)K(x,y)$ satisfies (2.2), then T^*b^{-1} is bounded from H^1 to L^1. Therefore, for such an operator T and a para-accretive function b, $T^*1 = 0$ means $\int Tf(x)dx = 0$ for all $f \in H^1$ and $Tb = 0$ means $\int T^*g(x)b(x)dx = 0$ for all $g \in H^1_b$, where $g \in H^1_b$ if and only if $bg \in H^1$. See [9, 14] for more details about the Hardy space H^1_b. Similarly, suppose that T is bounded on L^2 such that its kernel $K(x,y)$ satisfies the conditions (2.1) and (2.2), and $K(x,y)b^{-1}(y)$ satisfies the condition (2.3). Then T^* and Tb^{-1} are bounded from H^1 to L^1. Therefore, for such an operator T and a para-accretive function b, $T1 = 0$ means $\int T^*f(x)dx = 0$ for all $f \in H^1$ and $T^*b = 0$ means $\int Tg(x)b(x)dx = 0$ for all $g \in H^1_b$.

Definition 2.3. ([8]). Fix two exponents $0 < \beta \leq 1$ and $\gamma > 0$. Suppose that b is a para-accretive function. A function f defined on $\mathbb{R}^n$ is said to be a test function of type (β, γ, b) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ if

$$|f(x)| \leq C \frac{d^\gamma}{(d + |x - x_0|)^{n+\gamma}}, \tag{2.4}$$

$$|f(x)| - |f(x')| \leq C\Big(\frac{|x - x'|}{d + |x - x_0|}\Big)^\beta \frac{d^\gamma}{(d + |x - x_0|)^{n+\gamma}} \tag{2.5}$$

for $|x - x'| \leq (d + |x - x_0|)/2$, and

$$\int_{\mathbb{R}^n} f(x)b(x)dx = 0.$$

Remark 2.4. Replacing the condition (2.5) by

$$|f(x) - f(x')| \leq C\Big(\frac{|x - x'|}{d}\Big)^\beta \Big(\frac{d^\gamma}{(d + |x - x_0|)^{n+\gamma}} + \frac{d^\gamma}{(d + |x' - x_0|)^{n+\gamma}}\Big), \tag{2.6}$$

one obtains Meyer's smooth atoms (see [13]). Obviously, conditions (2.4) and (2.5) imply (2.6).

Denote by $\mathcal{M}^{(\beta,\gamma,b)}(x_0, d)$ the collection of all test functions of type (β, γ, b) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. For $f \in \mathcal{M}^{(\beta,\gamma,b)}(x_0, d)$, the norm of f in $\mathcal{M}^{(\beta,\gamma,b)}(x_0, d)$ is defined by

$$\|f\|_{\mathcal{M}^{(\beta,\gamma,b)}(x_0,d)} = \inf\{C : \text{(2.4) and (2.5) hold}\}.$$

We denote $\mathcal{M}^{(\beta,\gamma,b)}(0, 1)$ simply by $\mathcal{M}^{(\beta,\gamma,b)}$. Then $\mathcal{M}^{(\beta,\gamma,b)}$ is a Banach space under the norm $\|f\|_{\mathcal{M}^{(\beta,\gamma,b)}}$. The dual space $\big(\mathcal{M}^{(\beta,\gamma,b)}\big)'$ consists of all linear functionals $\mathcal{L}$ from $\mathcal{M}^{(\beta,\gamma,b)}$ to $\mathbb{C}$ satisfying

$$|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{M}^{(\beta,\gamma,b)}} \qquad \text{for all } f \in \mathcal{M}^{(\beta,\gamma,b)}.$$

We denote $\langle h, f\rangle$ the natural pairing of elements $h \in \big(\mathcal{M}^{(\beta,\gamma,b)}\big)'$ and $f \in \mathcal{M}^{(\beta,\gamma,b)}$. It is easy to check that for any $x_0 \in \mathbb{R}^n$ and $d > 0$, $\mathcal{M}^{(\beta,\gamma,b)}(x_0, d) = \mathcal{M}^{(\beta,\gamma,b)}$ with the equivalent norms. Thus, for all $h \in \big(\mathcal{M}^{(\beta,\gamma,b)}\big)'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{M}^{(\beta,\gamma,b)}(x_0, d)$ with any $x_0 \in \mathbb{R}^n$ and $d > 0$. As usual, we write

$$b\mathcal{M}^{(\beta,\gamma,b)} = \big\{f : f = bg \ \text{ for some } g \in \mathcal{M}^{(\beta,\gamma,b)}\big\}.$$

If $f \in b\mathcal{M}^{(\beta,\gamma,b)}$ and $f = bg$ for $g \in \mathcal{M}^{(\beta,\gamma,b)}$, then the norm of f is defined by $\|f\|_{b\mathcal{M}^{(\beta,\gamma,b)}} = \|g\|_{\mathcal{M}^{(\beta,\gamma,b)}}$.

To state the discrete Calderón reproducing formula, we need an approximation to the identity associated to a para-accretive function.

Definition 2.5. ([3, 8]). Let b be a para-accretive function. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is called an approximation to the identity associated to b if the kernels $S_k(x,y)$ of S_k are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that there exist constant C and some $0<\varepsilon\le 1$ satisfying, for all x, x', y, and $y' \in \mathbb{R}^n$,

(i) $|S_k(x,y)| \le C\dfrac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$;

(ii) $|S_k(x,y)-S_k(x',y)| \le C\Big(\dfrac{|x-x'|}{2^{-k}+|x-y|}\Big)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$

for $|x-x'|\le \frac{1}{2}(2^{-k}+|x-y|)$;

(iii) $|S_k(x,y)-S_k(x,y')| \le C\Big(\dfrac{|y-y'|}{2^{-k}+|x-y|}\Big)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$

for $|y-y'|\le \frac{1}{2}(2^{-k}+|x-y|)$;

(iv) $\big|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]\big|$

$$\le C\Big(\frac{|x-x'|}{2^{-k}+|x-y|}\Big)^{\varepsilon}\Big(\frac{|y-y'|}{2^{-k}+|x-y|}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$$

for $|x-x'|\le \frac{1}{2}(2^{-k}+|x-y|)$ and $|y-y'|\le \frac{1}{2}(2^{-k}+|x-y|)$;

(v) $\displaystyle\int_{\mathbb{R}^n} S_k(x,y)b(y)dy = 1$ for all $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$;

(vi) $\displaystyle\int_{\mathbb{R}^n} S_k(x,y)b(x)dx = 1$ for all $k\in\mathbb{Z}$ and $y\in\mathbb{R}^n$.

Remark 2.6. Note that we can regard the ε's in Definitions 2.1 and 2.5 to be the same by choosing the smaller one. Coifman constructed an approximation to the identity $\{S_k\}_{k\in\mathbb{Z}}$ such that $D_k(x,y)$, the kernel of $D_k = S_k - S_{k-1}$, satisfies $D_k(x,y)=0$ for $|x-y|>C2^{-k}$ (see [3, p. 16] and [8, p. 63]).

We now recall the definition and properties of A_p weights. We refer readers to [4, 6] for the details about A_p. For $1<p<\infty$, a locally integrable nonnegative function w on $\mathbb{R}^n$ is said to be in A_p if there exists $C>0$ such that

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\Big)^{p-1} \le C \quad \text{for any cube } Q\subseteq\mathbb{R}^n. \tag{2.7}$$

The class $w\in A_1$ consists of weights satisfying for some $C>0$ that

$$\frac{1}{|Q|}\int_Q w(x)dx \le C\cdot \operatorname*{ess\,inf}_{x\in Q} w(x) \quad \text{for any cube } Q\subseteq\mathbb{R}^n,$$

and $A_\infty := \cup_{1\le p<\infty} A_p$. If $w \in A_p$ for $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. If $w \in A_p$, $p \ge 1$, then there exists an absolute constant C such that $w(\lambda Q) \le C\lambda^{np} w(Q)$. A close relation to A_p is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w(y)^r dy\right)^{1/r} \le \frac{C}{|B|}\int_B w(y)dy \qquad \text{for any cube } Q \subseteq \mathbb{R}^n,$$

we say that w satisfies the *reverse Hölder condition of order* r and write $w \in RH_r$. It follows from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for $s < r$. It is known that $w \in A_\infty$ if and only if $w \in RH_r$ for some $r > 1$. Moreover, if $w \in RH_r$ for $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. Thus we write $r_w = \sup\{r > 1 : w \in RH_r\}$ to denote the *critical index of* w *for the reverse Hölder condition*. If $w \in A_p \cap RH_r$ with $p \ge 1$ and $r > 1$, then there exist constants $C_1, C_2 > 0$ such that

$$C_1\Big(\frac{|E|}{|I|}\Big)^p \le \frac{w(E)}{w(I)} \le C_2\Big(\frac{|E|}{|I|}\Big)^{(r-1)/r} \tag{2.8}$$

for any measurable subset E of a cube I.

To introduce weighted Hardy spaces associated to para-accretive functions, we need to establish the following weighted Plancherel-Pôlya-type inequalities.

Theorem 2.7. *Suppose that* $\{S_k\}_{k\in\mathbb{Z}}$ *and* $\{P_k\}_{k\in\mathbb{Z}}$ *are approximations to the identity associated to* b *defined in Definition 2.5. Set* $D_k = S_k - S_{k-1}$ *and* $E_k = P_k - P_{k-1}$. *For* $n/(n+\varepsilon) < p < \infty$, *if* $w \in A_{(n+\varepsilon)p/n}$, *then*

$$(i)\quad \Big\|\Big\{\sum_k\sum_{Q_k}\Big(\sup_{z\in Q_k}|E_k bf(z)|\Big)^2\chi_{Q_k}\Big\}^{1/2}\Big\|_{L^p_w}$$
$$\approx \Big\|\Big\{\sum_k\sum_{Q_k}\Big(\inf_{z\in Q_k}|D_k bf(z)|\Big)^2\chi_{Q_k}\Big\}^{1/2}\Big\|_{L^p_w} \qquad \textit{for } f \in (b\mathcal{M}^{(\beta,\gamma,b)})',$$

$$(ii)\quad \Big\|\Big\{\sum_k\sum_{Q_k}\Big(\sup_{z\in Q_k}|E_k f(z)|\Big)^2\chi_{Q_k}\Big\}^{1/2}\Big\|_{L^p_w}$$
$$\approx \Big\|\Big\{\sum_k\sum_{Q_k}\Big(\inf_{z\in Q_k}|D_k f(z)|\Big)^2\chi_{Q_k}\Big\}^{1/2}\Big\|_{L^p_w} \qquad \textit{for } f \in \big(\mathcal{M}^{(\beta,\gamma,b)}\big)',$$

where Q_k*'s are all dyadic cubes with the side length* 2^{-k-N} *for some fixed positive large* N.

We postpone the proof of Theorem 2.7 and display two discrete Calderón-type reproducing formulas in the followings, which play a crucial role in the proof of Theorem 2.7.

Lemma 2.8. ([9]). *Suppose that $\{S_k\}$ is an approximation to the identity associated to b defined in Definition 2.5 and $D_k = S_k - S_{k-1}$. Then there exists a family of operators $\{\widetilde{D}_k\}$ with kernel $\widetilde{D}_k(x,y)$ such that, for all $f \in (b\mathcal{M}^{(\beta,\gamma,b)})'$,*

$$f(x) = \sum_k \sum_{Q_k} D_k b f(y_{Q_k}) \int_{Q_k} \widetilde{D}_k(y,x) b(y) dy, \tag{2.9}$$

where Q_k's are all dyadic cubes with the side length 2^{-k-N} for some fixed positive large N, y_{Q_k} is any fixed point in Q_k, and the series converges in the sense that, for all $g \in b\mathcal{M}^{(\beta',\gamma')}$ with $\beta < \beta'$ and $\gamma < \gamma'$,

$$\lim_{M,J\to\infty} \Big\langle \sum_{|k|\le M} \sum_{dist(0,Q_k)\le J} D_k b f(y_{Q_k}) \int_{Q_k} \widetilde{D}_k(y,x) b(y) dy, g \Big\rangle = \langle f, g\rangle.$$

Moreover, $\widetilde{D}_k(x,y)$'s satisfy the following estimates: for $0 < \varepsilon' < \varepsilon$, where ε is the regularity exponent of S_k, there exists a constant $C > 0$ such that

$$|\widetilde{D}_k(x,y)| \le C \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}},$$

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x,y')| \le C \Big(\frac{|y-y'|}{2^{-k}+|x-y|}\Big)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + |x-y|)^{n+\varepsilon'}}$$
$$\text{for } \ |y-y'| \le (2^{-k}+|x-y|)/2,$$

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y) b(y)\, dy = 0 \quad \text{for } k \in \mathbb{Z} \ \text{ and } \ x \in \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} \widetilde{D}_k(x,y) b(x)\, dx = 0 \quad \text{for } k \in \mathbb{Z} \ \text{ and } \ y \in \mathbb{R}^n.$$

Lemma 2.9. ([9]). *Let S_k, $\widetilde{D}_k$, $\widetilde{D}_k(x,y)$, Q_k, and y_{Q_k} be given in Lemma 2.8. Then, for all $f \in \big(\mathcal{M}^{(\beta,\gamma,b)}\big)'$,*

$$f(x) = \sum_k \sum_{Q_k} D_k f(y_{Q_k}) \int_{Q_k} b(x) \widetilde{D}_k(y,x) b(y) dy,$$

where the series converges in the sense that, for all $g \in \mathcal{M}^{(\beta',\gamma')}$ with $\beta < \beta'$ and $\gamma < \gamma'$,

$$\lim_{M,J\to\infty} \Big\langle \sum_{|k|\le M} \sum_{dist(0,Q_k)\le J} D_k f(y_{Q_k}) \int_{Q_k} b(x) \widetilde{D}_k(y,x) b(y) dy, g \Big\rangle = \langle f, g\rangle.$$

The following weighted version of Fefferman-Stein vector-valued maximal inequality will be used as well.

Lemma 2.10. ([1]). *Let* $f = (f_1, f_2, \cdots)$ *be a sequence of functions on* $\mathbb{R}^n$. *If* $1 < p, r < \infty$, *there is a constant* $C_{n,p,r} > 0$ *such that*

$$\Big\| \Big(\sum_{k=1}^{\infty} |Mf(\cdot)|^r \Big)^{1/r} \Big\|_{L^p_w} \le C_{n,p,r} \Big\| \Big(\sum_{k=1}^{\infty} |f(\cdot)|^r \Big)^{1/r} \Big\|_{L^p_w}$$

if and only if $w \in A_p$, *where* M *is the Hardy-Littlewood maximal function.*

We are ready to demonstrate the weighted Plancherel-Pôlya-type inequalities.

Proof of Theorem 2.7. We prove (i) only and the proof of (ii) is similar. Given $f \in (b\mathcal{M}^{(\beta,\gamma,b)})'$, since $w \in A_{(n+\varepsilon)p/n}$, there exists q satisfying $1 < q < (n+\varepsilon)p/n$ such that $w \in A_q$. Set $r = p/q$. Choose ε' and ε'' satisfying $0 < \varepsilon'' < \varepsilon' < \varepsilon$ and $n/(n+\varepsilon'') < r$. By Lemma 2.8, f can be written as

$$f(x) = \sum_k \sum_{Q_k} D_k bf(y_{Q_k}) \int_{Q_k} \widetilde{D}_k(y, x) b(y) dy,$$

where Q_k's are all dyadic cubes with the side length 2^{-k-N} for some fixed positive large N and y_{Q_k} is any fixed point in Q_k. Thus,

$$E_j bf(x) = \sum_k \sum_{Q_k} D_k bf(y_{Q_k}) \int_{Q_k} E_j b\widetilde{D}_k(y, \cdot)(x) b(y) dy.$$

Using the inequality (see [11])

$$\begin{aligned} \big| E_j b\widetilde{D}_k(y, \cdot)(x) \big| &= \Big| \int E_j(x, z) b(z) \widetilde{D}_k(y, z) dz \Big| \\ &\le C 2^{-|j-k|\varepsilon''} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |x-y|)^{n+\varepsilon'}}, \end{aligned}$$

where $j \wedge k$ denotes $\min(j, k)$, we obtain

$$\begin{aligned} |E_j bf(x)| &\le C \sum_k \sum_{Q_k} D_k bf(y_{Q_k}) \int_{Q_k} 2^{-|j-k|\varepsilon''} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |x-y|)^{n+\varepsilon'}}\, dy \\ &\le C \sum_k \sum_{Q_k} 2^{-|j-k|\varepsilon''} 2^{-kn} \frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)} + |x-y_{Q_k}|)^{n+\varepsilon'}} |D_k bf(y_{Q_k})|. \end{aligned}$$

Thus

$$\begin{aligned}&\Big|\sup_{z\in Q_j} E_j bf(z)\Big|\chi_{Q_j}(x)\\&\le C\sum_k\sum_{Q_k} 2^{-|j-k|\varepsilon''}2^{-kn}\frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)}+|x-y_{Q_k}|)^{n+\varepsilon'}}|D_k bf(y_{Q_k})|\chi_{Q_j}(x).\end{aligned}$$

By an estimate in [5, p. 147-148], we have

$$\begin{aligned}&\sum_{Q_k}\frac{2^{-(j\wedge k)\varepsilon'}}{(2^{-(j\wedge k)}+|x-y_{Q_k}|)^{n+\varepsilon'}}|D_k bf(y_{Q_k})|\chi_{Q_j}(x)\\&\qquad\le C2^{(j\wedge k)n}2^{[k-(j\wedge k)]n/r}\bigg\{M\bigg(\sum_{Q_k}|D_k bf(y_{Q_k})|\chi_{Q_k}\bigg)^r\bigg\}^{1/r}(x)\end{aligned}$$

since $n/(n+\varepsilon')<n/(n+\varepsilon'')<r$. Noticing that

$$\sup_j\sum_k 2^{-|j-k|\varepsilon''}2^{-kn}2^{(k\wedge j)n}2^{[k-(k\wedge j)]n/r}<\infty,$$

and by Hölder's inequality we obtain

$$\begin{aligned}\sup_{z\in Q_j}|E_j bf(z)|^2\chi_{Q_j}(x)&\le C\sum_k 2^{-|j-k|\varepsilon''}2^{-kn}2^{(k\wedge j)n}2^{[k-(k\wedge j)]n/r}\\&\qquad\times\bigg\{M\bigg(\sum_{Q_k}|D_k bf(y_{Q_k})|\chi_{Q_k}\bigg)^r\bigg\}^{2/r}(x)\chi_{Q_j}(x).\end{aligned}$$

This yields

$$\begin{aligned}&\bigg\{\sum_j\sum_{Q_j}\sup_{z\in Q_j}|E_j bf(z)|^2\chi_{Q_j}(x)\bigg\}^{1/2}\\&\quad\le C\bigg\{\sum_j\sum_k 2^{-|j-k|\varepsilon''}2^{-kn}2^{(k\wedge j)n}2^{[k-(k\wedge j)]n/r}\\&\qquad\times\bigg[M\bigg(\sum_{Q_k}|D_k bf(y_{Q_k})|\chi_{Q_k}\bigg)^r\bigg]^{2/r}(x)\bigg\}^{1/2}\\&\quad\le C\bigg\{\sum_k\bigg[M\bigg(\sum_{Q_k}|D_k bf(y_{Q_k})|\chi_{Q_k}\bigg)^r\bigg]^{2/r}(x)\bigg\}^{1/2},\end{aligned}$$

where the last inequality follows from the fact that

$$\sup_k\sum_j 2^{-|j-k|\varepsilon''}2^{-kn}2^{(k\wedge j)n}2^{[k-(k\wedge j)]n/r}<\infty.$$

Since y_{Q_k} is any point in Q_k,

$$\begin{aligned}&\Big\{\sum_j\sum_{Q_j}\sup_{z\in Q_j}|E_jbf(z)|^2\chi_{Q_j}(x)\Big\}^{1/2}\\&\qquad\le C\Big\{\sum_k\Big[M\Big(\sum_{Q_k}\inf_{z\in Q_k}|D_kbf(z)|\chi_{Q_k}(x)\Big)^r\Big]^{2/r}\Big\}^{1/2}.\end{aligned}$$

Therefore, noticing that $r=p/q<2$ and using Lemma 2.10, we have

$$\begin{aligned}&\Big\|\Big\{\sum_j\sum_{Q_j}\sup_{z\in Q_j}|E_jbf(z)|^2\chi_{Q_j}(x)\Big\}^{1/2}\Big\|_{L^p_w}^p\\&\quad\le C\int_{\mathbb{R}^n}\Big\{\sum_k\Big[M\Big(\sum_{Q_k}\inf_{z\in Q_k}|D_kbf(z)|\chi_{Q_k}(x)\Big)^r\Big]^{2/r}\Big\}^{p/2}w(x)dx\\&\quad\le C\int_{\mathbb{R}^n}\Big\{\sum_k\Big[M\Big(\sum_{Q_k}\inf_{z\in Q_k}|D_kbf(z)|\chi_{Q_k}(x)\Big)^r\Big]^{2/r}\Big\}^{(r/2)q}w(x)dx\\&\quad\le C\int_{\mathbb{R}^n}\Big\{\sum_k\Big[\Big(\sum_{Q_k}\inf_{z\in Q_k}|D_kbf(z)|\chi_{Q_k}(x)\Big)^r\Big]^{2/r}\Big\}^{(r/2)q}w(x)dx\\&\quad\le C\Big\|\Big\{\sum_k\sum_{Q_k}\inf_{z\in Q_k}|D_kbf(z)|^2\chi_{Q_k}(x)\Big\}^{1/2}\Big\|_{L^p_w}^p.\end{aligned}$$

This completes the proof of Theorem 2.7. ■

We now introduce the g-functions and S-functions associated to a para-accretive function b.

Definition 2.11. ([9]). Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity associated to b defined in Definition 2.5 and $D_k=S_k-S_{k-1}$. Define the g-functions and S-functions by

$$g(f)(x):=\Big\{\sum_k|D_kf(x)|^2\Big\}^{1/2},\quad f\in\big(\mathcal{M}^{(\beta,\gamma,b)}\big)',$$

$$g_b(f)(x):=\Big\{\sum_k|D_kbf(x)|^2\Big\}^{1/2},\quad f\in\big(b\mathcal{M}^{(\beta,\gamma,b)}\big)',$$

$$S(f)(x):=\Big\{\sum_k\int_{|x-y|\le 2^{-k}}2^{kn}|D_kf(y)|^2dy\Big\}^{1/2},\quad f\in\big(\mathcal{M}^{(\beta,\gamma,b)}\big)',$$

$$S_b(f)(x):=\Big\{\sum_k\int_{|x-y|\le 2^{-k}}2^{kn}|D_kbf(y)|^2dy\Big\}^{1/2},\quad f\in\big(b\mathcal{M}^{(\beta,\gamma,b)}\big)'.$$

Similar to the classical case, we have the equivalent L^p-norms for g-functions and S-functions as follows.

Theorem 2.12. *Let* $n/(n+\varepsilon) < p < \infty$ *and* $w \in A_{(n+\varepsilon)p/n}$. *Then* $\|S(f)\|_{L^p_w} \approx \|g(f)\|_{L^p_w}$ *and* $\|S_b(f)\|_{L^p_w} \approx \|g_b(f)\|_{L^p_w}$.

Proof. We show the equivalence of $\|S_b(f)\|_{L^p_w}$ and $\|g_b(f)\|_{L^p_w}$ only, and the proof of $\|S(f)\|_{L^p_w} \approx \|g(f)\|_{L^p_w}$ is similar. By Theorem 2.7,

$$
\begin{aligned}
\|S_b(f)\|_{L^p_w} &= \Bigg\|\Bigg\{\sum_k\sum_{Q_k}\int_{|x-y|\le 2^{-k}} 2^{kn}|D_kbf(y)|^2\chi_{Q_k}(x)dy\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k\sum_{Q_k}\sup_{z\in cQ_k}|D_kbf(z)|^2\chi_{Q_k}(x)\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k\sum_{Q_k}\inf_{z\in cQ_k}|D_kbf(z)|^2\chi_{Q_k}(x)\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k|D_kbf(x)|^2\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&= C\,\|g_b(f)\|_{L^p_w}\,,
\end{aligned}
$$

where $C > 1$ is a fixed number depends on N, and on the other hand

$$
\begin{aligned}
\|g_b(f)\|_{L^p_w} &= \Bigg\|\Bigg\{\sum_k|D_kbf(x)|^2\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k\sum_{Q_k}\sup_{z\in Q_k}|D_kbf(z)|^2\chi_{Q_k}(x)\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k\sum_{Q_k}\inf_{z\in Q_k}|D_kbf(z)|^2\chi_{Q_k}(x)\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k\sum_{Q_k}\chi_{Q_k}(x)\int_{|x_y|\le 2^{-k}} 2^{kn}\inf_{z\in Q_k}|D_kbf(z)|^2dy\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&\le C\Bigg\|\Bigg\{\sum_k\sum_{Q_k}\chi_{Q_k}(x)\int_{|x_y|\le 2^{-k}} 2^{kn}|D_kbf(y)|^2dy\Bigg\}^{1/2}\Bigg\|_{L^p_w}\\
&= C\|S_b(f)\|_{L^p_w}.
\end{aligned}
$$

This completes the proof. ■

We now may introduce the weighted Hardy spaces associated to para-accretive functions.

Definition 2.13. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity associated to b defined in Definition 2.5 and $D_k = S_k - S_{k-1}$. For $n/(n+\varepsilon) < p \le 1$ and $w \in A_{(n+\varepsilon)p/n}$, we define the weighted Hardy space $H^p_{b,w}$ to be the collection of $f \in (b\mathcal{M}^{(\beta,\gamma,b)})'$ such that

$$\|f\|_{H^p_{b,w}} := \|g_b(f)\|_{L^p_w}\,.$$

Remark 2.14. By Theorem 2.7, we deduce the norm $\|\cdot\|_{H^p_{b,w}}$ to be independent of the choice of approximation to the identity. Furthermore, we may assume that $D_k(x,y)$ satisfies the property given in Remark 2.6; that is, $D_k(x,y) = 0$ for $|x-y| > C2^{-k}$.

As a consequence of Theorems 2.7 and 2.12, we have the following result.

Theorem 2.15. *Let $n/(n+\varepsilon) < p \le 1$ and $w \in A_{(n+\varepsilon)p/n}$. Then*

$$\|f\|_{H^p_{b,w}} \approx \|S_b(f)\|_{L^p_w} \approx \left\|\left\{\sum_k \sum_{Q_k} |D_k bf(y_{Q_k})|^2 \chi_{Q_k}(x)\right\}^{1/2}\right\|_{L^p_w},$$

where y_{Q_k} is any fixed point in Q_k.

3. Atomic Decomposition and Molecular Characterizations of $H^p_{b,w}$

In this section, we demonstrate the atomic decomposition and molecular characterizations for $H^p_{b,w}$.

Definition 3.1. Let $n/(n+\varepsilon) < p \le 1$, $w \in A_{(n+\varepsilon)p/n}$, and b be a para-accretive function. A $(p,2,w)$ *b-atom* a is a function on $\mathbb{R}^n$, which is supported on a cube Q and satisfies

$$\|a\|_{L^p_w} \le w(Q)^{1/2-1/p} \qquad \text{and} \qquad \int_{\mathbb{R}^n} a(x)b(x)dx = 0.$$

Theorem 3.2. *Let $n/(n+\varepsilon) < p \le 1$, $w \in A_{(n+\varepsilon)p/n}$, and b be a para-accretive function. Then $f \in H^p_{b,w}$ if and only if f can be represented as $f = \sum_k \lambda_k a_k$, where a_k's are $(p,2,w)$ b-atoms and $\sum_k |\lambda_k|^p < \infty$, and the series converges in the norm of $H^p_{b,w}$. Moreover, $\|f\|_{H^p_{b,w}} \approx \inf\{\sum_k |\lambda_k|^p\}^{1/p}$, where the infimum is taken over all decompositions of f into $(p,2,w)$ b-atoms.*

Proof. We first prove the "if" part. By [10] it suffices to check

$$\|g_b(a)\|_{L^p_w} \le C \quad \text{for all } \ (p,2,w)\ b\text{-atom } a,$$

where C is a constant independent of a. Let a be a $(p,2,w)$ b-atom whose support is contained in a cube Q centered at x_0. Write

$$\|g_b(a)\|_{L^p_w}^p \le \int_{\mathbb{R}^n} g_b(a)^p(x)w(x)dx = \Big(\int_{2Q} + \int_{(2Q)^c}\Big) g_b(a)^p(x)w(x)dx := I_1 + I_2.$$

By [7], S-function is bounded on L^2_w for $w \in A_2$. It follows from Theorem 2.12 that g-function is also bounded on L^2_w. Since function $b(x)a(x) \in L^2_w$, we have $\|g_b(a)(\cdot)\|_{L^2_w} = \|g(ba)(\cdot)\|_{L^2_w} \le C\|ba\|_{L^2_w} \le C\|a\|_{L^2_w}$. Therefore by Hölder's inequality and the size condition of a, we have

$$I_1 \le \Big(\int_{2Q} g_b(a)^2(x)w(x)dx\Big)^{p/2} w(2Q)^{1-p/2} \le \|g_b(a)\|_{L^2_w}^p w(2Q)^{1-p/2} \le C.$$

For $x \in (2Q)^c$, using the b-vanishing moment and size condition of a, the smoothness condition of $D_k = S_k - S_{k-1}$, and (2.7) (since $w \in A_2$), we have the following pointwise estimate of $D_k ba$

$$\begin{aligned}
|D_k ba(x)| &= \Big|\int_Q (D_k(x,y) - D_k(x,x_0))\, b(y)a(y)dy\Big| \\
&\le C\int_Q |D_k(x,y) - D_k(x,x_0)\|\, a(y)|dy \\
&\le C\frac{2^{-k\varepsilon}}{(2^{-k}+|x-x_0|)^{n+2\varepsilon}} \int_Q |y-x_0|^{\varepsilon}|a(y)|dy \\
&\le C\frac{2^{-k\varepsilon}}{(2^{-k}+|x-x_0|)^{n+2\varepsilon}} |Q|^{\varepsilon/n}\|a\|_{L^2_w}\Big(\int_Q w^{-1}(y)dy\Big)^{1/2} \\
&\le C|Q|^{1+\varepsilon/n} w(Q)^{-1/p} \frac{2^{-k\varepsilon}}{(2^{-k}+|x-x_0|)^{n+2\varepsilon}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g_b(a)(x) &= \Big\{\sum_k |D_k ba(x)|^2\Big\}^{1/2} \\
&\le C|Q|^{1+\varepsilon/n} w(Q)^{-1/p} \\
&\quad \times \Big\{\Big(\sum_{2^{-k}\le|x-x_0|} + \sum_{2^{-k}>|x-x_0|}\Big) \frac{2^{-2k\varepsilon}}{(2^{-k}+|x-x_0|)^{2n+4\varepsilon}}\Big\}^{1/2} \\
&\le C|Q|^{1+\varepsilon/n} w(Q)^{-1/p} |x-x_0|^{-n-\varepsilon}.
\end{aligned}$$

Noticing that $w \in A_q$ with $1 < q < (n+\varepsilon)p/n$, we have

$$
\begin{aligned}
I_2 &\le C|Q|^{(1+\varepsilon/n)p}\int_{(2Q)^c}|x-x_0|^{(-n-\varepsilon)p}w(Q)^{-1}w(x)dx\\
&= C|Q|^{(1+\varepsilon/n)p}\sum_{m=1}^{\infty}\int_{2^{m+1}Q\setminus 2^mQ}|x-x_0|^{(-n-\varepsilon)p}w(Q)^{-1}w(x)dx\\
&\le C|Q|^{(1+\varepsilon/n)p}\sum_{m=1}^{\infty}|2^{m+1}Q|^{(-1-\varepsilon/n)p}\frac{w(2^{m+1}Q)}{w(Q)}\\
&\le C|Q|^{(1+\varepsilon/n)p}\sum_{m=1}^{\infty}|Q|^{(-1-\varepsilon/n)p}2^{(m+1)n(-1-\varepsilon/n)p}\Big(\frac{|2^{m+1}Q|}{|Q|}\Big)^q\\
&\le C\sum_{m=1}^{\infty}2^{(m+1)n[(-1-\varepsilon/n)p+q]}\\
&\le C.
\end{aligned}
\tag{3.1}
$$

To see the "only if" part, we will use Chang and Fefferman's idea in [2]. Applying the same procedure as in developing the discrete Calderón reproducing formula (see the proof of [9, Theorem 2.11]) to (2.9), we get

$$
f(x)=\sum_k\sum_{Q_k}|Q_k|D_k(x,x_{Q_k})b(x_{Q_k})\widetilde{D}_kb(f)(x_{Q_k})
$$

in distribution sense, where Q_k's are all dyadic cubes with the side length 2^{-k-N} for some fixed positive large N, x_{Q_k} is any fixed point in Q_k. For $l\in\mathbb{Z}$, set $\Omega_l=\{x\in\mathbb{R}^n:\widetilde{g}_bf(x)>2^l\}$, where

$$
\widetilde{g}_bf(x)=\Big\{\sum_k\sum_{Q_k}\big|\widetilde{D}_kb(f)(x_{Q_k})\big|^2\chi_{Q_k}(x)\Big\}^{1/2},
$$

and

$$
\mathcal{B}_l=\{Q:Q \text{ is dyadic cube such that } w(Q\cap\Omega_l)>\frac{1}{2}w(Q) \text{ and } w(Q\cap\Omega_{l+1})\le\frac{1}{2}w(Q)\}.
$$

Thus

$$
f(x)=\sum_l\sum_{\widetilde{Q}\in\mathcal{B}_l}\Big(\sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}}|Q|D_k(x,x_Q)b(x_Q)\widetilde{D}_kb(f)(x_Q)\Big),
$$

where $d(Q)$ denotes the side length of dyadic cube Q. By Remark 2.14, we have $D_k(x,y)=0$ for $|x-y|>C2^{-k}$. Thus

$$
\operatorname{supp}\Big(\sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}}|Q|D_k(x,x_Q)b(x_Q)\widetilde{D}_kb(f)(x_Q)\Big)\subseteq 5^n\widetilde{Q}.
$$

On the other hand, noticing that w and w^{-1} both belong to A_2, we have

$$\begin{aligned}
&\Bigg\| \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} |Q|D_k(x,x_Q)b(x_Q)\widetilde{D}_kb(f)(x_Q)\Bigg\|_{L^2_w}\\
&\quad\le \sup_{\|h\|_{L^2_{w^{-1}}}=1}\left|\left\langle \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} |Q|D_k(\cdot,x_Q)b(x_Q)\widetilde{D}_kb(f)(x_Q),h(\cdot)\right\rangle\right|\\
&\quad\le \sup_{\|h\|_{L^2_{w^{-1}}}=1}\left| \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} |Q|\widetilde{D}_kb(f)(x_Q)b(x_Q)D_k(h)(x_Q)\right|\\
&\quad\le C\sup_{\|h\|_{L^2_{w^{-1}}}=1}\int_{\mathbb{R}^n} \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} \left|\widetilde{D}_kb(f)(x_Q)D_k(h)(x_Q)\right|\chi_Q(x)dx\\
&\quad\le C\sup_{\|h\|_{L^2_{w^{-1}}}=1}\left\|\left\{ \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} \left|\widetilde{D}_kb(f)(x_Q)\right|^2\chi_Q(x)\right\}^{1/2}\right\|_{L^2_w}\\
&\qquad\times\left\|\left\{ \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} \left|D_k(h)(x_Q)\right|^2\chi_Q(x)\right\}^{1/2}\right\|_{L^2_{w^{-1}}}\\
&\quad\le C\left\|\left\{ \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} \left|\widetilde{D}_kb(f)(x_Q)\right|^2\chi_Q(x)\right\}^{1/2}\right\|_{L^2_w}\\
&\quad=\lambda_{\widetilde{Q}}w(5^n\widetilde{Q})^{1/2-1/p},
\end{aligned}$$

where

$$\lambda_{\widetilde{Q}}=C\left\|\left\{ \sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} \left|\widetilde{D}_kb(f)(x_Q)\right|^2\chi_Q(x)\right\}^{1/2}\right\|_{L^2_w} w(5^n\widetilde{Q})^{1/p-1/2}. \tag{3.2}$$

Set

$$a_{\widetilde{Q}}=\frac{1}{\lambda_{\widetilde{Q}}}\sum_{\substack{Q\subseteq\widetilde{Q}\\ d(Q)=2^{-k-N}}} |Q|D_k(x,x_Q)b(x_Q)\widetilde{D}_kb(f)(x_Q).$$

Then we have $f=\sum_l\sum_{\widetilde{Q}\in\mathcal{B}_l}\lambda_{\widetilde{Q}}a_{\widetilde{Q}}$, where $a_{\widetilde{Q}}$ satisfies (i) supp $a_{\widetilde{Q}}\subseteq 5^n\widetilde{Q}$, (ii) $\|a_{\widetilde{Q}}\|_{L^2_w}\le w(5^n\widetilde{Q})^{1/2-1/p}$, (iii) $\int a_{\widetilde{Q}}(x)b(x)dx=0$. This means that $a_{\widetilde{Q}}$ is a $(p,2,w)$ b-atom. It follows from (3.2) that

$$\begin{aligned}
&\sum_l \sum_{\widetilde{Q}\in\mathcal{B}_l} |\lambda_{\widetilde{Q}}|^p \\
&\le C \sum_l \sum_{\widetilde{Q}\in\mathcal{B}_l} \left(\left\| \left\{ \sum_{\substack{Q\subseteq\widetilde{Q} \\ d(Q)=2^{-k-N}}} \left|\widetilde{D}_k b(f)(x_Q)\right|^2 \chi_Q(x) \right\}^{1/2} \right\|_{L^2_w}^2 \right)^{p/2} w(5^n\widetilde{Q})^{1-p/2} \qquad (3.3)\\
&\le C \sum_l \left(\sum_{\widetilde{Q}\in\mathcal{B}_l} w(5^n\widetilde{Q}) \right)^{1-p/2} \left(\sum_{\widetilde{Q}\in\mathcal{B}_l} \sum_{\substack{Q\subseteq\widetilde{Q} \\ d(Q)=2^{-k-N}}} w(Q)\left|\widetilde{D}_k b(f)(x_Q)\right|^2 \right)^{p/2}.
\end{aligned}$$

We claim that $\widetilde{Q} \in \mathcal{B}_l$ implies that $\widetilde{Q} \subseteq \widetilde{\Omega}_l$, where $\widetilde{\Omega}_l = \{x : M\chi_{\Omega_l}(x) > (1/2)^{r/(r-1)}\}$. In fact, if $x \in \widetilde{Q}$, then

$$M\chi_{\widetilde{\Omega}_l}(x) \ge \frac{|\widetilde{Q}\cap\widetilde{\Omega}_l|}{|\widetilde{Q}|} \ge \left(\frac{w(\widetilde{Q}\cap\widetilde{\Omega}_l)}{w(\widetilde{Q})}\right)^{r/(r-1)} > \left(\frac{1}{2}\right)^{r/(r-1)},$$

where $r > 1$ such that $w \in RH_r$. Therefore, $\sum_{\widetilde{Q}\in\mathcal{B}_l} w(C\widetilde{Q}) \le Cw(\widetilde{\Omega}_l) \le Cw(\Omega_l)$ since M is of weak type $(1,1)$. Noticing that for $Q \in \mathcal{B}_l$, $w((\widetilde{\Omega}_l \setminus \Omega_{l+1}) \cap Q) = w(\widetilde{\Omega}_l \cap Q) - w(\Omega_{l+1} \cap Q) \ge w(Q) - \frac{1}{2}w(Q) = \frac{1}{2}w(Q)$, we have

$$\begin{aligned}
\int_{\widetilde{\Omega}_l\setminus\Omega_{l+1}} \widetilde{g}_b f(x)^2 w(x)dx &= \int_{\widetilde{\Omega}_l\setminus\Omega_{l+1}} \sum_k \sum_Q \left|\widetilde{D}_k bf(x_Q)\right|^2 \chi_Q(x) w(x)dx \\
&\ge \int_{\widetilde{\Omega}_l\setminus\Omega_{l+1}} \sum_{Q\in\mathcal{B}_l} \left|\widetilde{D}_k bf(x_Q)\right|^2 \chi_Q(x) w(x)dx \\
&= \sum_{Q\in\mathcal{B}_l} \left|\widetilde{D}_k bf(x_Q)\right|^2 w\Big(\big(\widetilde{\Omega}_l \setminus \Omega_{l+1}\big) \cap Q\Big) \\
&\ge \sum_{Q\in\mathcal{B}_l} \frac{1}{2} w(Q) \left|\widetilde{D}_k bf(x_Q)\right|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{\widetilde{Q}\in\mathcal{B}_l} \sum_{Q\subseteq\widetilde{Q}} \left|\widetilde{D}_k bf(x_Q)\right|^2 w(Q) &= \sum_{Q\in\mathcal{B}_l} \left|\widetilde{D}_k bf(x_Q)\right|^2 w(Q) \\
&\le 2 \int_{\widetilde{\Omega}_l\setminus\Omega_{l+1}} \widetilde{g}_b f(x)^2 w(x)dx \\
&\le \left(2^{l+1}\right)^2 w(\widetilde{\Omega}_l) \\
&\le C2^{2l} w(\Omega_l).
\end{aligned}$$

So by (3.3) we have

$$\sum_l \sum_{\widetilde{Q}\in\mathcal{B}_l} |\lambda_{\widetilde{Q}}|^p \le C\sum_l w(\Omega_l)^{1-p/2}\big(2^{2l}w(\Omega_l)\big)^{p/2}$$
$$= C\sum_l 2^{lp}w(\Omega_l)$$
$$\le C\,\|\widetilde{g}_b f\|^p_{L^p_w}$$
$$\le C\,\|f\|^p_{H^p_{b,w}}.$$

This completes the proof of Theorem 3.2. ■

We now introduce the weighted b-molecules. The idea of weighted molecules is duo to [12].

Definition 3.3. *Let $n/(n+\varepsilon) < p \le 1$ and $w \in A_{(n+\varepsilon)p/n}$ with critical index r_w for the reverse Hölder condition. Set $\delta > \max\{1/(r_w-1), 1/p-1\}$, $a_0 = 1-1/p+\delta$, and $b_0 = 1/2+\delta$. A $(p,2,\delta,w)$ b-molecule centered at $x_0 \in \mathbb{R}^n$ is a function $M \in L^2_w$ satisfying*

(*i*) $M(x)w(I^{x_0}_{|x-x_0|})^{b_0} \in L^2_w$, *where $I^{x_0}_{|x-x_0|}$ denotes the cube centered at x_0 with side length $2|x-x_0|$,*

(*ii*) $\|M\|^{a_0/b_0}_{L^2_w}\cdot\|M(\cdot)w(I^{x_0}_{|\cdot-x_0|})^{b_0}\|^{1-a_0/b_0}_{L^2_w} \equiv \mathfrak{N}_w(M) < \infty$,

(*iii*) $\displaystyle\int_{\mathbb{R}^n} M(x)b(x)dx = 0.$

Remark 3.4. Every $(p,2,w)$ b-atom a is a $(p,2,\delta,w)$ b-molecule for $\delta > \max\{1/(r_w-1), 1/p-1\}$, and $\mathfrak{N}_w(a) \le C$ where C is a constant independent of f. This follows from b-vanishing moment of a and the fact that if supp$(a) \subseteq I^{x_0}_R$, then $\|a\|_{L^p_w} \le w(I^{x_0}_R)^{1/2-1/p}$ and

$$\big\|a(\cdot)w\big(I^{x_0}_{|\cdot-x_0|}\big)^{b_0}\big\|_{L^2_w} = \Big(\int_{I^{x_0}_R} |a(x)|^2 w\big(I^{x_0}_{|x-x_0|}\big)^{2b_0} w(x)dx\Big)^{1/2}$$
$$\le w\big(I^{x_0}_{\sqrt{n}R}\big)^{b_0} w(I^{x_0}_R)^{1/2-1/p}$$
$$\le Cw(I^{x_0}_R)^{a_0}.$$

Theorem 3.5. *Let $n/(n+\varepsilon) < p \le 1$ and $w \in A_{(n+\varepsilon)p/n}$ with critical index r_w for the reverse Hölder condition. If M be a $(p,2,\delta,w)$ b-molecule for $\delta > \max\{1/(r_w-1), 1/p-1\}$, then M is in $H^p_{b,w}$ and $\|M\|_{H^p_{b,w}} \le C\mathfrak{N}_w(M)$, where the constant C is independent of the molecule M.*

Proof. Set $M_1(x) = M(x)b(x)$. Then M_1 satisfies (i′) $M_1(x)w(I^{x_0}_{|x-x_0|})^{b_0} \in L^2_w$,

(ii′) $\|M_1\|^{a_0/b_0}_{L^2_w} \cdot \|M_1(\cdot)w(I^{x_0}_{|\cdot-x_0|})^{b_0}\|^{1-a_0/b_0}_{L^2_w} \equiv \mathfrak{N}_w(M_1) \approx \mathfrak{N}_w(M)$,

(iii′) $\int_{\mathbb{R}^n} M_1(x)dx = 0$.

Without loss of generality, we may assume that M_1 is centered at 0 and $\mathfrak{N}_w(M_1) = 1$. Define σ by setting $w(I_\sigma)^{1/p-1/2} = \|M_1\|^{-1}_{L^2_w}$, where $I_\sigma = I^0_\sigma$. Consider the sets

$$E_0 = \{x \in \mathbb{R}^n : |x| < \sigma\},\ \ E_k = \{x \in \mathbb{R}^n : 2^{k-1}\sigma \le |x| < 2^k\sigma\}\ \text{ for } k = 1, 2, \cdots.$$

Set $M_{1k} = M_1\chi_{E_k}$, $P_{1k}(x) = \frac{1}{|E_k|}\int_{\mathbb{R}^n} M_{1k}(y)dy \cdot \chi_{E_k}(x)$ for $k = 0, 1, 2, \cdots$, where χ_{E_k} is the characteristic function of E_k. Then

$$M_1(x) = \sum_{k=0}^{\infty} M_{1k}(x) = \sum_{k=0}^{\infty} \big(M_{1k}(x) - P_{1k}(x)\big) + \sum_{k=0}^{\infty} P_{1k}(x).$$

Observing that $\sum_{k=0}^{\infty}\int_{E_k} M_1(x)dx = \int_{\mathbb{R}^n} M_1(x)dx = 0$, and using Abel's summation formula, we write

$$\begin{aligned}
\sum_{k=0}^{\infty} P_{1k}(x) &= \sum_{k=0}^{\infty}\int_{E_k} M_1(y)dy\frac{\chi_{E_k}(x)}{|E_k|}\\
&= \sum_{k=0}^{\infty}\bigg(\sum_{j=k}^{\infty}\int_{E_j} M_1(y)dy - \sum_{j=k+1}^{\infty}\int_{E_j} M_1(y)dy\bigg)\frac{\chi_{E_k}(x)}{|E_k|}\\
&= \sum_{k=0}^{\infty}\bigg(\sum_{j=k+1}^{\infty}\int_{E_j} M_1(y)dy\bigg)\bigg(\frac{\chi_{E_{k+1}}(x)}{|E_{k+1}|} - \frac{\chi_{E_k}(x)}{|E_k|}\bigg)\\
&= \sum_{k=0}^{\infty}\int_{|y|\ge 2^k\sigma} M_1(y)dy\bigg(\frac{\chi_{E_{k+1}}(x)}{|E_{k+1}|} - \frac{\chi_{E_k}(x)}{|E_k|}\bigg)\\
&:= \sum_{k=0}^{\infty}\Phi_k(x).
\end{aligned}$$

Thus

$$M_1(x) = \sum_{k=0}^{\infty}\big(M_{1k}(x) - P_{1k}(x)\big) + \sum_{k=0}^{\infty}\Phi_k(x).$$

Since the above equation holds in L^2_w and hence holds in almost everywhere in $\mathbb{R}^n$, so we have

$$M(x) = \frac{M_1(x)}{b(x)} = \sum_{k=0}^{\infty}\frac{\big(M_{1k}(x) - P_{1k}(x)\big)}{b(x)} + \sum_{k=0}^{\infty}\frac{\Phi_k(x)}{b(x)}. \tag{3.4}$$

By the definition of M_{1k} and P_{1k}, $(M_{1k} - P_{1k})/b$ has b-vanishing moment, and is supported at $I_{2^k\sigma}$. Noticing that $w \in A_2$ we have

$$\begin{aligned}\|P_{1k}\|_{L^2_w} &= \frac{w(E_k)^{1/2}}{|E_k|}\Big|\int_{E_k} M_{1k}(y)dy\Big| \\ &\leq \|M_{1k}\|_{L^2_w}\frac{w(E_k)^{1/2}}{|E_k|}\Big(\int_{E_k} w(y)^{-1}dy\Big)^{1/2} \\ &\leq C\|M_{1k}\|_{L^2_w}.\end{aligned}$$

Thus

$$\Big\|\frac{M_{1k} - P_{1k}}{b}\Big\|_{L^2_w} \leq C\|M_{1k}\|_{L^2_w},$$

where we use the fact that the inverse of a para-accretive function belongs to L^∞ in the last estimate. Notice that $\mathfrak{N}_w(M_1) = 1$ and $w(I_\sigma)^{1/p-1/2} = \|M_1\|^{-1}_{L^2_w}$ imply $\|M_1(\cdot)w(I_{|\cdot|})^{b_0}\|_{L^2_w} = w(I_\sigma)^{a_0}$. From the choice of δ, we are able to choose $1 < r < r_w$ such that $\delta > 1/(r-1) > 1/(r_w - 1)$. By (2.8), we have, for $k = 1, 2, \cdots$,

$$\begin{aligned}\|M_{1k}\|_{L^2_w} &\leq C\Big\|M_{1k}(\cdot)\Big(\frac{w(I_{|\cdot|})}{w(I_{2^k\sigma})}\Big)^{b_0}\Big\|_{L^2_w} \\ &\leq Cw(I_\sigma)^{a_0}w(I_{2^k\sigma})^{-b_0} \qquad (3.5)\\ &\leq C2^{-kna_0(r-1)/r}w(I_{2^k\sigma})^{1/2-1/p},\end{aligned}$$

and for $k = 0$,

$$\|M_{10}\|_{L^2_w} \leq \|M_1\|_{L^2_w} \leq Cw(I_{2^k\sigma})^{1/2-1/p}.$$

Hence, for $k = 0, 1, 2, \cdots$,

$$\Big\|\frac{M_{1k} - P_{1k}}{b}\Big\|_{L^2_w} \leq C2^{-kna_0(r-1)/r}w(I_{2^k\sigma})^{1/2-1/p}.$$

It follows that, for $k = 0, 1, 2, \cdots$,

$$C^{-1}2^{kna_0(r-1)/r}\frac{M_{1k}(x) - P_{1k}(x)}{b(x)} := \alpha_k(x)$$

ia a $(p, 2, w)$ b-atom supported at $I_{2^k\sigma}$. In other words,

$$\frac{M_{1k}(x) - P_{1k}(x)}{b(x)} = \lambda_k\alpha_k(x),$$

where α_k is a $(p, 2, w)$ b-atom supported at $I_{2^k\sigma}$ and $\lambda_k = C2^{-kna_0(r-1)/r}$. Since $na_0p(r-1)/r > 0$, $\sum_{k=0}^\infty |\lambda|^p \leq C\sum_{k=0}^\infty 2^{-kna_0p(r-1)/r} < \infty$. By Theorem 3.2,

$$\sum_{k=0}^{\infty} \frac{M_{1k}(x) - P_{1k}(x)}{b(x)} \in H^p_{b,w}$$

with its $H^p_{b,w}$ norm no more than $(\sum_{k=0}^{\infty} |\lambda_k|^p)^{1/p} \le C < \infty$.

Let us treat $\sum_{k=0}^{\infty} \frac{\Phi_k(x)}{b(x)}$. First, obviously $\frac{\Phi_k(x)}{b(x)}$ has b-vanishing moment. Noticing that $w \in A_2$, by Hölder's inequality and (3.5), we have

$$\begin{aligned}
\Big| \int_{|x| \ge 2^k \sigma} M_1(y) dy \Big| &= \Big| \sum_{j=k+1}^{\infty} \int_{E_j} M_{1j}(y) dy \Big| \\
&\le \sum_{j=k+1}^{\infty} \|M_{1j}\|_{L^2_w} \big(w^{-1}(I_{2^j\sigma})\big)^{-1/2} \\
&\le C \sum_{j=k+1}^{\infty} 2^{-na_0 j(r-1)/r} w(I_{2^j\sigma})^{-1/p} |I_{2^j\sigma}| \\
&= C\sigma^n w(I_{2^{k+1}\sigma})^{-1/p} \sum_{j=k+1}^{\infty} 2^{-nj\big(a_0(r-1)/r - 1\big)} \Big(\frac{w(I_{2^{k+1}\sigma})}{w(I_{2^j\sigma})}\Big)^{1/p} \\
&\le C\sigma^n w(I_{2^{j+1}\sigma})^{-1/p} 2^{(k+1)np^{-1}(r-1)/r} \\
&\qquad \times \sum_{j=k+1}^{\infty} 2^{-nj\big(a_0(r-1)/r - 1 + p^{-1}(r-1)/r\big)} \\
&\le C(2^{k+1}\sigma)^n w(I_{2^{j+1}\sigma})^{-1/p} 2^{-(k+1)na_0(r-1)/r},
\end{aligned}$$

since $a_0(r-1)/r - 1 + p^{-1}(r-1)/r = (1+\delta)(r-1)/r - 1 > 0$ by the choice of δ. Thus

$$\Big| \frac{\Phi_k(x)}{b(x)} \Big| \le C w(I_{2^{j+1}\sigma})^{-1/p} 2^{-(k+1)na_0(r-1)/r}.$$

Since supp$\frac{\Phi_k(x)}{b(x)} \subseteq I_{2^{k+1}\sigma}$, we have

$$\Big\| \frac{\Phi_k(\cdot)}{b(\cdot)} \Big\|_{L^2_w} \le C 2^{-(k+1)na_0(r-1)/r} w(I_{2^{j+1}\sigma})^{1/2-1/p}.$$

It yields $\frac{\Phi_k(x)}{b(x)} = \mu_k \beta_k(x)$, where $\mu_k = C2^{-(k+1)na_0(r-1)/r}$ and $\beta_k(x)$ is $(p, 2, w)$ b-atom supported at $I_{2^{k+1}\sigma}$. Since $\sum_{k=0}^{\infty} |\mu_k|^p \le C \sum_{k=0}^{\infty} 2^{-(k+1)na_0 p(r-1)/r} \le C < \infty$, by Theorem 3.2,

$$\frac{\Phi_k(x)}{b(x)} \in H^p_{b,w}$$

with its $H^p_{b,w}$ norm no more than $\big(\sum_{k=0}^{\infty} |\mu_k|^p\big)^{1/p} \le C < \infty$. So by (3.4), $M \in H^p_{b,w}$ and $\|M\|_{H^p_{b,w}} \le C < \infty$. This completes the proof of Theorem 3.5. ■

4. Boundedness of Calderón-zygmund Operators on L^p_w and $H^p_{b,w}$

We give applications to the boundedness of Calderón-Zygmund operators.

Theorem 4.1. *Let T be a Calderón-Zygmund operator given in Definition 2.1. For $n/(n+\varepsilon) < p \le 1$ and $w \in A_{(n+\varepsilon)p/n}$, define the operator T_b by*

$$T_b(f)(x) = \int_{\mathbb{R}^n} K(x,y)b(y)f(y)dy.$$

Then T_b is bounded from $H^p_{b,w}$ to L^p_w.

Proof. By atomic decomposition of $H^p_{b,w}$, it suffices to show for any $(p,2,w)$ b-atom a, we have $\|T_b(a)\|_{L^p_w} \le C$, where C is a constant independent of a. Suppose a is supported on a cube Q with center x_Q. We write

$$\big\|T_b(a)\big\|^p_{L^p_w} = \int_{\mathbb{R}^n} \big|T_b(a)(x)\big|^p w(x)dx = \int_{2Q} \quad + \int_{(2Q)^c} \quad := I_1 + I_2.$$

For I_1, by Hölder's inequality, the L^2_w boundedness of T_b (since $w \in A_2$, see [4]), and the size condition of a, we obtain

$$I_1 \le \Big(\int_{2Q} \big|T_b(a)(x)\big|^2 w(x)dx\Big)^{p/2}\Big(\int_{2Q} w(x)dx\Big)^{1-p/2} \le C\|a\|^p_{L^2_w} w(Q)^{1-p/2} \le C.$$

Let us treat I_2. If $x \in (2Q)^c$, by the b-vanishing moment of a and condition (2.3), we have

$$\begin{aligned}\big|T_b(a)(x)\big| &= \Big|\int_{\mathbb{R}^n} \big(K(x,y) - K(x,x_Q)\big)b(y)a(y)dy\Big| \\ &\le C\int_Q \big|(K(x,y) - K(x,x_Q)\big||a(y)|dy \\ &\le C\int_Q \frac{|y-x_Q|^\varepsilon}{|x-x_Q|^{n+\varepsilon}}|a(y)|dy \\ &\le C\frac{|Q|^{\varepsilon/n}}{|x-x_Q|^{n+\varepsilon}}\|a\|_{L^2_w}\big(w^{-1}(Q)\big)^{1/2} \\ &\le C\frac{|Q|^{\varepsilon/n+1}}{|x-x_Q|^{n+\varepsilon}}w(Q)^{-1/2}\|a\|_{L^2_w} \\ &\le C\frac{|Q|^{\varepsilon/n+1}}{|x-x_Q|^{n+\varepsilon}}w(Q)^{-1/p},\end{aligned} \tag{4.1}$$

where the next to last inequality is obtained since $w \in A_2$. Thus by (3.1)

$$
\begin{aligned}
I_2 &= \int_{(2Q)^c} \big|T_b(a)(x)\big|^p w(x)dx \\
&\le C|Q|^{\varepsilon p/n+p} w(Q)^{-1} \int_{(2Q)^c} \frac{1}{|x-x_Q|^{(n+\varepsilon)p}} w(x)dx \\
&\le C.
\end{aligned}
$$

This completes the proof of Theorem 4.1. ∎

Theorem 4.2. *Suppose that T is a Calderón-Zygmund operator given in Definition 2.1. Let $n/(n+\varepsilon) < p \le 1$ and $w \in A_{(n+\varepsilon)p/n}$ with critical index r_w for the reverse Hölder condition such that $r_w > (n+\varepsilon)/(n+\varepsilon-nq)$.*

(i) *If $T^*b = 0$, then T is bounded from H^p_w to $H^p_{b,w}$.*

(ii) *If $T^*1 = 0$, then T_b is bounded from $H^p_{b,w}$ to H^p_w.*

(iii) *If $T^*b = 0$, then T_b is bounded on $H^p_{b,w}$.*

Proof. We only prove (i), since the proof of (ii) and (iii) are similar. Observe that $1 < q < (n+\varepsilon)p/n$ implies $1/p-1 < \frac{n+\varepsilon}{nq}-1$, and $r_w > (n+\varepsilon)/(n+\varepsilon-nq)$ implies $(r_w-1)^{-1} < \frac{n+\varepsilon}{nq}-1$. So we can choose δ such that $\max\{(r_w-1)^{-1}, 1/p-1\} < \delta < \frac{n+\varepsilon}{nq}-1$. By the atomic and molecular decomposition theory established in the above section, it suffices to verify that, for every $(p,2,w)$ atom in H^p_w, Ta is a $(p,2,\delta,w)$ b-molecule and $\mathfrak{N}_w(Ta) \le C$ with C independent of a.

Assume $\operatorname{supp} a \subseteq Q$, where Q is a cube centered at x_Q. Set $a_0 = 1-1/p+\delta$ and $b_0 = 1/2+\delta$. Since $T^*b = 0$ implies $\int_{\mathbb{R}^n} Ta(x)b(x)dx = 0$, so we need only to check Ta satisfies $\mathfrak{N}_w(Ta) = \|Ta\|_{L^2_w}^{a_0/b_0} \cdot \|Ta(\cdot)w(I^{x_Q}_{|\cdot-x_Q|})^{b_0}\|_{L^2_w}^{1-a_0/b_0} \le C < \infty$. We write

$$
\begin{aligned}
\Big\|Ta(\cdot)w(I^{x_Q}_{|\cdot-x_Q|})^{b_0}\Big\|^2_{L^2_w} &= \int_{\mathbb{R}^n} \big|Ta(x)\big|^2 w\big(I^{x_Q}_{|x-x_Q|}\big)^{2b_0} w(x)dx \\
&= \int_{2Q} + \int_{(2Q)^c} \\
&:= I_1 + I_2.
\end{aligned}
$$

By the L^2_w boundedness of T and the size condition of a, we have

$$
I_1 \le Cw(2Q)^{2+\delta}\|Ta\|^2_{L^2_w} \le Cw(Q)^{2+\delta}\|Ta\|^2_{L^2_w} \le Cw(Q)^{2a_0}.
$$

For $x \in (I_{2R})^c$, same estimate to (4.1) leads

$$
\big|T(a)(x)\big| \le C\frac{|Q|^{\varepsilon/n+1}}{|x-x_Q|^{n+\varepsilon}} w(Q)^{-1/p}.
$$

Observe that it follows from the choice of δ that

$$2(n+\varepsilon)-(2b_0+1)nq=2(n+\varepsilon)-(2+2\delta)nq>0.$$

Thus, by the fact that $w\in A_q$, we get

$$\begin{aligned}
I_2 &= \int_{(2Q)^c}\big|Ta(x)\big|^2 w\big(I^{x_Q}_{|x-x_Q|}\big)^{2b_0}w(x)dx\\
&\le C|Q|^{2(\varepsilon/n+1)}w(Q)^{-2/p}\int_{(2Q)^c}\frac{1}{|x-x_Q|^{2(n+\varepsilon)}}w\big(I^{x_Q}_{|x-x_Q|}\big)^{2b_0}w(x)dx\\
&\le C|Q|^{2(\varepsilon/n+1)}w(Q)^{-2/p}\sum_{m=1}^{\infty}\int_{2^{m+1}Q\setminus 2^mQ}\frac{1}{|x-x_Q|^{2(n+\varepsilon)}}w\big(I^{x_Q}_{|x-x_Q|}\big)^{2b_0}w(x)dx\\
&\le Cw(Q)^{-2/p}\sum_{m=1}^{\infty}2^{-2m(n+\varepsilon)}w(2^{m+1}Q)^{2b_0+1}\\
&\le Cw(Q)^{-2/p}w(Q)^{2b_0+1}\sum_{m=1}^{\infty}2^{-2m(n+\varepsilon)}\Big(\frac{w(2^{m+1}Q)}{w(Q)}\Big)^{2b_0+1}\\
&\le Cw(Q)^{2a_0}\sum_{m=1}^{\infty}2^{-m\big(2(n+\varepsilon)-(2b_0+1)nq\big)}\\
&\le Cw(Q)^{2a_0}.
\end{aligned}$$

By the L^2_w boundedness of T and the size condition of atom a, we have

$$\begin{aligned}
\mathfrak{N}_w(Ta) &= \|Ta\|_{L^2_w}^{a_0/b_0}\cdot\|Ta(\cdot)w\big(I^{x_Q}_{|\cdot-x_Q|}\big)^{b_0}\|_{L^2_w}^{1-a_0/b_0}\\
&\le C\|a\|_{L^2_w}^{a_0/b_0}w(Q)^{a_0(1-a_0/b_0)}\\
&\le C.
\end{aligned}$$

This completes the proof of Theorem 4.2. ∎

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