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# STABILITY OF EXACT PENALTY FOR NONCONVEX INEQUALITY-CONSTRAINED MINIMIZATION PROBLEMS

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**Abstract.** In this paper we use the penalty approach in order to study inequality-constrained minimization problems with locally Lipschitz objective and constraint functions in Banach spaces. A penalty function is said to have the generalized exact penalty property if there is a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. In this paper we show that the generalized exact penalty property is stable under perturbations of objective functions, constraint functions and the right-hand side of constraints.

## 1. INTRODUCTION

Penalty methods are an important and useful tool in constrained optimization. See, for example, [2-5, 9, 12] and the references mentioned there. The notion of exact penalization was introduced by Eremin [7] and Zangwill [17] for use in the development of algorithms for nonlinear constrained optimization. Since that time exact penalty functions have continued to play a key role in the theory of mathematical programming [8, 10, 11, 13, 15, 19-21]. For more discussions and various applications of exact penalization to various constrained optimization problems see [2, 3, 5, 12].

We use the penalty approach in order to study inequality-constrained minimization problems with locally Lipschitzian constraints in Banach spaces. A penalty function is said to have the exact penalty property [2, 3, 5, 12] if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem.

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In this paper we will establish the exact penalty property for a large class of inequality-constrained minimization problems

$$(P_i) \qquad f(x) \to \min \text{ subject to } x \in A$$

where

$$A = \{x \in X : \; g_i(x) \le c_i \text{ for all } i = 1, \dots, n\}.$$

Here $X$ is a Banach space, $c_i$, $i = 1, \dots, n$ are real numbers, and the constraint functions $g_i$, $i = 1, \dots, n$ and the objective function $f$ are locally Lipschitz.

We associate with the inequality-constrained minimization problem above the corresponding family of unconstrained minimization problems

$$f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \to \min, \; z \in X$$

where $\gamma > 0$ is a penalty. In this paper we establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. This novel approach in the penalty type methods was used in [19-21]. In the present paper we obtain a generalization of the results of [19-21]. We study the stability of the generalized exact penalty property under perturbations of the functions $f$ and $g_1, \dots, g_n$ and of the parameters $c_1, \dots, c_n$. The stability of the generalized exact penalty property is crucial in practice. One reason is that in practice we deal with a problem which is a perturbation of the problem we wish to consider. Another reason is that the computations introduce numerical errors.

Consider a minimization problem $h(z) \to \min$, $z \in X$ where $h : X \to R^1$ is a lower semicontinuous bounded from below function. If the space $X$ is infinite-dimensional, then the existence of solutions of the problem is not guaranteed and in this situation we consider $\delta$-approximate solutions. Namely, $x \in X$ is a $\delta$-approximate solution of the problem $h(z) \to \min$, $z \in X$, where $\delta > 0$, if $h(x) \le \inf\{h(z) : \; z \in X\} + \delta$.

In [19-21] and in this paper we are interested in approximate solutions of the unconstrained penalized problem and in approximate solutions of the corresponding constrained problem. Under certain assumptions which hold for a large class of problems we show the existence of a constant $\Lambda_0 > 0$ such that the following property holds:

For each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ which depends only on $\epsilon$ such that if $x$ is a $\delta(\epsilon)$-approximate solution of the unconstrained penalized problem whose penalty coefficient is larger than $\Lambda_0$, then there exists an $\epsilon$-approximate solution $y$ of the corresponding constrained problem such that $||y - x|| \le \epsilon$.

This property implies that any exact solution of the unconstrained penalized problem whose penalty coefficient is larger than $\Lambda_0$, is an exact solution of the corresponding constrained problem. Indeed, let $x$ be a solution of the unconstrained penalized problem whose penalty coefficient is larger than $\Lambda_0$. Then for any $\epsilon > 0$ the point $x$ is also a $\delta(\epsilon)$-approximate solution of the same unconstrained penalized problem and in view of the property above there is an $\epsilon$-approximate solution $y_\epsilon$ of the corresponding constrained problem such that $||x - y_\epsilon|| \leq \epsilon$. Since $\epsilon$ is an arbitrary positive number we can easily deduce that $x$ is an exact solution of the corresponding constrained problem. Therefore our results also includes the classical penalty result as a special case.

It should be mentioned that if one uses methods in order to solve optimization problems these methods usually provide only approximate solutions of the problems. Therefore our results are important and useful even when optimization problems have exact solutions.

As we have already mentioned the main result of the present paper is a generalization of the results of [19-21]. In [19] we considered the problem $(P_i)$ with one constraint function ($n = 1$) and established a very simple sufficient condition for the exact penalty property. It was shown that the problem $f(x) \to \min$ subject to $g(x) \leq c$ possesses the exact penalty if the real number $c$ is not a critical value of the function $g$. In other words the set $g^{-1}(c)$ does not contain a critical point of the function $g$. Note that in [19] we used the notion of a critical point of a Lipschitz function introduced in [18]. The result of [19] was generalized in [20] for the problem $(P_i)$ with an arbitrary number of constraints $n$. Moreover, in [20] we showed the stability of the generalized exact penalty property under perturbations of the objective functions $f$. We considered a family of inequality-constrained problems of type $(P_i)$ with given real numbers $c_1, \dots, c_n$, given locally Lipschitz constraint functions $g_1, \dots, g_n$ and with objective functions $f$ which are close (in a certain natural sense) to a given fonction $f_0$. In [20] we showed that all the constrained minimization problems belonging to this family possess the generalized exact penalty property with the same penalty coefficient which depends only on $f_0, g_1 \dots, g_n, c_1, \dots, c_n$. Another generalization of the result of [19] was obtained in [21]. In [21] we assumed that $g_0$ is a locally Lipschitz function defined on $X$, $f_0 : X \to R^1$ is a function which is Lipschitz on all bounded subsets of $X$ and which satisfies a growth condition, and that for a real number $c_0$ which is not a critical value of $g_0$, the set $g_0^{-1}(c_0)$ is nonempty. We considered a family of constrained minimization problems $f(x) \to \min$ subject to $g(x) \leq c$ where a triple $(f, g, c)$ is close to the triple $(f_0, g_0, c_0)$ in a certain natural sense. We showed that all the constrained minimization problems belonging to this family possess the generalized exact penalty property with the same penalty coefficient which depends only on $f_0, g_0, c_0$. Note that the proofs in [20, 21] are based on tools of variational analysis [4, 12, 16]. In [20] in order to generalize the results of [19] we introduced a notion of a critical point of a Lipshitz mapping with respect to a parameter $\kappa \in (0, 1)$. In the proof of the stability result of [21] we used the methods and techniques of [18]. In the present paper combining the methods and techiques of [20, 21] we generalize their results for inequality-constrained problems of type $(P_i)$ with an arbitrary number of constraints $n$. We establish stability of the generalized exact penalty property under perturbations of objective functions, constraint functions and the right-hand side of constraints.

More precisely, we consider a family of constrained minimization problems of type $(P_i)$ with an objective function close to a given function $f$, with constraint functions close to given functions $g_1, \dots, g_n$ and with the right-hand side of constraints close to given constants $c_1, \dots, c_n$ in a certain natural sense. Under certain conditions on $f, g_1, \dots, g_n, c_1, \dots, c_n$ we show that all the constrained minimization problems belonging to this family possess the generalized exact penalty property with the same penalty coefficient which depends only on $f$, $g_1, \dots, g_n$, $c_1, \dots, c_n$.

## 2. Preliminaries

Let $(X, ||\cdot||)$ be a Banach space and let $(X^*, ||\cdot||_*)$ be its dual space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||x - y|| \leq r\}.$$

Assume that $f : U \to R^1$ be a Lipschitz function which is defined on a nonempty open set $U \subset X$. For each $x \in U$ let

$$f^0(x, h) = \limsup_{t \to 0^+, y \to x} [f(y + th) - f(y)]/t, \ h \in X$$

be the Clarke generalized directional derivative of $f$ at the point $x$ [4], let

$$\partial f(x) = \{l \in X^* : \ f^0(x, h) \geq l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of $f$ at $x$ [4] and set

$$\Xi_f(x) = \inf\{f^0(x, h) : \ h \in X \text{ and } ||h| = 1\}$$

[18].

A point $x \in X$ is called a critical point of $f$ if $0 \in \partial f(x)$ [18].

A real number $c \in R^1$ is called a critical value of $f$ if there is a critical point $x \in U$ of $f$ such that $f(x) = c$.

In order to consider a constrained minimization problem with several constraints we need to use a notion of a critical point for a Lipschitz mapping $F : X \to R^n$ introduced in [20].

Assume that $n$ is a natural number, $U$ is a nonempty open subset of $X$ and that $F = (f_1, \dots, f_n) : U \to R^n$ is a locally Lipschitz mapping.

Let $\kappa \in (0, 1)$. For each $x \in U$ set [20]

$$\Xi_{F,\kappa}(x) = \inf\{||\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i2}\eta_{i2})|| : \tag{2.1}$$

$$\eta_{i1},\ \eta_{i2} \in \partial f_i(x),\ \alpha_{i1},\ \alpha_{i2} \in [0, 1],\ i = 1, \dots, n$$

and there is $j \in \{1, \dots, n\}$ such that $\alpha_{j1}\alpha_{j2} = 0$ and $|\alpha_{j1}| + |\alpha_{j2}| \geq \kappa\}$.

It is known [4, Chapter 2, Sect. 2.3] that for each $x \in U$ and all $i = 1, \dots, n$,

$$\partial(-f_i)(x) = -\partial f_i(x). \tag{2.2}$$

This equality implies that

$$\Xi_{-F,\kappa}(x) = \Xi_{F,\kappa}(x) \text{ for each } x \in U. \tag{2.3}$$

In the sequel we assume that $U = X$.

A point $x \in X$ is called a critical point of $F$ with respect to $\kappa$ if $\Xi_{F,\kappa}(x) = 0$ [20].

A vector $c = (c_1, \dots, c_n) \in R^n$ is called a critical value of $F$ with respect to $\kappa$ if there is a critical point $x \in X$ of $F$ with respect to $\kappa$ such that $F(x) = c$.

**Remark 2.1.** Let $n = 1$. Then $x \in X$ is a critical point of $F$ with respect to $\kappa$ if and only if $0 \in \partial F(x)$. Therefore $x$ is a critical point of $F$ in our sense if and only if $x$ is a critical point of $F$ the sense of [19]. It is clear that in this case the notion of a critical point does not depend on $\kappa$.

**Remark 2.2.** Assume that $f_i \in C^1$, $i = 1, \dots, n$ and $Df_i(x)$ is the Frechet derivative of $f_i$ at $x \in X$, $i = 1, \dots, n$. If $x \in X$ is a critical point of $F$ with respect to $\kappa$, then $Df_i(x)$, $i = 1, \dots, n$ are linear dependent.

The following proposition was proved in [20, Proposition 1.1].

**Proposition 2.1.** *Assume that* $\{x_k\}_{k=1}^{\infty} \subset X$ , $x = lim_{k\to\infty} x_k$ *in the norm topology and that* $\liminf_{k\to\infty} \Xi_{F,\kappa}(x_k) = 0$. *Then* $\Xi_{F,\kappa}(x) = 0$.

Let $M$ be a nonempty subset of $X$. We say that the mapping $F : X \to R^n$ satisfies Palais-Smale (P-S) condition on $M$ with respect to $\kappa$ if for each bounded with respect to the norm topology sequence $\{x_i\}_{i=1}^{\infty} \subset M$ such that $\{F(x_i)\}_{i=1}^{\infty}$ is bounded and $\liminf_{i\to\infty} \Xi_{F,\kappa}(x_i) = 0$ there exists a convergent subsequence of $\{x_i\}_{i=1}^{\infty}$ in $X$ with the norm topology [1, 14, 18].

For each function $h : X \to R^1$ and each nonempty set $A \subset X$ put

$$\inf(h) = \inf\{h(z) : \ z \in X\}, \ \inf(h; A) = \inf\{h(z) : \ z \in A\}.$$

For each $x \in X$ and each $B \subset X$ put

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

We assume that the sum over empty set iz zero.

## 3. Main Results

Denote by $\mathcal{M}$ the set of all continuous functions $h : X \to R^1$. We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\begin{aligned} &\mathcal{E}(\mathcal{M}, q, \epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(x) - g(x)| \le \epsilon \text{ for all } x \in B(0, M)\} \\ &\cap\{(f, g) \in \mathcal{M} \times \mathcal{M} : \ |(f - g)(x) - (f - g)(y)| \le q||x - y|| \\ &\qquad \text{for each } x, y \in B(0, M)\}, \end{aligned} \tag{3.1}$$

where $M, q, \epsilon$ are positive numbers. It is not difficult to see that this uniform space is metrizable and complete.

Let $n$ be a natural number, $f \in \mathcal{M}$, $G = (g_1, \dots, g_n)$ with $g_i \in \mathcal{M}$ for all $i = 1, \dots, n$ and let $c = (c_1, \dots, c_n) \in R^n$.

Put

$$A(G, c) = \{x \in X : \ g_i(x) \le c_i \text{ for all } i = 1, \dots, n\} \tag{3.2}$$

and consider the following constrained minimization problem

$$f(x) \to \min \ \text{ subject to } x \in A(G, c). \tag{P}$$

We associate with the problem (P) the corresponding family of unconstrained minimization problems

$$f(x) + \sum_{i=1}^{n} \lambda_i \max\{g_i(x) - c_i, 0\} \to \min, \ x \in X, \tag{$P_\lambda$}$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$.

For each $\kappa \in (0, 1)$ set

$$\begin{aligned} \Omega_\kappa = \{&x = (x_1, \dots, x_n) \in R^n : \\ &x_i \ge \kappa \text{ for all } i = 1, \dots, n \text{ and } \max_{i=1,\dots,n} x_i = 1\}. \end{aligned} \tag{3.3}$$

Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t \to \infty} \phi(t) = \infty \tag{3.4}$$

and $\bar{a}$ be a positive number. Denote by $\mathcal{M}_\phi$ the set of all functions $h \in \mathcal{M}$ such that

$$h(x) \geq \phi(||x||) - \bar{a} \text{ for all } \in X. \tag{3.5}$$

Assume that

$$\bar{f} \in \mathcal{M}_\phi \tag{3.6}$$

is Lipschitz on all bounded subsets of $X$, $\bar{G} = (\bar{g}_1, \dots, \bar{g}_n) : X \to R^1$ is a locally Lipschitz mapping and that $\bar{c} = (\bar{c}_1, \dots, \bar{c}_n) \in R^n$.

We assume that $A(\bar{G}, \bar{c}) \neq \emptyset$ and fix

$$\theta \in A(\bar{G}, \bar{c}). \tag{3.7}$$

In view of (3.4) there exists a number $M_0$ such that

$$M_0 > 2 + ||\theta|| \text{ and } \phi(M_0 - 2) > \bar{f}(\theta) + \bar{a} + 4. \tag{3.8}$$

For each $x \in A(\bar{G}, \bar{c})$ put

$$I(x) = \{i \in \{1, \dots, n\} : \ \bar{c}_i = \bar{g}_i(x)\}. \tag{3.9}$$

Fix $\kappa \in (0, 1)$. In this paper we use the following assumptions.

$(A1)$ If $x \in A(\bar{G}, \bar{c})$, $q \geq 1$ is the cardinality of a subset $\{i_1, \dots, i_q\}$ of $I(x)$ with $i_1 < i_2 < \dots < i_q$ and if $x$ is a critical point of the mapping

$$(\bar{g}_{i_1}, \dots, \bar{g}_{i_q}) : X \to R^q$$

with respect to $\kappa$, then $\bar{f}(x) > \inf(\bar{f}; A(\bar{G}, \bar{c}))$.

$(A2)$ There is $\gamma_* > 0$ such that for each finite strictly increasing sequence of natural numbers $\{i_1, \dots, i_q\}$ which satisfies $\{i_1, \dots, i_q\} \subset \{1, \dots, n\}$ the mapping $(\bar{g}_{i_1}, \dots, \bar{g}_{i_q}) : \ X \to R^q$ satisfies (P-S) condition on the set

$$\cap_{j \in \{i_1, \dots, i_q\}} (\bar{g}_j^{-1}([\bar{c}_j - \gamma_*, \bar{c}_j + \gamma_*]))$$

with respect to $\kappa$.

$(A3)$ For each $\epsilon > 0$ there is $x_\epsilon \in A(\bar{G}, \bar{c})$ such that $\bar{f}(x_\epsilon) \leq \inf(\bar{f}; A(\bar{G}, \bar{c})) + \epsilon$ and if $I(x_\epsilon) \neq \emptyset$, then zero does not belong to the convex hull of the set

$$\cup_{i \in I(x_\epsilon)} \partial \bar{g}_i(x_\epsilon).$$

In the present paper we establish the existence of exact penalty under assumptions (A1)-(A3). Usually this existence is related to calmness of the perturbed constraint mapping. Here we use the assumptions of the different nature. Note that (A1) holds if any solution of the optimization problem is not a critical point of the correspoding constraint mapping. Assumption (A2) is a version of the classical Palais-Smale condition. Assumption (A3) holds if there is a solution of the optimization problem which is not a critical point of the corresponding constaint maping.

The following theorem is our main result.

**Theorem 3.1.** *Let* (*A1*), (*A2*) *and* (*A3*) *hold and let* $q > 0$*. Then there exist positive numbers* $\Lambda_0, r > 0$ *such that for each* $\epsilon > 0$ *there exists* $\delta \in (0, \epsilon)$ *such that the following assertion holds:*

*If* $f \in \mathcal{M}_\phi$ *satisfies*

$$(f, \bar{f}) \in \mathcal{E}(M_0, q, r),$$

*if* $G = (g_1, \dots, g_n) : \ X \to R^n$ *satisfies*

$$g_i \in \mu \text{ and } (g_i, \bar{g}_i) \in \mathcal{E}(M_0, r, r) \text{ for all } i = 1, \dots, n,$$

*if* $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa, \ \lambda \geq \Lambda_0, \ c = (c_1, \dots, c_n) \in R^n$ *satisfies*

$$|\bar{c}_i - c_i| \leq r \text{ for all } i = 1, \dots, n$$

*and if* $x \in X$ *satisfies*

$$f(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\} \leq \inf\{f(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : z \in X\} + \delta,$$

*then there is* $y \in A(G, c)$ *such that*

$$||x - y|| \leq \epsilon \text{ and } f(y) \leq \inf(f; A(G, c)) + \epsilon.$$

Theorem 3.1 easily implies the following result.

**Theorem 3.2.** *Let (A1), (A2) and (A3) hold and let* $q > 0$*. Then there exist positive numbers* $\Lambda_0, r$ *such that for each* $f \in \mathcal{M}_\phi$ *satisfying* $(f, \bar{f}) \in \mathcal{E}(M_0, q, r)$*, each mapping* $G = (g_1, \dots, g_n) : X \to R^n$ *which satisfies*

$$g_i \in \mathcal{M} \text{ and } (g_i, \bar{g}_i) \in \mathcal{E}(M_0, r, r) \text{ for all } i = 1, \dots, n, \tag{3.10}$$

*each* $c = (c_1, \dots, c_n) \in R^n$ *satisfying*

$$|\bar{c}_i - c_i| \leq r \text{ for all } i = 1, \dots, n, \tag{3.11}$$

*each* $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$*, each* $\lambda \geq \Lambda_0$ *and each sequence* $\{x_k\}_{k=1}^\infty \subset X$ *which satisfies*

$$\lim_{k\to\infty} [f(x_k) + \sum_{i=1}^n \lambda\gamma_i \max\{g_i(x_k) - c_i, 0\}]$$
$$= \inf\{f(z) + \sum_{i=1}^n \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : \ z \in X\}$$

*there is a sequence* $\{y_k\}_{k=1}^\infty \subset A(G,c)$ *such that*

$$\lim_{k\to\infty} ||y_k - x_k|| = 0 \text{ and } \lim_{k\to\infty} f(y_k) = \inf(f; A(G,c)).$$

**Corollary 3.1.** *Let* (*A1*)*,* (*A2*) *and* (*A3*) *hold and let* $q > 0$*. Then there exist positive numbers* $\Lambda_0, r$ *such that if* $f \in \mathcal{M}_\phi$ *satisfies* $(f, \bar{f}) \in \mathcal{E}(M_0, q, r)$*, if a mapping* $G = (g_1, \dots, g_n) : X \to R^n$ *satisfies (3.10), if* $c = (c_1, \dots, c_n) \in R^n$ *satisfies (3.11), if* $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa, \ \lambda \geq \Lambda_0$ *and if* $x \in X$ *satisfies*

$$f(x) + \sum_{i=1}^n \lambda\gamma_i \max\{g_i(x) - c_i, 0\}$$
$$= \inf\{f(z) + \sum_{i=1}^n \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : \ z \in X\},$$

*then* $x \in A(G,c)$ *and* $f(x) = \inf(f; A(G,c))$*.*

## 4. Proof of Theorem 3.1

For each $f \in \mathcal{M}_\phi$, each $G = (g_1, \dots, g_n) : X \to R^n$, each $c = (c_1, \dots, c_n) \in R^n$ and each $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$ define for all $z \in X$

$$\psi_{\lambda,c}^{(f,G)}(z) = f(z) + \sum_{i=1}^n \lambda_i \max\{g_i(z) - c_i, 0\}. \tag{4.1}$$

We show that there exist positive numbers $\Lambda_0, r$ such that the following property holds:

(P1) For each $\epsilon \in (0,1)$ there exists $\delta \in (0, \epsilon)$ such that for each $f \in \mathcal{M}_\phi$ satisfying

$$(f, \bar{f}) \in \mathcal{E}(M_0, q, r),$$

each $G = (g_1, \dots, g_n) : X \to R^n$ satisfying

$$g_i \in \mathcal{M} \text{ and } (g_i, \bar{g}_i) \in \mathcal{E}(M_0, r, r) \text{ for all } i = 1, \dots, n,$$

each $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$, each $c = (c_1, \dots, c_n) \in R^n$ satisfying

$$|c_i - \bar{c}_i| \leq r, \ i = 1, \dots, n$$

and each $x \in X$ satisfying

$$\psi^{(f,G)}_{\lambda\gamma,c}(x) \leq \inf(\psi^{(f,G)}_{\lambda\gamma,c}) + \delta$$

we have

$$\{y \in B(x, \epsilon) \cap A(G, c) : \ \psi^{(f,G)}_{\lambda\gamma,c}(y) \leq \psi^{(f,G)}_{\lambda\gamma,c}(x)\} \neq \emptyset.$$

It is not difficult to see that (P1) implies the validity of Theorem 3.1.

Let us assume that there are no $\Lambda_0 > 0, r > 0$ such that the property (P1) holds. Then for each natural number $k$ there exist $\epsilon_k \in (0, 1)$, $f^{(k)} \in \mathcal{M}_\phi$ satisfying

$$(f_k, \bar{f}) \in \mathcal{E}(M_0, q, k^{-1}), \tag{4.2}$$

$G^{(k)} = (g_1^{(k)}, \dots, g_n^{(k)}) : X \to R^n$ satisfying

$$g_i^{(k)} \in \mathcal{M} \text{ and } (g_i^{(k)}, \bar{g}_i) \in \mathcal{E}(M_0, k^{-1}, k^{-1}) \text{ for all } \ i = 1, \dots, n, \tag{4.3}$$

$$\gamma^{(k)} = (\gamma_1^{(k)}, \dots, \gamma_n^{(k)}) \in \Omega_\kappa, \ \lambda_k \geq k, \tag{4.4}$$

$c^{(k)} = (c_1^{(k)}, \dots, c_n^{(k)}) \in R^n$ satisfying

$$|c_i^{(k)} - \bar{c}_i| \leq k^{-1}, \ i = 1, \dots, n, \tag{4.5}$$

and $x_k \in X$ such that

$$\psi^{(f^{(k)},G^{(k)})}_{\lambda_k\gamma^{(k)},c^{(k)}}(x_k) \leq \inf(\psi^{(f^{(k)},G^{(k)})}_{\lambda_k\gamma^{(k)},c^{(k)}}) + 2^{-1}\epsilon_k k^{-2}, \tag{4.6}$$

$$\{y \in B(x_k, \epsilon_k) \cap A(G^{(k)}, c^{(k)}) : \ \psi^{(f^{(k)},G^{(k)})}_{\lambda_k\gamma^{(k)},c^{(k)}}(y) \leq \psi^{(f^{(k)},G^{(k)})}_{\lambda_k\gamma^k,c^{(k)}}(x_k)\} = \emptyset. \tag{4.7}$$

For each natural number $k$ set

$$\psi_k = \psi^{(f^{(k)},G^{(k)})}_{\lambda_k\gamma^{(k)},c^{(k)}}. \tag{4.8}$$

Set

$$\bar{\psi} = \psi^{(\bar{f},\bar{G})}_{\lambda_k\gamma^{(k)},c^{(k)}}. \tag{4.9}$$

Let $k$ be a natural number. It follows from (4.6) and Ekeland's variational principle [6] that there exists $y_k \in X$ such that

$$\psi_k(y_k) \leq \psi_k(x_k), \tag{4.10}$$

$$||y_k - x_k|| \le (2k)^{-1}\epsilon_k, \tag{4.11}$$

$$\psi_k(y_k) \le \psi_k(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{4.12}$$

By (4.7), (4.8), (4.10) and (4.11)

$$y_k \notin A(G^{(k)}, c^{(k)}) \text{ for all natural numbres } k. \tag{4.13}$$

For each natural number $k$ set

$$\begin{aligned} I_k &= \{i \in \{1, \dots, n\} : \ g_i^{(k)}(y_k) = c_i^{(k)}\}, \\ I_{k+} &= \{i \in \{1, \dots, n\} : \ g_i^{(k)}(y_k) > c_i^{(k)}\}, \\ I_{k-} &= \{i \in \{1, \dots, n\} : \ g_i^{(k)}(y_k) < c_i^{(k)}\}. \end{aligned} \tag{4.14}$$

By (4.13), (4.14) and (3.2),

$$I_{k+} \neq \emptyset \text{ for all integers } k \ge 1. \tag{4.15}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that for all natural numbers $k$,

$$I_k = I_1, \ I_{k+} = I_{1+}, \ I_{k-} = I_{1-}. \tag{4.16}$$

We continue the proof with the two steps.

**Step 1.** We will show that for all suffciently large natural numbers $k$

$$A(G^{(k)}, c^{(k)}) \neq \emptyset, \ ||y_k|| \le M_0 - 2$$

and that

$$\limsup_{k\to\infty} f^{(k)}(y_k) \le \limsup_{k\to\infty} \inf(f^{(k)}; A(G^{(k)}, c^{(k)})) \le \inf(\bar{f}; A(\bar{G}, \bar{c})).$$

Let $\delta_0 \in (0, 2^{-1})$. By (A3) there exists

$$z_0 \in A(\bar{G}, \bar{c}) \tag{4.17}$$

such that:

$$\bar{f}(z_0) \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + \delta_0; \tag{4.18}$$

(4.19) if $I(z_0) \neq \emptyset,$ then $0$ does not belong to the convex hull of the set $\cup_{i \in I(z_0)} \partial \bar{g}_i(z_0)$.

By (4.17), (4.18) and (3.7),

$$\bar{f}(z_0) \le \bar{f}(\theta) + 1. \tag{4.20}$$

In view of (4.20), (3.6), (3.5) and (3.8),

$$||z_0|| \le M_0 - 2. \tag{4.21}$$

Define $z_1 \in X$ as follows.

$$\text{If } I(z_0) = \emptyset, \text{ then set } z_1 = z_0. \tag{4.22}$$

Assume that

$$I(z_0) \neq \emptyset. \tag{4.23}$$

Choose $\delta_1 \in (0, 1)$ such that

$$\bar{c}_i > \bar{g}_i(z_0) + 4\delta_1 \text{ for all integers } i \in \{1, \dots, n\} \setminus I(z_0). \tag{4.24}$$

By (4.19) and (4.23) there exists

$$\eta \in X \text{ such that } ||\eta|| = 1 \text{ and } \delta_2 \in (0, 1) \tag{4.25}$$

such that

$$l(\eta) \le -2\delta_2 \text{ for all } l \in \cup_{i \in I(z_0)} \partial \bar{g}_i(z_0). \tag{4.26}$$

In view of (4.26),

$$\bar{g}_i^0(z_0, \eta) \le -2\delta_2 \text{ for all } i \in I(z_0). \tag{4.27}$$

Since the function $\bar{f}$ is Lipschitz on bounded subsets of $X$ and the functions $\bar{g}_i^0(\cdot, \eta)$, $i = 1, \dots, n$ are upper semicontinuous it follows from (4.24) and (4.27) that there exist a number $\delta_3 \in (0, \min\{1, \delta_1\})$ such that

$$\bar{g}_i^0(z, \eta) \le -(3/2)\delta_2 \text{ for all } i \in I(z_0) \text{ and all } z \in B(z_0, \delta_3), \tag{4.28}$$

$$\bar{c}_i > \bar{g}_i(z) + 3\delta_1 \text{ for all } i \in \{1, \dots, n\} \setminus I(z_0) \text{ and all } z \in B(z_0, \delta_3), \tag{4.29}$$

$$|\bar{f}(z) - \bar{f}(z_0)| \le \delta_0 \text{ for all } z \in B(z_0, \delta_3). \tag{4.30}$$

Put

$$z_1 = z_0 + \delta_3 \eta. \tag{4.31}$$

By (4.31), (4.29) and (4.25),

$$\bar{c}_i > \bar{g}_i(z_1) + 3\delta_1 \text{ for all } i \in \{1, \dots, n\} \setminus I(z_0). \tag{4.32}$$

Let $j \in I(z_0)$. By the mean value theorem [4, Theorem 2.3.7], (4.25) and (4.28) there exist

$$s \in [0, \delta_3] \text{ and } l \in \partial \bar{g}_j(z_0 + s\eta)$$

such that

$$\bar{g}_j(z_0+\delta_3\eta)-\bar{g}_j(z_0) = l(\delta_3\eta) \leq \bar{g}_j^0(z+s\eta, \delta_3\eta) = \delta_3\bar{g}_j^0(z_0+s\eta, \eta) \leq \delta_3(-3/2)\delta_2.$$

Combined with (3.9) and (4.31) this implies that

$$\bar{g}_j(z_1) \leq \bar{c}_j - (3/2)\delta_2\delta_3 \text{ for all } j \in I(z_0). \tag{4.33}$$

Relations (4.32) and (4.33) imply that

$$\bar{g}_j(z_1) \leq \bar{c}_j - (3/2)\delta_2\delta_3 \text{ for all } j \in \{1, \dots, n\}. \tag{4.34}$$

By (4.30), (4.31), (4.25) and (4.18),

$$\bar{f}(z_1) \leq \bar{f}(z_0) + \delta_0 \leq \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0. \tag{4.35}$$

In view of (4.31), (4.25) and (4.21),

$$||z_1|| \leq ||z_0|| + \delta_3 \leq M_0 - 1. \tag{4.36}$$

Now we conclude that in both cases which were considered separately ($I(z_0) = \emptyset$; $I(z_0) \neq \emptyset$) we have defined $z_1 \in X$ such that

$$\bar{g}_j(z_1) < \bar{c}_j, \ j = 1, \dots, n. \tag{4.37}$$

$$\bar{f}(z_1) \leq \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0, \tag{4.38}$$

$$||z_1|| \leq M_0 - 1 \tag{4.39}$$

(see (4.34)-(4.36), (4.22), (4.21), (4.17) and (4.18)). It follows from (4.37), (4.39), (4.3), (4.5), (3.2) and (3.1) that there exists a natural number $k_0$ such that

$$z_1 \in A(G^{(k)}, c^{(k)}) \text{ for all integers } k \geq k_0. \tag{4.40}$$

In view of (3.5), (4.8), (4.1), (4.10), (4.6), (4.40), (4.39), (4.2) and (4.48) for any integer $k \geq k_0$

$$
\begin{aligned}
&\phi(||y_k||) - \bar{a} \le f^{(k)}(y_k) \le \psi_k(y_k) \le \psi_k(x_k) \le \inf(\psi_k) + (2k^2)^{-1} \\
&\le \inf(\psi_k; A(G^{(k)}, c^{(k)})) + (2k^2)^{-1} = \inf(f^{(k)}; A(G^{(k)}, c^{(k)})) + 2^{-1}k^{-2} \\
&\le f^{(k)}(z_1) + 2^{-1}k^{-2} \le \bar{f}(z_1) + k^{-1} + 2^{-1}k^{-2} \\
&\le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0 + 2k^{-1}.
\end{aligned} \tag{4.41}
$$

By (4.41), the inequality $\delta_0 < 1/2$, (3.7) and (3.8) for all integers $k \ge k_0$

$$\phi(y_k) - \bar{a} \le \bar{f}(\theta) + 2,\ ||y_k|| \le M_0 - 2. \tag{4.42}$$

It follows from (4.30) and (4.42) that for all sufficiently large natural numbers $k$

$$A(G^{(k)}, c^{(k)}) \ne \emptyset,\ ||y_k|| \ \le M_0 - 2. \tag{4.43}$$

By (4.41),

$$\limsup_{k\to\infty} f^{(k)}(y_k) \le \limsup_{k\to\infty} \inf(f^{(k)}, A(G^{(k)}, c^{(k)})) \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0.$$

Since $\delta_0$ is an arbitrary element of the interval $(0, 1/2)$ we conclude that

$$\limsup_{k\to\infty} f^{(k)}(y_k) \le \limsup_{k\to\infty} \inf(f^{(k)}, A(G^{(k)}, c^{(k)})) \le \inf(\bar{f}; A(\bar{G}, \bar{c})). \tag{4.44}$$

**Step 2.** In this step we will complete the proof of the theorem. It follows from (4.41), the inequality $\delta_0 \in (0, 1/2)$, (4.8), (4.1), the inclusion $f^{(k)} \in \mathcal{M}_\phi$ and (3.5) that for each integer $k \ge k_0$ and each $i \in I_{1+}$

$$-\bar{a} + \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\} \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2.$$

Together with (4.4) and (3.3) this implies for each integer $k \ge k_0$ and each $i \in I_{1+}$

$$g_i^{(k)}(y_k) - c_i^{(k)} = \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\} \le k^{-1}\kappa^{-1}(\inf(\bar{f}; A(\bar{G}, \bar{c})) + 2 + \bar{a}). \tag{4.45}$$

Then for all sufficiently large natural numbers $k$

$$0 \le g_i^{(k)}(y_k) - c_i^{(k)} \le \gamma_*/2 \text{ for all } i \in I_{1+}.$$

Together with (4.14), (4.16), (4.3), (4.5) and (4.43) this implies that for all sufficiently large natural numbers $k$

$$-\gamma_* \le \bar{g}_i(y_k) - \bar{c}_i \le \gamma_* \text{ for all } i \in I_{1+} \cup I_1. \tag{4.46}$$

Since $\bar{f}$ is Lipschitz on bounded subsets of $X$ there exists a number $L_0 > 1$ such that

$$|\bar{f}(u_1) - \bar{f}(u_2)| \le L_0||u_1 - u_2|| \text{ for each } u_1, u_2 \in B(0, M_0). \tag{4.47}$$

Let $k \geq k_0$ be an integer. It follows from (4.42), (4.14) and (4.16) that there exists an open neighborhood $V$ of $y_k$ in $X$ such that for each $y \in V$

$$g_i^{(k)}(y) > c_i^{(k)} \text{ for all } i \in I_{1+},\ g_i^{k)}(y) < c_i^{(k)} \text{ for all } i \in I_{1-},$$
$$V \subset B(0, M_0 - 1). \tag{4.48}$$

It follows from (4.14), (4.16), (4.1), (4.8) and (4.48) that for each $z \in V$

$$\begin{aligned}
&f^{(k)}(y_k)+\lambda_k \sum_{i\in I_{1+}} \gamma_i^{(k)}(g_i^{(k)}(y_k) - c_i^{(k)})+\lambda_k \sum_{i\in I_1} \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}\\
&= f^{(k)}(y_k) + \sum_{i=1}^{n} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}\\
&= \psi^{(f^k,G^{(k)})}_{\lambda_k\gamma^{(k)},c_k} = \psi_k(y_k) \leq \psi_k(z) + k^{-1}||z - y_k||\\
&= \psi^{(f^{(k)},G^{(k)})}_{\lambda_k\gamma^{(k)},c^{(k)}}(z) + k^{-1}||z - y_k||\\
&= f^{(k)}(z) + \lambda_k \sum_{i\in I_{1+}} \gamma_i^{(k)}(g_i^{(k)}(z) - c_i^{(k)})\\
&\quad +\lambda_k \sum_{i\in I_1} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\} + k^{-1}||z - y_k||.
\end{aligned}$$

By the relation above, (4.48) and the properties of Clarke's generalized gradient [4, Chapter 2, Sect. 2.3],

$$\begin{aligned}
&0 \in \partial f^{(k)}(y_k) + \lambda_k \sum_{i\in I_{1+}} \gamma_i^{(k)} \partial g_i^{(k)}(y_k)\\
&+\lambda_k \sum_{k,i\in I_1} \gamma_i^{(k)}(\cup\{\alpha\partial g_i^{(k)}(y_k) : \alpha\in[0, 1]\}) + k^{-1}\{l\in X^* : \ ||l||_* \leq 1\}.
\end{aligned} \tag{4.49}$$

In view of the properties of Clarke's generalized gradient [4, Chapter 2, Sect. 2.3], (4.42), (4.2) and (3.1),

$$\begin{aligned}
\partial f^{(k)}(y_k) &= \partial(\bar{f} + (f^{(k)} - \bar{f}))(y_k) \subset \partial\bar{f}(y_k) + \partial(f^{(k)} - \bar{f})(y_k)\\
&\subset \partial\bar{f}(y_k) + q\{l \in X^* : \ ||l||_* \leq 1\}.
\end{aligned} \tag{4.50}$$

By the properties of Clarke's generalized gradient [4, Chapter 2, Sect. 2.3], (4.42), (4.3) and (3.1) for all $i \in I_1 \cup I_{1+}$,

$$\begin{aligned}
\partial g_i^{(k)}(y_k) &= \partial(\bar{g}_i + (g_i^{(k)} - \bar{g}_i))(y_k) \subset \partial\bar{g}_i(y_k) + \partial(g_i^{(k)} - \bar{g}_i)(y_k)\\
&\subset \partial\bar{g}_i(y_k) + k^{-1}\{l \in X^* : \ ||l||_* \leq 1\}.
\end{aligned} \tag{4.51}$$

Relations (4.49), (4.4), (4.50), (4.51), (3.3) and (4.4),

$$
\begin{aligned}
&0 \in \lambda_k^{-1}\partial f^{(k)}(y_k) + \sum_{i \in I_{1+}} \gamma_i^{(k)} \partial g_i^{(k)}(y_k) + \sum_{i \in I_1} \gamma_i^{(k)} (\cup\{\alpha \partial g_i^{(k)}(y_k) : \\
&\quad \alpha \in [0,1]\}) + k^{-2}\{l \in X^* : \ ||l||_* \le 1\} \\
&\quad \subset \lambda_k^{-1}\partial \bar{f}(y_k) + k^{-1} q\{l \in X^* : \ ||l||_* \le 1\} \\
&\quad + \sum_{i \in I_{1+}} \gamma_i^{(k)} [\partial \bar{g}_i(y_k) + k^{-1}\{l \in X^* : \ ||l||_* \le 1\}] \\
&\quad + \sum_{i \in I_1} \gamma_i^{(k)} (\cup\{\alpha \partial \bar{g}_i(y_k) + \alpha k^{-1}\{l \in X^* : \ ||l||_* \le 1\} : \ \alpha \in [0,1]\}) \\
&\quad + k^{-2}\{l \in X^* : \ ||l||_* \le 1\} \\
&\quad \subset \lambda_k^{-1}\partial \bar{f}(y_k) + \sum_{i \in I_{1+}} \gamma_i^{(k)} \partial \bar{g}_i(y_k) + \sum_{i \in I_1} \gamma_i^{(k)} (\cup\{\alpha \partial \bar{g}_i(y_k) : \alpha \in [0,1]\}) \\
&\quad + (q/k + n/k + n/k + k^{-2})\{l \in X^* : \ ||l||_* \le 1\}.
\end{aligned} \tag{4.52}
$$

It follows from (4.52) that there exists $l_* \in X^*$ satisfying

$$
\begin{aligned}
&||l_*||_* \le 1, \\
&l_0 \in \partial \bar{f}(y_k), \ l_i \in \partial \bar{g}_i(y_k), \ i \in I_{1+} \cup I_1, \ \alpha_i \in [0,1], \ i \in I_1
\end{aligned} \tag{4.53}
$$

such that

$$
0 = k^{-1}(q + 2n + k^{-1}) l_* + \lambda_k^{-1} l_0 + \sum_{i \in I_{1+}} \gamma_i^{(k)} l_i + \sum_{i \in I_1} \alpha_i \gamma_i^{(k)} l_i.
$$

Combined with (4.53), (4.42) and (4.47) this implies that

$$
|| \sum_{i \in I_{1+}} \gamma_i^{(k)} l_i + \sum_{i \in I_1} \alpha_i \gamma_i^{(k)} l_i|| \le k^{-1}(q + 2n + 1) + k^{-1} L_0. \tag{4.54}
$$

In view of (4.15) and (4.16) there exists a finite strictly increasing sequence of natural numbers $i_1 < \cdots < i_q$, where $q \ge 1$ is an integer, such that

$$
\{i_1, \ldots, i_q\} = I_{1+} \cup I_1.
$$

Consider a mapping $G = (\bar{g}_{i_1}, \ldots, \bar{g}_{i_q}) : X \to R^q$. By (4.54), (4.53), (2.1), (4.15), (4.16), (3.3) and (4.4),

$$
\Xi_{G,\kappa}(y_k) \le k^{-1}(q + 2n + 1 + L_0) \text{ for each natural number } k \ge k_0. \tag{4.55}
$$

It follows from (4.55), (A2), (4.46) and (4.42) that there exists a subsequence $\{y_{k_p}\}_{p=1}^{\infty}$ of the sequence $\{y_k\}_{k=1}^{\infty}$ which converges to $y_* \in X$ in the norm topology:

$$\lim_{p\to\infty} ||y_{k_p} - y_*|| = 0. \tag{4.56}$$

By (4.55), (4.56) and Proposition 2.1

$$\Xi_{G,\kappa}(y_*) = 0. \tag{4.57}$$

In view of (4.56), (4.5), (4.3), (4.42), (4.45), (4.14) and (4.16) for $s \in \{1, \dots, q\}$

$$\bar{g}_{i_s}(y_*) - \bar{c}_{i_s} = \lim_{p\to\infty} (\bar{g}_{i_s}(y_{k_p}) - c_{i_s}^{(k_p)}) = \lim_{p\to\infty} (g_{i_s}^{(k_p}(y_{k_p}) - c_{i_s}^{(k_p)}) = 0. \tag{4.58}$$

For any

$$j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_q\},$$

we have $j \in I_{1-}$ and by (4.14), (4.16), (4.56), (4.42) and (4.3),

$$\bar{g}_j(y_*) = \lim_{p\to\infty} \bar{g}_j(y_{k_p)} = \lim_{p\to\infty} g_j^{(k_p})(y_{k_p}) \le \lim_{p\to\infty} c_j^{(k_p)} = \bar{c}_j.$$

Together with (4.58) this implies that

$$y_* \in A(\bar{G}, \bar{c}). \tag{4.59}$$

In view of (4.58),

$$\{i_1, \dots, i_q\} \subset I(y_*). \tag{4.60}$$

By (4.56), (4.42), (4.2) and (4.44)

$$\bar{f}(y_*) = \lim_{p\to\infty} \bar{f}(y_{k_p}) = \lim_{p\to\infty} f^{k_p}(y_{k_p}) \le \inf(\bar{f}; A(\bar{G}, \bar{c})).$$

Combined with (4.59), (4.57) and (4.60) this implies that

$$y_* \in A(\bar{G}, \bar{c}),\ \bar{f}(y_*) = \inf(\bar{f}; A(\bar{G}, \bar{c})),\ \Xi_{G,\kappa}(y_*) = 0.$$

Together with (4.60) this contradicts (A1). The contradiction we have reached proves that there exist $\Lambda_0, r > 0$ such that the property (P1) holds. This completes the proof of Theorem 3.1.

## References

1. T. Q. Bao and B. S. Mordukhovich, Variational principles for set-valued mappings with applications to multiobjective optimization, *Control and Cybernetics*, **36** (2007).

2. D. Boukari and A. V. Fiacco, Survey of penalty, exact-penalty and multiplier methods from 1968 to 1993 *Optimization*, **32** (1995), 301-334.

3. J. V. Burke, An exact penalization viewpoint of constrained optimization, *SIAM J. Control Optim.*, **29** (1991), 968-998.

4. F. H. Clarke, *Optimization and Nonsmooth Analysis*, Willey Interscience, 1983.

5. G. Di Pillo and L. Grippo, Exact penalty functions in constrained optimization, *SIAM J. Control Optim.*, **27** (1989), 1333-1360.

6. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324-353.

7. I. I. Eremin, The penalty method in convex programming, *Soviet Math. Dokl.*, **8** (1966), 459-462.

8. S.-P. Han and O. L. Mangasarian, Exact penalty function in nonlinear programming, *Math. Programming*, **17** (1979), 251-269.

9. J.-B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms*, Springer, Berlin, 1993.

10. D. G. Luenberger, Control problems with kinks, *IEEE Trans. Automat. Control*, **15** (1970), 570-575.

11. B. S. Mordukhovich, Penalty functions and necessary conditions for the extremum in nonsmooth and nonconvex optimization problems, *Uspekhi Math. Nauk*, **36** (1981), 215-216.

12. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications*, Springer, Berlin, 2006.

13. J. Outrata, A note on the usage of nondifferentiable exact penalties in some special optimization problems, *Kybernetika*, **24** (1988), 251-258.

14. R. Palais, Lusternik-Schnirelman theory of Banach manifolds, *Topology*, **5** (1966), 115-132.

15. R. T. Rockafellar, Penalty methods and augmented Lagrangians in nonlinear programming, Fifth Conference on Optimization Techniques (Rome, 1973), Part I. *Lecture Notes in Comput. Sci.*, Springer, Berlin, **3** (1973), 418-425.

16. R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

17. W. I. Zangwill, Nonlinear programming via penalty functions, *Management Sci.*, **13** (1967), 344-358.

18. A. J. Zaslavski, On critical points of Lipschitz functions on smooth manifolds, *Siberian Math. J.*, **22** (1981), 63-68.

19. A. J. Zaslavski, A sufficient condition for exact penalty in constrained optimization, *SIAM Journal on Optimization*, **16** (2005), 250-262.

20. A. J. Zaslavski, Existence of exact penalty for optimization problems with mixed constraints in Banach spaces, *J. Math. Anal. Appl.*, **324** (2006), 669-681.

21. A. J. Zaslavski, *Existence of exact penalty and its stability for nonconvex constrained optimization problems in Banach spaces*, Set-Valued Analysis, in press.

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