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HAUSDORFF NORMS OF RETRACTIONS IN BANACH SPACES OF CONTINUOUS FUNCTIONS

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Abstract. We construct retractions with positive lower Hausdorff norms and small Hausdorff norms in Banach spaces of real continuous functions which domains are not necessarily bounded or finite dimensional. Moreover, we give precise formulas for the lower Hausdorff norms and the Hausdorff norms of such maps.

1. INTRODUCTION

Let X be a Banach space, and let

$$B(X) = \{x \in X : \|x\| \leq 1\} \quad \text{and} \quad S(X) = \{x \in X : \|x\| = 1\}.$$

It is well known that there exists a continuous mapping (retraction) $R : B(X) \to S(X)$ satisfying $Rx = x$ for all $x \in S(X)$ if and only if the space X has infinite dimension.

Recall that the *Hausdorff measure of noncompactness* $\gamma_X(A)$ of a bounded subset A of X is the infimum of all $\varepsilon > 0$ such that A has a finite ε-net in X (see [2]).

Assume X is infinite dimensional and set $\inf \phi = \infty$. Throughout this paper whenever X is a Banach space of real continuous function we meant it endowed with the sup norm $\|\cdot\|_\infty$. Given a retraction $R : B(X) \to S(X)$ the quantitative characteristics

$$\underline{\gamma}_X(R) = \sup\{0 \leq k \leq 1 : \gamma_X(RA) \geq k\gamma_X(A) \text{ for every } A \subseteq B(X)\},$$

$$\gamma_X(R) = \inf\{k \geq 1 : \gamma_X(RA) \leq k\gamma_X(A) \text{ for every } A \subseteq B(X)\}$$

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are the *lower Hausdorff norm* ($\underline{\gamma}_X$-norm, for short) and the *Hausdorff norm* (γ_X-norm, for short) of R, respectively. It is of interest in problems of nonlinear analysis (see, for example, [1, 5, 7]) the estimate of $\underline{\gamma}_X(R)$ for a given retraction R and, in connection with the Hausdorff norm, the evaluation of the following quantitative characteristic

$$W(X) = \inf\{k \geq 1 : \exists \text{ a retraction } R : B(X) \to S(X) \text{ with } \gamma_X(R) \leq k\},$$

called the *Wośko constant* of the space X. The constant $W(X)$ was introduced by Wośko in [11], where it is proved that $W(\mathcal{C}[0,1]) = 1$. The same result has been extended in [3] and [9] to other Banach spaces of real continuous functions. On the other hand we observe that there is not a unified method to evaluate $W(X)$, most of the evaluations have required individual constructions in each space X (see, for example, [1, 4, 10]).

Let K be a set in a normed space E containing the closed unit ball $B(E)$ and denote by $\mathcal{B}C_{B(E)}(K)$ the Banach space of all real bounded functions that are *continuous* on K and *uniformly continuous* on $B(E)$. In the first section we prove that in the space $\mathcal{B}C_{B(E)}(K)$ for any $u > 0$ there exists a retraction R_u of the closed unit ball onto its boundary with lower Hausdorff norm

$$\underline{\gamma}_{\mathcal{B}C_{B(E)}(K)}(R_u) = \begin{cases} \dfrac{1}{2}, & \text{if } u \leq 4 \\ \dfrac{2}{u}, & \text{if } u > 4 \end{cases}$$

and Hausdorff norm $\gamma_{\mathcal{B}C_{B(E)}(K)}(R_u) = \frac{u+8}{u}$. The latter equality gives that the Wośko constant $W(\mathcal{B}C_{B(E)}(K)) = 1$.

In the second section we deal with the Hilbert cube P. We show that in the Banach space $\mathcal{C}(P)$, of all real valued continuous functions defined on P, for any $u > 0$ there exists a retraction R_u of the closed unit ball onto its boundary with lower Hausdorff norm

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, & \text{if } u \leq 4 \\ \dfrac{4}{u}, & \text{if } u > 4 \end{cases}$$

and Hausdorff norm $\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}$. Hence $W(\mathcal{C}(P)) = 1$.
We observe that a retraction R which has positive $\underline{\gamma}_X$-norm is a proper map, i.e., the preimage $R^{-1}M$ of any compact set $M \subseteq X$ is compact. Thus all retraction R_u we construct are proper maps.

2. Retractions in the Space $\mathcal{BC}_{B(E)}(K)$

Let A and S, with $A \subseteq S$, be nonempty sets of a topological space. In the following we will denote by $\mathcal{BC}(S)$ and $\mathcal{BCU}(S)$ the Banach spaces of all real functions defined on S which are, respectively, bounded and continuous, bounded and uniformly continuous. Moreover we denote by $\mathcal{BC}_A(S)$ the Banach space of all real bounded functions that are continuous on S and uniformly continuous on A.

Let $(E, \|\cdot\|)$ be a normed space and K a subset of E containing the closed unit ball $B(E)$. Now for each $a \in [0,1]$ we introduce the maps $\lambda_a, \lambda^a : E \to E$ by

$$\lambda_a(x) = \begin{cases} \dfrac{2}{1+a}\, x, & \text{if } \|x\| \le \dfrac{1+a}{2} \\ \dfrac{x}{\|x\|}, & \text{if } \dfrac{1+a}{2} < \|x\| \le 1 \\ x, & \text{if } \|x\| > 1 \end{cases}$$

and

$$\lambda^a(x) = \begin{cases} \dfrac{1+a}{2}\, x, & \text{if } \|x\| \le 1 \\ x, & \text{if } \|x\| > 1. \end{cases}$$

Moreover for $f \in \mathcal{BC}_{B(E)}(K)$, we set

$$\begin{aligned} A_f &\doteqdot \left\{ f \circ (\lambda_a)_{|K} : a \in [0,1] \right\}, \\ A^f &\doteqdot \left\{ f \circ (\lambda^a)_{|K} : a \in [0,1] \right\}. \end{aligned}$$

Observe that $A_f \subseteq \mathcal{BC}_{B(E)}(K)$ and $A^f \subseteq \mathcal{L}_\infty(K)$, where $\mathcal{L}_\infty(K)$ is the space of all real essentially bounded functions defined on K.

We begin with the following two lemmas.

Lemma 2.1. *Let* $a \in [0,1]$.

(*i*) $\|\lambda_a(x) - \lambda_a(y)\| \le \frac{4}{1+a} \|x - y\|$, *for all* $x, y \in E$

(*ii*) *Let* (a_m) *be a sequence in* $[0,1]$ *converging to* a. *Then* $\|\lambda_{a_m} - \lambda_a\|_\infty \to 0$ *and* $\|\lambda^{a_m} - \lambda^a\|_\infty \to 0$.

Proof. (a) Let $x, y \in E$. Observe that

$$\left\| \frac{x}{\alpha} - \frac{y}{\beta} \right\| \le \frac{2}{\beta} \|x - y\|, \tag{1}$$

for $\alpha, \beta \in (0, \infty)$ with $\|x\| \le \alpha \le \beta \le \|y\|$.
Now if $\|x\|, \|y\| \le \frac{1+a}{2}$ then $\|\lambda_a(x) - \lambda_a(y)\| = \frac{2}{1+a} \|x - y\|$. If $\|x\|, \|y\| >$

1 then $\|\lambda_a(x) - \lambda_a(y)\| = \|x - y\|$. Moreover, using (1) it is easy to get the following implications:

$$\begin{aligned}
\|x\| \leq \frac{1+a}{2}, \quad \frac{1+a}{2} \leq \|y\| \leq 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq \frac{4}{1+a}\|x-y\|, \\
\|x\| \leq \frac{1+a}{2}, \quad \|y\| > 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq 2\|x-y\|, \\
\frac{1+a}{2} \leq \|x\| \leq 1, \quad \|y\| > 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq 2\|x-y\|, \\
\frac{1+a}{2} \leq \|x\| \leq 1, \quad \frac{1+a}{2} \leq \|y\| \leq 1 \quad &\Rightarrow \quad \|\lambda_a(x) - \lambda_a(y)\| \leq \frac{4}{1+a}\|x-y\|.
\end{aligned}$$

(b) Enough to check that $\|\lambda_a(x) - \lambda_b(x)\| \leq |a-b|$ and $\left\|\lambda^a(x) - \lambda^b(x)\right\| \leq \frac{1}{2}|a-b|$. ■

Lemma 2.2. *For all* $f \in \mathcal{BC}_{B(E)}(K)$*, the sets* A_f *and* A^f *are compact in* $\mathcal{BC}_{B(E)}(K)$ *and in* $\mathcal{L}_\infty(K)$*, respectively.*

Proof. Let (a_m) be a sequence of elements of $[0,1]$ converging to a. In order to prove the compactness of the set A_f we will show that

$$\left\| f \circ (\lambda_{a_m})_{|K} - f \circ (\lambda_a)_{|K} \right\|_\infty \to 0.$$

Let $\varepsilon > 0$. Since f is uniformly continuous on $B(E)$, there is $\delta > 0$ such that

$$|f(y) - f(z)| \leq \varepsilon, \tag{2}$$

for all $y, z \in B(E)$ with $\|y - z\| \leq \delta$. By Lemma 2.1 (b) there is $\overline{m} \in N$ such that

$$\|\lambda_{a_m}(x) - \lambda_a(x)\| \leq \delta, \tag{3}$$

for all $m \geq \overline{m}$ and $x \in B(E)$. By (2) and (3) we have

$$\left\| f \circ (\lambda_{a_m})_{|K} - f \circ (\lambda_a)_{|K} \right\|_\infty = \max_{x \in B(E)} |f(\lambda_{a_m}(x)) - f(\lambda_a(x))| \leq \varepsilon,$$

for all $m \geq \overline{m}$. Hence the thesis. The proof of the compactness of the set A^f in $\mathcal{L}_\infty(K)$ is similar. ■

In virtue of (a) of Lemma 2.1 we can define the map $Q : B(\mathcal{BC}_{B(E)}(K)) \to B(\mathcal{BC}_{B(E)}(K))$ as follows

$$Qf(x) := \begin{cases} f\left(\dfrac{2}{1+\|f\|_\infty} x\right), & \text{if } \|x\| \leq \dfrac{1+\|f\|_\infty}{2} \\ f\left(\dfrac{x}{\|x\|}\right), & \text{if } \dfrac{1+\|f\|_\infty}{2} < \|x\| \leq 1 \\ f(x), & \text{if } \|x\| > 1 \end{cases}$$

Then $\|Qf\|_\infty = \|f\|_\infty$ for all $f \in B(\mathcal{BC}_{B(E)}(K))$ and $Qf = f$ for all $f \in S(\mathcal{BC}_{B(E)}(K))$. Observe that the mapping Q can be written as

$$Qf = f \circ (\lambda_{\|f\|_\infty})_{|K}. \tag{4}$$

Proposition 2.3. *The map Q is continuous.*

Proof. Let (f_m) be a sequence in $B(\mathcal{BC}_{B(E)}(K))$ such that $\|f_m - f\|_\infty \to 0$. Let $\varepsilon > 0$. Then there exists $m_1 \in N$ such that $\|f_m - f\|_\infty \leq \frac{\varepsilon}{2}$ for all $m \geq m_1$. By the uniform continuity of f on $B(E)$ there is $\delta > 0$ such that if $x, y \in B(E)$ with $\|x - y\| \leq \delta$ then

$$|f(x) - f(y)| \leq \frac{\varepsilon}{2}.$$

Since $\|f_m\|_\infty \to \|f\|_\infty$, Lemma 2.1(b) implies $\left\|\lambda_{\|f_m\|_\infty} - \lambda_{\|f\|_\infty}\right\|_\infty \to 0$. Hence there is $m_2 \in N$ such that

$$\left\|\lambda_{\|f_m\|_\infty} - \lambda_{\|f\|_\infty}\right\|_\infty \leq \delta,$$

for all $m \geq m_2$. Therefore

$$\left| f\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f\|_\infty}\right)_{|K}(x)\right| \leq \frac{\varepsilon}{2},$$

for all $x \in K$ and $m \geq m_2$. Then, for any $x \in K$ and $m \geq \max\{m_1, m_2\}$, we have

$$\begin{aligned}
&\left| f_m\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f\|_\infty}\right)(x)\right| \\
\leq &\left| f_m\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x)\right| + \left| f\left(\lambda_{\|f_m\|_\infty}\right)_{|K}(x) - f\left(\lambda_{\|f\|_\infty}\right)_{|K}(x)\right| \\
\leq &\|f_m - f\|_\infty + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

So we obtain $\|Qf_m - Qf\|_\infty \to 0$. ∎

By the following proposition we give lower and upper estimates of the Hausdorff measure of noncompactness of QA for a set A in $B(\mathcal{BC}_{B(E)}(K))$.

Proposition 2.4. *Let A be a subset of $B(\mathcal{BC}_{B(E)}(K))$. Then*

$$\frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A).$$

Proof. Let $A \subseteq B(\mathcal{BC}_{B(E)}(K))$. First we prove $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A)$. Fix $\alpha > \gamma_{\mathcal{BC}_{B(E)}(K)}(A)$ and let $\{f_1, ..., f_n\}$ be an α-net of A in $\mathcal{BC}_{B(E)}(K)$. By Lemma 2.2 the set $\cup_{i=1}^{n} A_{f_i}$ is compact in $\mathcal{BC}_{B(E)}(K)$. Hence given $\delta > 0$ we can choose a δ-net $\{g_1, ..., g_m\}$ of $\cup_{i=1}^{n} A_{f_i}$ in $\mathcal{BC}_{B(E)}(K)$.
We show that $\{g_1, ..., g_m\}$ is a $(\alpha + \delta)$-net of QA in $\mathcal{BC}_{B(E)}(K)$. To this end for $g \in QA$ let $f \in A$ such that $Qf = g$ and fix $i \in \{1, ..., n\}$ such that $\|f - f_i\|_\infty \leq \alpha$. Since $f_i \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} \in A_{f_i}$ we can find $j \in \{1, ..., m\}$ such that $\left\|\left((f_i \circ \lambda_{\|f\|_\infty}\right)_{|K} - g_j\right\|_\infty \leq \delta$. Then

$$\begin{aligned}
&\|g_j - Qf\|_\infty \\
&\leq \left\|g_j - f_i \circ \left(\lambda_{\|f\|_\infty}\right)_{|K}\right\|_\infty + \left\|f_i \circ \left(\lambda_{\|f\|_\infty}\right)_{|K} - f \circ \left(\lambda_{\|f\|_\infty}\right)_{|K}\right\|_\infty \\
&\leq \delta + \left\|(f_i - f) \circ \left(\lambda_{\|f\|_\infty}\right)_{|K}\right\|_\infty \leq \delta + \alpha.
\end{aligned}$$

Therefore $\gamma_{\mathcal{BC}_{B(E)}(K)}(K)(QA) \leq \alpha + \delta$, so $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A)$. We now prove $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA) \geq \frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A)$. Fix $\beta > \gamma_{\mathcal{BC}_{B(E)}(K)}(QA)$ and let $\{h_1, ..., h_s\}$ be a β-net for QA in $\mathcal{BC}_{B(E)}(K)$. By Lemma 2.2 the set $\cup_{i=1}^{s} A^{h_i}$ is compact in $\mathcal{L}_\infty(K)$. Therefore given $\delta > 0$ we can choose a δ-net $\{p_1, ..., p_k\}$ for $\cup_{i=1}^{s} A^{h_i}$ in $\mathcal{L}_\infty(K)$. We now show that $\{p_1, ..., p_k\}$ is a $(\beta+\delta)$-net for A in $\mathcal{L}_\infty(K)$. Let $f \in A$. Fix $l \in \{1, ..., s\}$ such that $\|Qf_l - h_l\|_\infty \leq \beta$. Since $h_l \circ (\lambda^{\|f\|_\infty})_{|K} \in A^{h_l}$ we can find $m \in \{1, ..., k\}$ such that $\left\|h_l \circ (\lambda^{\|f\|_\infty})_{|K} - p_m\right\|_\infty \leq \delta$. Then

$$\begin{aligned}
&\|f - p_m\|_\infty \\
\leq\ & \left\|f - h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right\|_\infty + \left\|h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K} - p_m\right\|_\infty \\
\leq\ & \left\|\left(f - h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right) \circ \left(\varphi_{\|f\|_\infty}\right)_{|K}\right\|_\infty + \delta \\
=\ & \sup_{\left\{x \in K : \|x\| \in \left[0, \frac{1+\|f\|_\infty}{2}\right] \cup (1,\infty)\right\}} \left|\left(\left(f - h_l \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right) \circ \left(\lambda_{\|f\|_\infty}\right)_{|K}\right)(x)\right| + \delta \\
=\ & \sup_{\left\{x \in K : \|x\| \in \left[0, \frac{1+\|f\|_\infty}{2}\right] \cup (1,\infty)\right\}} \left|\left(f \circ \left(\lambda^{\|f\|_\infty}\right)_{|K}\right)(x) - h_l(x)\right| + \delta \\
\leq\ & \|Qf - h_l\|_\infty + \delta \leq \beta + \delta.
\end{aligned}$$

Therefore $\gamma_{\mathcal{L}_\infty(K)}(A) \leq \beta + \delta$, so we obtain $\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq 2(\beta + \delta)$ and consequently $\frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(QA)$. ∎

Remark 2.5. Observe that for all $f \in \mathcal{BCU}(B(E))$ the set A^f is compact in $\mathcal{BCU}(B(E))$. Therefore

$$\gamma_{\mathcal{BCU}(B(E))}(QA) = \gamma_{\mathcal{BCU}(B(E))}(A),$$

for all $A \subseteq B(\mathcal{BCU}(B(E)))$.

Next, for a given $u \in (0, +\infty)$, we define a map $P_u : B(\mathcal{BC}_{B(E)}(K)) \to \mathcal{BC}_{B(E)}(K)$ by

$$(P_u f)(x) = \begin{cases} 0, & \text{if } \|x\| \leq \dfrac{1 + \|f\|_\infty}{2} \text{ or } \|x\| > 1, \\ u\left(\|x\| - \dfrac{1 + \|f\|_\infty}{2}\right), & \text{if } \dfrac{1 + \|f\|_\infty}{2} \leq \|x\| \leq \dfrac{3 + \|f\|_\infty}{4}, \\ -u(\|x\| - 1), & \text{if } \dfrac{3 + \|f\|_\infty}{4} \leq \|x\| \leq 1. \end{cases}$$

Proposition 2.6. *The map P_u is compact, i.e., P_u is continuous and $P_u B(\mathcal{BC}_{B(E)}(K))$ is relatively compact.*

Proof. First we prove that the map P_u is continuous. Observe that if $f, g \in B(\mathcal{BC}_{B(E)}(K))$ with $\|f\|_\infty \leq \|g\|_\infty$ and $\frac{1+\|g\|_\infty}{2} \leq \frac{3+\|f\|_\infty}{4}$ we have

$$\|P_u f - P_u g\|_\infty = u \frac{\|g\|_\infty - \|f\|_\infty}{2} . \tag{5}$$

Let now (f_m) be a sequence in $B(\mathcal{BC}_{B(E)}(K))$ converging to f. Then $\|f_m\|_\infty \to \|f\|_\infty$. Moreover if $\|f\|_\infty = 1$ we have

$$\|P_u f_m - P_u f\|_\infty = \frac{u}{4}(1 - \|f_m\|). \tag{6}$$

On the other hand, if $\|f\|_\infty \neq 1$ there is $\overline{m} \in N$ such that for all $m \geq \overline{m}$

$$\|f_m\|_\infty \leq \|f\|_\infty \Longrightarrow \frac{1 + \|f\|_\infty}{2} \leq \frac{3 + \|f_m\|_\infty}{4},$$

and

$$\|f\|_\infty \leq \|f_m\|_\infty \Longrightarrow \frac{1 + \|f_m\|_\infty}{2} \leq \frac{3 + \|f\|_\infty}{4}.$$

Thus by (5), if $\|f\|_\infty \neq 1$, it follows that

$$\|P_u f_m - P_u f\|_\infty = \frac{u}{2} \left| \|f_m\|_\infty - \|f\|_\infty \right| . \tag{7}$$

for all $m \geq \overline{m}$. By (6) and (7) it follows $\|P_u f_m - P_u f\|_\infty \to 0$. To complete the proof it remains to show that $P_u B(\mathcal{BC}_{B(E)}(K))$ is compact. Let (g_m) be a sequence in $P_u B(\mathcal{BC}_{B(E)}(K))$ and let (f_m) be a sequence in $B(\mathcal{BC}_{B(E)}(K))$ with $P_u f_m = g_m$. Without loss of generality we can assume that $\|f_m\|_\infty \to \|f\|_\infty$ where $f \in B(\mathcal{BC}_{B(E)}(K))$ and that (7) holds for all m. Then

$$\|g_m - P_u f\|_\infty = \|P_u f_m - P_u f\|_\infty = \frac{u}{2}\left|\|f_m\|_\infty - \|f\|_\infty\right| \to 0,$$

which completes the proof. ∎

Now we are in the position to prove the main result of this Section.

Theorem 2.7. *Let K be a set in a normed space E such that $B(E) \subseteq K$. For any $u > 0$ there is a retraction $R_u : B\left(\mathcal{BC}_{B(E)}(K)\right) \to S\left(\mathcal{BC}_{B(E)}(K)\right)$ such that*

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \begin{cases} \dfrac{1}{2}, & \textit{if } u \leq 4 \\ \dfrac{2}{u}, & \textit{if } u > 4. \end{cases}$$

and

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{u+8}{u}.$$

In particular we have that $W(\mathcal{BC}_{B(E)}(K)) = 1$.

Proof. Let $u \in (0, +\infty)$. Define a map $T_u : B(\mathcal{BC}_{B(E)}(K)) \to \mathcal{BC}_{B(E)}(K)$ by $T_u f = Qf + P_u f$. Since P_u is compact, Proposition 2.4 implies

$$\frac{1}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(T_u A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(A), \tag{8}$$

for any $A \subseteq B(\mathcal{BC}_{B(E)}(K))$. We have $T_u f = f$ for all $f \in S(\mathcal{BC}_{B(E)}(K))$ and for any $f \in B(\mathcal{BC}_{B(E)}(K))$ we find

$$\|T_u f\|_\infty \geq \min_{f \in B(\mathcal{BC}_{B(E)}(K))} \max\left\{\|f\|_\infty, \frac{u}{4}(1 - \|f\|_\infty) - \|f\|_\infty\right\} \geq \frac{u}{u+8}.$$

We define a retraction $R_u : B(\mathcal{BC}_{B(E)}(K)) \to S(\mathcal{BC}_{B(E)}(K))$ by setting

$$R_u f = \frac{T_u f}{\|T_u f\|_\infty}.$$

Then

$$R_u A \subseteq \left[0, \frac{u+8}{u}\right] \cdot T_u A,$$

therefore using the monotonicity property of the Hausdorff measure of noncompactness and (8) we get

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}\left(\left[0, \frac{u+8}{u}\right] \cdot T_u A\right) \leq \frac{u+8}{u}\gamma_{\mathcal{BC}_{B(E)}(K)}(A).$$

The latter inequality together with (12) of the Example 2.13 implies

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{u+8}{u}. \tag{9}$$

On the other hand,

$$\|T_u f\|_\infty \leq \max\left\{1, \frac{u}{4}\right\} = \begin{cases} 1, & \text{if } u \leq 4, \\ \frac{u}{4}, & \text{if } u > 4. \end{cases}$$

Fix $u > 4$. We have $T_u A \subseteq \left[0, \frac{u}{4}\right] \cdot R_u A$. Again by the monotonicity property of the Hausdorff measure of noncompactness, and using the left hand-side of (8), we obtain

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq 2\gamma_{\mathcal{BC}_{B(E)}(K)}\left(\left[0, \frac{u}{4}\right] \cdot R_u A\right) = \frac{u}{2}\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A).$$

Then for all $u > 4$ we have

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{2}{u}. \tag{10}$$

If $u \leq 4$, we have $T_u A \subseteq [0, 1] \cdot R_u A$. So $\gamma_{\mathcal{BC}_{B(E)}(K)}(A) \leq 2\ \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A)$ hence

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{1}{2}. \tag{11}$$

By Example 2.14 we obtain

$$\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \begin{cases} \frac{1}{2}, & \text{if } u \leq 4 \\ \frac{2}{u}, & \text{if } u > 4. \end{cases}$$

Finally by (9), since $\lim_{u\to\infty} \frac{u+8}{u} = 1$, we infer $W(\mathcal{BC}_{B(E)}(K)) = 1$. ∎

The following example shows that, given a set K in an infinite dimensional normed space E with $B(E) \subseteq K$, the map Q is not anymore continuous when considered from $B(\mathcal{BC}(K))$ into itself. Therefore our construction does not work in the case of the Banach space $\mathcal{BC}(K)$.

Example 2.8. Let (x_k) be a sequence of elements of $B(\mathcal{BC}(K))$ such that $\|x_k\| = \frac{1}{2}$ $(k = 1, 2, ...)$ and $\|x_i - x_j\| \geq \frac{1}{4}$ for all $i, j \in N$ with $i \neq j$. Moreover let (a_k) be a monotone increasing sequence of elements of $(0, 1)$ such that $a_k \to 1$. Set $y_k = \frac{2}{1+a_k} x_k$ $(k = 1, 2, ...)$. We have that $\|y_k\| = \frac{1}{1+a_k} < 1$ and $\|x_k\| = \frac{1}{2} < \frac{1}{1+a_k}$ $(k = 1, 2, ...)$. Set $S = \{x_k : k = 1, 2, ...\} \cup \{y_k : k = 1, 2, ...\}$. Then S is a closed subset of K and the map $f : S \to R$ defined by $f(x_k) = 1$ and $f(y_k) = 0$ $(k = 1, 2, ...)$ is continuous. By the Dugundji's theorem there is a continuous extension $\widetilde{f} : K \to [0, 1]$ of f. We have that $\|f\|_\infty = \left\|\widetilde{f}\right\|_\infty = 1$ and $\widetilde{f}$ is not uniformly continuous on K. In fact, fixed $\varepsilon \in (0, 1)$, since $\|x_k - y_k\| \to 0$, for all $\delta > 0$ there is $\overline{k}$ such that $\left\|x_{\overline{k}} - y_{\overline{k}}\right\| \leq \delta$ and $\left|f(x_{\overline{k}}) - f(y_{\overline{k}})\right| = 1 > \varepsilon$. Now we show that the map Q is not continuous. Put $\widetilde{f}_k = a_k\ \widetilde{f}$ $\ (k = 1, 2, ...)$. Then $\left\|\widetilde{f}_k - \widetilde{f}\right\|_\infty \to 0$ but $\left\|Q\widetilde{f}_k - Q\widetilde{f}\right\|_\infty = 1$ $(k = 1, 2, ...)$. In fact

$$\begin{aligned} 1 &\geq \left\|Q\widetilde{f}_k - Q\widetilde{f}\right\|_\infty \\ &\geq \left|Q\widetilde{f}_k(x_k) - Q\widetilde{f}(x_k)\right| = \left|\widetilde{f}_k\left(\lambda_{\|f_k\|_\infty}(x_k)\right) - \widetilde{f}\left(\lambda_{\|f\|_\infty}(x_k)\right)\right| \\ &= \left|\widetilde{f}_k\left(\lambda_{a_k}(x_k)\right) - \widetilde{f}(x_k)\right| = \left|a_k\widetilde{f}\left(\frac{2}{1+a_k}x_k\right) - \widetilde{f}(x_k)\right| = 1. \end{aligned}$$

Since the measure of noncompactness of a set is invariant under isometries, the following corollary enlarges the class of spaces for which Theorem 2.7 holds.

Corollary 2.9. *Let M be a metric space. Assume that there are a set K in a normed space E containing $B(E)$ and an homeomorphism $h : K \to M$ such that both the restrictions $h_{/B(E)}$ and $h^{-1}_{/h(B(E))}$ are lipschitz, where h^{-1} is the inverse homeomorphism of h. Then for any $u > 0$ there is a retraction R_u : $B\left(\mathcal{BC}_{h(B(E))}(M)\right) \to S\left(\mathcal{BC}_{h(B(E))}(M)\right)$ such that*

$$\underline{\gamma}_{\mathcal{BC}_{h(B(E))}(M)}(R_u) = \begin{cases} \dfrac{1}{2}, & \text{if } u \leq 4 \\ \dfrac{2}{u}, & \text{if } u > 4 \end{cases}$$

and $\gamma_{\mathcal{BC}_{h(B(E))}(M)}(R_u) = \frac{u+8}{u}$. In particular, we have $W(\mathcal{BC}_{h(B(E))}(M)) = 1$.

Proof. It is enough to observe that $i : \mathcal{BC}_{B(E)}(K) \to \mathcal{BC}_{h(B(E))}(M)$ defined by $i(f) = f \circ h$ is an isometry. ∎

Corollary 2.10. *Let $x_0 \in E$, $r > 0$ and K be a set in E such that $B_{x_0,r}(E) := \{x \in E : \|x - x_0\| \leq r\} \subseteq K$. For any $u > 0$ there is a retraction R_u :*

$B(\mathcal{BC}_{B_{x_0,r}(E)}(K)) \to S(\mathcal{BC}_{B_{\{x_0,r\}}(E)}(K))$ *such that*

$$\underline{\gamma}_{\mathcal{BC}_{B_{x_0,r}(E)}(K)}(R_u) = \begin{cases} \frac{1}{2}, & \textit{if } \ u \leq 4 \\ \frac{2}{u}, & \textit{if } \ u > 4 \end{cases}$$

and $\gamma_{\mathcal{BC}_{B_{x_0,r}(E)}(K)}(R_u) = \frac{u+8}{u}$. *In particular,* $W(\mathcal{BC}_{B_{x_0,r}(E)}(K)) = 1$.

Proof. It follows by Corollary 2.9 when we consider $h(x) = \frac{1}{r}(x - x_0)$ for all $x \in K$. ■

Remark 2.11. As a particular case of Theorem 2.7, if E is a finite dimensional normed space, then for any $u > 0$ there is a retraction $R_u : B(\mathcal{BC}(E)) \to S(\mathcal{BC}(E))$ such that

$$\underline{\gamma}_{\mathcal{BC}(E)}(R_u) = \begin{cases} \dfrac{1}{2}, & \text{if } \ u \leq 4 \\ \dfrac{2}{u}, & \text{if } \ u > 4 \end{cases}$$

and $\gamma_{\mathcal{BC}(E)}(R_u) = \frac{u+8}{u}$.

This result with Corollary 2.10 yields $W(\mathcal{C}(K)) = 1$ ([9, Theorem 10]) when K is a convex compact set in E with nonempty interior, and also yields $W(\mathcal{BC}(R)) = 1$ ([3, Theorem 2.4]).

The following corollary covers the case of the space $\mathcal{BCU}(E))$. By repeating the proof of Theorem 2.7, taking into account Remark 2.5, and by slight modifications of Examples 2.13 and 2.14 we have a different evaluation of $\underline{\gamma}_{\mathcal{BCU}(E))}(R_u)$.

Corollary 2.12. *For any* $u > 0$ *there is a retraction* $R_u : B\left(\mathcal{BCU}(E)\right) \to S(\mathcal{BCU}(E))$ *such that*

$$\underline{\gamma}_{\mathcal{BCU}(E)}(R_u) = \begin{cases} 1, & \textit{if } \ u \leq 4 \\ \dfrac{4}{u}, & \textit{if } \ u > 4 \end{cases}$$

and $\gamma_{\mathcal{BCU}(E)}(R_u) = \frac{u+8}{u}$. *In particular,* $W(\mathcal{B}CU(E)(K)) = 1$

In connection with Theorem 2.7 we have the following Examples 2.13 and 2.14.

Example 2.13. Let K be a set in a normed space E such that $B(E) \subseteq K$. Define the maps $f_n : K \to R$ $(n = 3, 4, ...)$ by

$$f_n(x) = \begin{cases} \dfrac{u}{u+8}, & \text{if } \ \|x\| < \dfrac{1}{2} - \dfrac{1}{n}, \\ -n\dfrac{u}{u+8}\left(\|x\| - \dfrac{1}{2}\right), & \text{if } \ \dfrac{1}{2} - \dfrac{1}{n} \leq \|x\| < \dfrac{1}{2} + \dfrac{1}{n}, \\ -\dfrac{u}{u+8}, & \text{if } \ \|x\| \geq \dfrac{1}{2} + \dfrac{1}{n}. \end{cases}$$

Then the following are the expression for Qf_n and, given $u > 0$, that for $P_u f_n$ respectively

$$Qf_n(x) = \begin{cases} \dfrac{u}{u+8}, & \text{if } \|x\| < \dfrac{1+\frac{u}{u+8}}{2}(\dfrac{1}{2}-\dfrac{1}{n}), \\ -2n\dfrac{\frac{u}{u+8}}{1+\frac{u}{u+8}}\left(\|x\| - \dfrac{1+\frac{u}{u+8}}{2}\dfrac{1}{2}\right), & \text{if } \dfrac{1+\frac{u}{u+8}}{2}(\dfrac{1}{2}-\dfrac{1}{n}) \le \|x\| \\ & \quad < \dfrac{1+\frac{u}{u+8}}{2}(\dfrac{1}{2}+\dfrac{1}{n}), \\ -\dfrac{u}{u+8}, & \text{if } \|x\| \ge \dfrac{1+\frac{u}{u+8}}{2}(\dfrac{1}{2}+\dfrac{1}{n}); \end{cases}$$

$$(P_u f_n)(x) = \begin{cases} 0, & \text{if } \|x\| \le \dfrac{1+\frac{u}{u+8}}{2} \text{ or } \|x\| > 1, \\ u\left(\|x\| - \dfrac{1+\frac{u}{u+8}}{2}\right), & \text{if } \dfrac{1+\frac{u}{u+8}}{2} \le \|x\| < \dfrac{3+\frac{u}{u+8}}{4}, \\ -u\left(\|x\|-1\right), & \text{if } \dfrac{3+\frac{u}{u+8}}{4} \le \|x\| \le 1. \end{cases}$$

Hence we obtain

$$\|Qf_n + P_u f_n\|_\infty = \max\left\{\|f_n\|_\infty,\ \frac{u}{4}(1-\|f_n\|) - \|f_n\|\right\} = \frac{u}{u+8}.$$

Setting $A = \{f_n : n \ge 3\}$, we have

$$R_u A = \{R_u f_n : n \ge 3\} = \frac{u+8}{u}(Q+P_u)A,$$

and

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u(A)) = \frac{u+8}{u}\gamma_{\mathcal{BC}_{B(E)}(K)}(A). \tag{12}$$

Example 2.14. Let K be a set in a normed space E. Without loss of generality we may assume $B_2(E) \subseteq K$. Define the maps $f_{c,n} : K \to R$ $(n = 3, 4, ...)$ by

$$f_{c,n}(x) = \begin{cases} -c, & \text{if } \|x\| < 1 - \dfrac{1}{n}, \\ nc(\|x\|-1), & \text{if } 1-\frac{1}{n} \le \|x\| < 1+\frac{1}{n}, \\ c, & \text{if } \|x\| \ge 1 + \dfrac{1}{n}. \end{cases}$$

We have that

$$Qf_{c,n}(x) = \begin{cases} -c, & \text{if } \ \|x\| < \dfrac{1+c}{2}(1-\dfrac{1}{n}), \\ 2n\dfrac{c}{1+c}(\|x\| - \dfrac{1+c}{2}), & \text{if } \ \dfrac{1+c}{2}(1-\dfrac{1}{n}) \leq \|x\| < \dfrac{1+c}{2}, \\ 0, & \text{if } \ \dfrac{1+c}{2} \leq \|x\| < 1, \\ nc(\|x\|-1), & \text{if } \ 1 \leq \|x\| < 1+\dfrac{1}{n}, \\ c, & \text{if } \ \|x\| \geq 1+\dfrac{1}{n}. \end{cases}$$

Set $A_c = \{f_{c,n} : n \geq 3\}$. Then $\gamma_{\mathcal{BC}_{B(E)}(K)}(A_c) = c$ and $\gamma_{\mathcal{BC}_{B(E)}(K)}(QA_c) = \frac{c}{2}$. Let $u > 0$. We find

$$P_u f_n(x) = \begin{cases} 0, & \text{if } \ \|x\| < \dfrac{1+c}{2} \text{ or } \|x\| > 1, \\ u(\|x\| - \dfrac{1+c}{2}), & \text{if } \ \dfrac{1+c}{2} \leq \|x\| < \dfrac{3+c}{4}, \\ -u(\|x\|-1), & \text{if } \ \dfrac{3+c}{4} \leq \|x\| < 1. \end{cases}$$

Thus $\|P_u f_n\|_\infty = u\frac{1-c}{4}$. Moreover if $c \leq \frac{u}{u+4}$ then $\|Qf_n + P_u f_n\|_\infty = \max\{c, u\frac{1-c}{4}\} = u\frac{1-c}{4}$. Hence

$$\gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_c) = \frac{4}{u(1-c)}\gamma_{\mathcal{BC}_{B(E)}(K)}(QA_c) = \frac{2}{u(1-c)}\gamma_{\mathcal{BC}_{B(E)}(K)}(A_c).$$

Now by (10), $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{2}{u}$. Suppose $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{2}{u} + \sigma$ with $\sigma > 0$. Fix $\overline{c}$ such that $\frac{2}{u(1-\overline{c})} < \frac{2}{u} + \sigma$, then

$$\begin{aligned} \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}) &= \frac{2}{u(1-\overline{c})}\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \\ &< \left(\frac{2}{u} + \sigma\right)\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}), \end{aligned}$$

which is a contradiction. So that $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{2}{u}$. For each $u \leq 4$, by (11) we have $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) \geq \frac{1}{2}$. Suppose $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{1}{2} + \sigma$ with $\sigma > 0$. Fix $\overline{c}$ such that $\frac{2}{u(1-\overline{c})} < \frac{1}{2} + \sigma$, then

$$\begin{aligned} \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}) &= \frac{2}{u(1-\overline{c})}\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \\ &< (\frac{1}{2} + \sigma)\gamma_{\mathcal{BC}_{B(E)}(K)}(A_{\overline{c}}) \leq \gamma_{\mathcal{BC}_{B(E)}(K)}(R_u A_{\overline{c}}), \end{aligned}$$

which is a contradiction so that $\underline{\gamma}_{\mathcal{BC}_{B(E)}(K)}(R_u) = \frac{1}{2}$.

3. Retractions in the Space $\mathcal{C}(P)$

Let l_2 be the real Hilbert space, with the usual norm $\|\cdot\|_2$ and canonical basis (e_n). Denote by

$$P = \{x = (x_n) \in l_2 : |x_n| \leq \frac{1}{n} \quad (n = 1, 2, ...)\}$$

the Hilbert cube. We consider the Banach space $\mathcal{C}(P)$ of all real continuous function defined on P.

For $a \in [0, 1]$ define the maps φ_a and $\varphi^a : P \to P$ by

$$\varphi_a(t) := (\lambda_a(t_1), t_2, ..., t_n, ...),$$
$$\varphi^a(t) := (\lambda^a(t_1), t_2, ..., t_n, ...),$$

where by λ_a we continue to denote the restriction of λ_a to the interval $[-1, 1]$, i.e.

$$\lambda_a(t_1) := \begin{cases} -1, & \text{if } t_1 \in \left[-1, -\frac{1+a}{2}\right), \\ \frac{2}{1+a}t_1, & \text{if } t_1 \in \left[-\frac{1+a}{2}, \frac{1+a}{2}\right), \\ 1, & \text{if } t_1 \in \left[\frac{1+a}{2}, 1\right], \end{cases}$$

and

$$\lambda^a(t_1) := \frac{1+a}{2}t_1, \qquad t_1 \in [-1, 1].$$

Moreover for $f \in \mathcal{C}(P)$ set

$$B_f = \{f \circ \varphi_a : a \in [0, 1]\},$$
$$B^f = \{f \circ \varphi^a : a \in [0, 1]\}.$$

Lemma 3.1. *Let $a \in [0, 1]$. The maps φ_a and φ^a are continuous. Moreover, if (a_m) is a sequence in $[0, 1]$ converging to a, then $\|\varphi_{a_m} - \varphi_a\|_\infty \to 0$ and $\|\varphi^{a_m} - \varphi^a\|_\infty \to 0$.*

Proof. Clearly φ_a and φ^a are continuous. Let $\varepsilon > 0$. Then there is $\overline{m}$ such that $|a_m - a| \leq \varepsilon$ for all $m \geq \overline{m}$. It is easy to verify that $|\lambda_{a_m}(t_1) - \lambda_a(t_1)| \leq |a_m - a|$ for any $t_1 \in [0, 1]$, hence we obtain that

$$\|\varphi_{a_m}(t) - \varphi_a(t)\|_2^2 = |\lambda_{a_m}(t_1) - \lambda_a(t_1)|^2 \leq \varepsilon^2,$$

for all $m \geq \overline{m}$ and for all $t \in P$. Thus $\|\varphi_{a_m} - \varphi_a\|_\infty \to 0$. The proof of the continuity of the maps φ^a is similar, taking into account that $|\lambda^{a_m}(t_1) - \lambda^a(t_1)| \leq \frac{1}{2}|a_m - a|$ for any $t_1 \in [0, 1]$. ■

Lemma 3.2. *Let* $f \in \mathcal{C}(P)$. *The sets* B_f *and* B^f *are compact in* $\mathcal{C}(P)$.

Proof. Let (a_m) be a sequence of elements of $[0,1]$ converging to a. In order to prove the compactness of the set B_f it will be sufficient to show that

$$\|f \circ \varphi_{a_m} - f \circ \varphi_a\|_\infty \to 0. \tag{13}$$

Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathcal{C}(P)$, there is $\delta > 0$ such that for all $s, t \in P$ such that $\|t - s\|_2 \leq \delta$ we have

$$|f(t) - f(s)| \leq \varepsilon. \tag{14}$$

By Lemma 3.1 (c) we can choose $\overline{m}$ such that

$$\|\varphi_{a_m}(t) - \varphi_a(t)\|_2 \leq \delta, \tag{15}$$

for any $m \geq \overline{m}$ and $t \in P$. By (14) and (15) it follows that

$$\|f \circ \varphi_{a_m} - f \circ \varphi_a\|_\infty = \max_{t \in P} |f(\varphi_{a_m}(t)) - f(\varphi_a(t))| \leq \varepsilon\ ,$$

for all $m \geq \overline{m}$. Hence we get (13). The proof of the compactness of the set B^f in $\mathcal{C}(P)$ is similar. ■

As no confusion may arise, we denote by Q and P_u the maps we use to construct the retractions in the space $\mathcal{C}(P)$. In virtue of 3.1 (a) we define the map $Q : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ as follows

$$Qf := f \circ \varphi_{\|f\|_\infty}. \tag{16}$$

It is easy to see that $\|Qf\|_\infty = \|f\|_\infty$ for all $f \in B(\mathcal{C}(P))$ and $Qf = f$ for all $f \in S(\mathcal{C}(P))$. Let $u \in (0, +\infty)$. Moreover, we define the map $P_u : B(\mathcal{C}(P)) \to \mathcal{C}(P)$ by

$$(P_u f)(t) := \begin{cases} u(t_1 + 1), & \text{if } t_1 \in \left[-1, -\dfrac{3 + \|f\|_\infty}{4}\right), \\ -u(t_1 + \dfrac{1 + \|f\|_\infty}{2}), & \text{if } t_1 \in \left[-\dfrac{3 + \|f\|_\infty}{4}, -\dfrac{1 + \|f\|_\infty}{2}\right), \\ 0, & \text{if } t_1 \in \left[-\dfrac{1 + \|f\|_\infty}{2}, \dfrac{1 + \|f\|_\infty}{2}\right], \\ u(t_1 - \dfrac{1 + \|f\|_\infty}{2}), & \text{if } t_1 \in \left]\dfrac{1 + \|f\|_\infty}{2}, \dfrac{3 + \|f\|_\infty}{4}\right], \\ -u(t_1 - 1), & \text{if } t_1 \in \left]\dfrac{3 + \|f\|_\infty}{4}, 1\right]. \end{cases}$$

The proofs of the following propositions are similar to the proof of Propositions 2.3 and 2.6, hence are omitted.

Proposition 3.3. *The map Q is continuous.*

Proposition 3.4. *The map P_u is compact.*

Next result gives a precise estimate of the Hausdorff measure of noncompactness of QA for $A \subseteq B(\mathcal{C}(P))$.

Proposition 3.5. *Let A be a subset of $B(\mathcal{C}(P))$. Then $\gamma_{\mathcal{C}(P)}(QA) = \gamma_{\mathcal{C}(P)}(A)$.*

Proof. Let $A \subseteq B(\mathcal{C}(P))$. By a proof analogous to that of Proposition 2.4 we find $\gamma_{\mathcal{C}(P)}(QA) \leq \gamma_{\mathcal{C}(P)}(A)$. Moreover, taking into account that for each $f \in \mathcal{C}(P)$ the set B^f is compact in $\mathcal{C}(P)$ (Proposition 3.2), we find $\gamma_{\mathcal{C}(P)}(A) \leq \gamma_{\mathcal{C}(P)}(QA)$. ∎

The next theorem is the main result of the Section.

Theorem 3.6. *For any $u > 0$ there is a retraction $R_u : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ with*

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, & \textit{if } u \leq 4 \\ \dfrac{4}{u}, & \textit{if } u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}.$$

In particular, we have $W(\mathcal{C}(P)) = 1$.

Proof. Let $u \in (0, +\infty)$. Define the map $T_u : B(\mathcal{C}(P)) \to \mathcal{C}(P)$ by $T_u f = Qf + P_u f$. Since P_u is compact, Proposition 3.5 implies

$$\gamma_{\mathcal{C}(P)}(T_u A) = \gamma_{\mathcal{C}(P)}(A), \tag{17}$$

for any $A \subseteq B(\mathcal{C}(P))$. Then we define $R_u : B(\mathcal{C}(P)) \to S(\mathcal{C}(P))$ by

$$R_u f = \frac{T_u f}{\|T_u f\|_\infty}.$$

Next, adapting the proof of Theorem 2.7 and Example 2.13 to this setting we find

$$\gamma_{\mathcal{C}(P)}(R_u) = \frac{u+8}{u}. \tag{18}$$

Also, taking into account (17) and adapting Example 2.14 we obtain

$$\underline{\gamma}_{\mathcal{C}(P)}(R_u) = \begin{cases} 1, & \text{if } u \leq 4 \\ \dfrac{4}{u}, & \text{if } u > 4. \end{cases}$$

Finally, since $\lim_{u\to\infty}\frac{u+8}{u}=1$, we have $W(\mathcal{C}(P))=1$. ■

We point out that our construction does not work in the case of the Banach space $\mathcal{B}C(l_2)$ of all real bounded and continuous functions defined on l_2. In fact, the following example shows the map Q defined in (16) is not anymore continuous when considered from $B(\mathcal{B}C(l_2))$ into itself.

Example 3.7. Set $I_k=\left[\sum_{i=1}^{2k}\frac{1}{i},\ \sum_{i=1}^{2k+1}\frac{1}{i}\right)$ and $J_k=\left[\sum_{i=1}^{2k+1}\frac{1}{i},\ \sum_{i=1}^{2k+2}\frac{1}{i}\right)$, for each $k=1,2,...$
Then let $f:l_2\to R$ defined by

$$f(t)=\begin{cases}0, & \text{if}\quad \|t\|_2^2\in\left[0,\ \dfrac{3}{2}\right),\\ (2k+1)(\|t\|_2^2-\displaystyle\sum_{i=1}^{2k}\frac{1}{i}), & \text{if}\quad \|t\|_2^2\in I_k, \qquad (k=1,2,...).\\ -(2k+2)(\|t\|_2^2-\displaystyle\sum_{i=1}^{2k+2}\frac{1}{i}), & \text{if}\quad \|t\|_2^2\in J_k.\end{cases}$$

The map f is bounded, continuous and $\|f\|_\infty=1$ but f is not uniformly continuous. In fact, let $0<\epsilon<1$. Given $\delta>0$ find $k\in N$ such that $\frac{1}{2k+1}\le\delta$ and choose $t,\ s\in l_2$ such that $\|t\|_2^2=\sum_{i=1}^{2k}\frac{1}{i}$ and $\|s\|_2^2=\sum_{i=1}^{2k+1}\frac{1}{i}$. Then $f(t)=0$ and $f(s)=1$ so that $|f(t)-f(s)|=1>\varepsilon$.
Consider now the sequence (f_m), where $f_m:l_2\to R$ is defined by $f_m=(1-\frac{1}{m})f$ $(m=1,2,...)$. Then $\|f_m\|_\infty=1-\frac{1}{m}$ and $\|f_m-f\|_\infty=\frac{1}{m}\to 0$. We now show that

$$\left\|f\circ\varphi_{1-\frac{1}{p}}-f\circ\varphi_{1-\frac{1}{q}}\right\|_\infty=1,$$

for all $p,q\in N$. Suppose $q<p$ so that $0<\frac{1+(1-\frac{1}{q})}{2}<\frac{1+(1-\frac{1}{p})}{2}$ and $\frac{2}{1+(1-\frac{1}{q})}>\frac{2}{1+(1-\frac{1}{p})}$. Set

$$\delta_{q,p}=(\frac{2}{1+(1-\frac{1}{q})})^2-(\frac{2}{1+(1-\frac{1}{p})})^2>0.$$

Fix $\overline{k}\in N$such that $(t_{1,\overline{k}})^2=\frac{1}{\delta_{q,p}(2\overline{k}+1)}$ and $t_{1,\overline{k}}\in\left(0,\frac{1+(1-\frac{1}{q})}{2}\right)$. Set

$$t^{(\overline{k})}=(t_{1,\overline{k}},\ t(\sum_{i=1}^{2\overline{k}}\frac{1}{i}-(t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{p})})^2)^{\frac{1}{2}},\ 0,\ 0,...).$$

Since

$$\left\|\varphi_{1-\frac{1}{p}}(t^{(\overline{k})})\right\|_2^2 = (t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{p})})^2 + \sum_{i=1}^{2\overline{k}} \frac{1}{i} - (t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{p})})^2 = \sum_{i=1}^{2\overline{k}} \frac{1}{i},$$

we have

$$f(\varphi_{1-\frac{1}{p}}(t^{(\overline{k})})) = f(\sum_{i=1}^{2\overline{k}} \frac{1}{i}) = 0.$$

On the other hand

$$\begin{aligned}\left\|\varphi_{1-\frac{1}{q}}(t^{(\overline{k})})\right\|_2^2 &= (t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{q})})^2 + \sum_{i=1}^{2\overline{k}} \frac{1}{i} - (t_{1,\overline{k}})^2(\frac{2}{1+(1-\frac{1}{p})})^2 \\ &= \sum_{i=1}^{2\overline{k}} \frac{1}{i} + \frac{1}{\delta_{q,p}(2\overline{k}+1)}((\frac{2}{1+(1-\frac{1}{q})})^2 - (\frac{2}{1+(1-\frac{1}{p})})^2) = \sum_{i=1}^{2\overline{k}+1} \frac{1}{i},\end{aligned}$$

implies

$$f(\varphi_{1-\frac{1}{q}}(t^{(\overline{k})})) = f(\sum_{i=1}^{2\overline{k}+1} \frac{1}{i}) = 1.$$

Thus $\left|f(\varphi_{1-\frac{1}{p}}(t^{(\overline{k})})) - f(\varphi_{1-\frac{1}{q}}(t^{(\overline{k})}))\right| = 1$ so that $\left\|f \circ \varphi_{1-\frac{1}{p}} - f \circ \varphi_{1-\frac{1}{q}}\right\|_\infty = 1$. Suppose $Q : B(\mathcal{BC}(l_2)) \to B(\mathcal{BC}(l_2))$ continuous. Then if $\|f_m - f\|_\infty \to 0$ we have that

$$\|Qf_m - Qf\|_\infty = \left\|f_m \circ \varphi_{\|f_m\|_\infty} - f \circ \varphi_{\|f\|_\infty}\right\|_\infty = \left\|f_m \circ \varphi_{1-\frac{1}{m}} - f\right\|_\infty \to 0.$$

Let $\varepsilon > 0$. Fix $p, q \in N$ such that

$$\|f - f_p\|_\infty + \|Qf_p - Qf_q\|_\infty + \|f_q - f\|_\infty < 1.$$

Then

$$\begin{aligned}1 &= \left\|f \circ \varphi_{1-\frac{1}{p}} - f \circ \varphi_{1-\frac{1}{q}}\right\|_\infty \le \left\|f \circ \varphi_{1-\frac{1}{p}} - f_p \circ \varphi_{\|f_p\|_\infty}\right\|_\infty \\ &\quad + \left\|f_p \circ \varphi_{\|f_p\|_\infty} - f_q \circ \varphi_{\|f_q\|_\infty}\right\|_\infty + \left\|f_q \circ \varphi_{\|f_q\|_\infty} - f \circ \varphi_{1-\frac{1}{q}}\right\|_\infty \\ &= \left\|f \circ \varphi_{1-\frac{1}{p}} - f_p \circ \varphi_{1-\frac{1}{p}}\right\|_\infty + \|Qf_p - Qf_q\|_\infty + \left\|f_q \circ \varphi_{1-\frac{1}{q}} - f \circ \varphi_{1-\frac{1}{q}}\right\|_\infty \\ &= \|f - f_p\|_\infty + \|Qf_p - Qf_q\|_\infty + \|f_q - f\|_\infty < 1,\end{aligned}$$

which is a contradiction. Observe that the set $B^f = \{f \circ \varphi^a : a \in [0,1]\}$ is not compact.

Since the Hausdorff measure of noncompactness of a set is invariant under isometries and the Banach space $\mathcal{C}(K)$ of all real continuous functions defined on a Hausdorff space K homeomorphic to the Hilbert cube is isometric to $\mathcal{C}(P)$, we get the following corollary.

Corollary 3.8. *Let K be a Hausdorff space homeomorphic to the Hilbert cube P. For any $u > 0$ there is a retraction $R_u : B(\mathcal{C}(K)) \to S(\mathcal{C}(K))$ with*

$$\underline{\gamma}_{\mathcal{C}(K)}(R_u) = \begin{cases} 1, & \textit{if } \ u \leq 4 \\ \dfrac{4}{u}, & \textit{if } \ u > 4 \end{cases}$$

and

$$\gamma_{\mathcal{C}(K)}(R_u) = \frac{u+8}{u}.$$

In particular, we have $W(\mathcal{C}(K)) = 1$.

Remark 3.9. The previous Corollary applies in two particular important cases.
(i) Every infinite dimensional compact convex subset K of a normed space is homeomorphic to the Hilbert cube P (see [8]).
(ii) Let K be a metrizable infinite dimensional compact convex set in a topological linear space and assume K is an absolute retract, then K is homeomorphic to the Hilbert cube P (see [6]).

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