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STRONGLY CONTINUOUS GROUPS, SIMILARITY AND NUMERICAL RANGE ON A HILBERT SPACE

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Abstract. It is shown that iB generates a strongly continuous group of exponential type ω on a Hilbert space if and only if for all $\alpha > \omega$, B is similar to an operator with spectrum and numerical range contained in the horizontal strip $\{z \in \mathbf{C} \,|\, |Im(z)| < \alpha\}$.

1. Introduction

It is a well-known result due to Sz-Nagy (see, for example, [5, Theorem 4.8.1]) that, if iB generates a bounded strongly continuous group on a Hilbert space, then B is similar to a self-adjoint operator. This is equivalent to saying that B is similar to an operator whose spectrum and numerical range are contained in the real line. In this paper, a similar result for generators of arbitrary strongly continuous groups on a Hilbert space is proven. It is shown that iB generates a strongly continuous group of exponential type ω if and only if for all $\alpha > \omega$, B is similar to an operator whose spectrum and numerical range are contained in the horizontal strip $\{z \in \mathbf{C} \,|\, |Im(z)| < \alpha\}$ (Theorem 2.4).

These results are closely related to Halmos's conjecture about operators that are similar to a contraction; see Remark 2.5. The paper concludes with some open questions.

Throughout, all operators are linear, on a Hilbert space H, with inner product $< >$. Denote by $\mathcal{D}(G)$ the domain of the operator G, by $\rho(G)$ its resolvent set, and by $\sigma(G)$ its spectrum. The space of bounded operators from H to itself will be written as $B(H)$.

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An operator B is *similar* to an operator G if there exists $S \in B(H)$, with $0 \in \rho(S)$, such that $B = SGS^{-1}$.

Definition 1.1. The *numerical range* of an operator G is

$$\{< Gx, x > \mid x \in \mathcal{D}(G), \|x\| = 1\}.$$

Definition 1.2. If Ω is an open subset of the complex plane, whose closure is not the entire plane, all eigenvalues of the operator A are contained in Ω and $\sigma(A) \subseteq \overline{\Omega}$, then an $H^\infty(\Omega)$ *functional calculus* for A is a continuous algebra homomorphism, $f \mapsto f(A)$, from $H^\infty(\Omega)$ into $B(H)$, such that $f_0(A) = I$ ($f_0(z) \equiv 1$) and $(z \mapsto (\lambda - z)^{-1})(A) = (\lambda - A)^{-1}$, for all $\lambda \notin \overline{\Omega}$.

Of particular interest in this paper will be horizontal strips

$$H_\alpha \equiv \{z \in \mathbf{C} \mid |Im(z)| < \alpha\}$$

and sectors

$$S_\alpha \equiv \{re^{i\phi} \mid |\phi| < \alpha, r > 0\},$$

for $\alpha > 0$.

Definition 1.3. If A generates a strongly continuous group (semigroup), it will be denoted by $\{e^{tA}\}_{t\in\mathbf{R}}(\{e^{tA}\}_{t\geq 0})$. See [7], [10] or [11] for basic material on strongly continuous semigroups.

A strongly continuous group $\{e^{tA}\}_{t\in\mathbf{R}}$ is of *exponential type* ω if, for all $\alpha > \omega$, there exists a constant M_α such that

$$\|e^{tA}\| \leq M_\alpha e^{|t|\alpha}, \;\; \forall t \in \mathbf{R}.$$

Definition 1.4. Suppose iB generates a strongly continuous group of exponential type $\omega < \frac{\pi}{2}$. As in [2] and [3, Chapter XXII] (see also [4, Example 4.1]), define $-e^B$ to be the generator of the regularized semigroup

$$W(t) \equiv \int_{\partial H_\gamma} e^{-te^w} \, (w - B)^{-1} \, \frac{dw}{2\pi i(i\frac{\pi}{2} - w)^2} \;\; (t \geq 0),$$

where $\omega < \gamma < \frac{\pi}{2}$. Note that the integral converges because $\|(w - B)^{-1}\|$ is bounded in $|Im(w)| \geq \gamma$. See [2] or [3] for basic material on regularized semigroups.

By [4, Example 4.1(7)], e^B is injective.

2. Main Results.

The proof of Theorem 2.4 is short, relying on two results. First, Le Merdy recently showed ([8]) that a strongly continuous holomorphic semigroup is similar to a semigroup of contractions if and only if the generator has an $H^\infty(-S_\theta)$ functional calculus, for some $\theta < \frac{\pi}{2}$. Second, it is shown in [2] that the generator of a strongly continuous group of exponential type ω automatically has an $H^\infty(iH_\alpha)$ functional calculus, for any $\alpha > \omega$. Finally, the exponential function, as in Definition 1.4, allows us to switch from strips H_α to sectors S_α.

The following result is actually stated in terms of bounded imaginary powers; see [9] for the equivalence with H^∞ functional calculi.

Lemma 2.1([8, Theorem 1.1]) *Suppose $-A$ is injective and generates a bounded strongly continuous analytic semigroup $\{e^{-tA}\}_{t\geq 0}$. Then $\{e^{-tA}\}_{t\geq 0}$ is similar to a semigroup of contractions if and only if A has an $H^\infty(S_{\frac{\pi}{2}})$ functional calculus.*

Lemma 2.2 *The following are equivalent, if $\omega \geq 0$.*

(a) *iB generates a strongly continuous group of exponential type ω.*

(b) *B has an $H^\infty(H_\alpha)$ functional calculus, for all $\alpha > \omega$.*

If $\omega < \frac{\pi}{2}$, then these are equivalent to

(c) e^B has an $H^\infty(S_\alpha)$ functional calculus, for all $\alpha > \omega$.

Then

$$f(e^B) \equiv (f \circ g)(B) \ \ (f \in H^\infty(S_\alpha)),$$

where $g(z) \equiv e^z$.

Proof: The equivalence of (a) and (b) is in [2, Theorem 3.5]. The equivalence of (b) and (c) is in [2, Corollary 2.9], and its proof. ∎

The following is well known (in fact, the growth condition at infinity may be removed); a simple proof is included for completeness, and to introduce Remark 2.5.

Lemma 2.3. *Suppose $-A$ generates a strongly continuous semigroup of contractions. Then A has an $\mathcal{A}(S_{\frac{\pi}{2}})$ functional calculus, with*

$$\|f(A)\| \leq \|f\|_{H^\infty(S_{\frac{\pi}{2}})} \ \forall f \in \mathcal{A}(\mathcal{S}_{\frac{\pi}{\in}}),$$

where

$$\mathcal{A}(S_{\frac{\pi}{2}}) \equiv \{f \in H^\infty(S_{\frac{\pi}{2}}) \cap C(\overline{S_{\frac{\pi}{2}}}) \mid \lim_{|z|\to\infty, z\in \overline{S_{\frac{\pi}{2}}}} f(z) \text{ exists } \}.$$

Proof: Let $T \equiv (A-1)(A+1)^{-1}$. Then T is a contraction, thus by Von Neumann's inequality (see, for example, [1, Proposition X.1.7]) has an $\mathcal{A}(\mathcal{D}) \equiv \mathcal{H}^\infty(\mathcal{D}) \cap \mathcal{C}(\overline{\mathcal{D}})$ functional calculus, where D is the unit disc, with

$$\|h(T)\| \leq \|h\|_{\mathcal{A}(D)} \ \ \forall h \in \mathcal{A}(D).$$

Let $k(z) \equiv (1+z)(1-z)^{-1}$, mapping D onto $S_{\frac{\pi}{2}}$. Then $A = k(T) \equiv (1+T)(1-T)^{-1}$, so

$$f(A) \equiv (f \circ k)(T) \ \ (f \in \mathcal{A}(S_{\frac{\pi}{2}}))$$

defines an $\mathcal{A}(S_{\frac{\pi}{2}})$ functional calculus for A, with

$$\|f(A)\| = \|(f\circ k)(T)\| \leq \|(f \circ k)\|_{\mathcal{A}(D)} = \|f\|_{\mathcal{A}(S_{\frac{\pi}{2}})},$$

for all $f \in \mathcal{A}(S_{\frac{\pi}{2}})$, as desired. ∎

Theorem 2.4. *The following are equivalent, if $\omega \geq 0$.*

(a) *iB generates a strongly continuous group of exponential type ω.*

(b) *For any $\alpha > \omega$, B is similar to an operator B_α such that iB_α generates a strongly continuous group with $\|e^{itB_\alpha}\| \leq e^{\alpha|t|}, \forall t \in \mathbf{R}$.*

(c) *For any $\alpha > \omega$, B is similar to an operator whose spectrum and numerical range are contained in H_α.*

(d) *B has an $H^\infty(H_\alpha)$ functional calculus, for all $\alpha > \omega$.*

Proof: It is well known (see [7], [10] or [11]) that (b) and (c) are equivalent, and it is clear that (b) $\to$ (a). The equivalence of (d) and (a) is in Lemma 2.2. All that remains is (a) $\to$ (c).

Since spectrum is preserved under similarity, it is sufficient to show that, for any $\alpha > \omega$, B is similar to an operator B_α with numerical range contained in $\overline{H_\alpha}$.

Fix $\alpha > \omega$, and let $r \equiv \frac{\pi}{2\alpha}$. Then $i(rB)$ generates a strongly continuous group of exponential type $r\omega$. Since $r\omega < \frac{\pi}{2}$, by Lemma 2.2 there exists $\phi < \frac{\pi}{2}$ such that e^{rB} has an $H^\infty(S_\phi)$ functional calculus. By Lemma 2.1, there exists $S_\alpha \in B(H)$ such that $0 \in \rho(S_\alpha)$ and

$$\|S_\alpha e^{-te^{rB}} S_\alpha^{-1}\| \leq 1, \ \ \forall t \geq 0.$$

Lemma 2.3 implies that e^{rB} has an $\mathcal{A}(\mathcal{S}_{\frac{\pi}{2}})$ functional calculus, with

$$\|S_\alpha f(e^{rB})S_\alpha^{-1}\| \leq \|f\|_{H^\infty(S_{\frac{\pi}{2}})}, \ \ \forall f \in \mathcal{A}(S_{\frac{\pi}{2}}).$$

Define iB_α as the generator of the strongly continuous group

$$e^{itB_\alpha} \equiv S_\alpha e^{itB} S_\alpha^{-1} \ \ (t \in \mathbf{R}).$$

For $\lambda \in \mathbf{R}$, with $|\lambda| > \alpha$, define

$$g_\lambda(z) \equiv r(ir\lambda - \ln z)^{-1} \ \ (z \in S_{\frac{\pi}{2}}).$$

Then

$$\begin{aligned}
\|(i\lambda - B_\alpha)^{-1}\| &= \|S_\alpha r(ir\lambda - rB)^{-1}S_\alpha^{-1}\| \\
&= \|S_\alpha \left((z \mapsto r(ir\lambda - z)^{-1})(rB)\right) S_\alpha^{-1}\| \\
&= \|S_\alpha g_\lambda(e^{rB})S_\alpha^{-1}\| \\
&\leq \|g_\lambda\|_{H^\infty(S_{\frac{\pi}{2}})} \\
&= r(r|\lambda| - \tfrac{\pi}{2})^{-1} = (|\lambda| - \alpha)^{-1}.
\end{aligned}$$

This implies that the numerical range of B_α is contained in $\overline{H_\alpha}$, as desired. ∎

Remark 2.5. Halmos's sixth problem ([6]) asks if every polynomially bounded operator is similar to a contraction. An operator T is *polynomially bounded* if there exists a constant M such that

$$\|p(T)\| \leq M \sup_{z \in D} |p(z)|,$$

for all polynomials p, where D is the unit disc.

By Von Neumann's inequality, the converse is true. Note that T is polynomially bounded if and only if T has an $\mathcal{A}(\mathcal{D}) \equiv \mathcal{H}^\infty(\mathcal{D}) \cap \mathcal{C}(\overline{\mathcal{D}})$ functional calculus.

As mentioned in the proof of Lemma 2.3 (see also the assertion before Lemma 2.3), by applying the Cayley transform $k(z) \equiv (1+z)(1-z)^{-1}$, Halmos's problem is equivalent to asking if, whenever A is densely defined and has an $H^\infty(S_{\frac{\pi}{2}}) \cap C(\overline{S_{\frac{\pi}{2}}})$ functional calculus, is $-A$ similar to an operator that generates a strongly continuous semigroup of contractions? See [8] for a strong result in this direction.

By Lemma 2.2, and some additional argument, Halmos's problem may be reformulated in a third way, as follows. If B has an $H^\infty(H_\alpha) \cap C(\overline{H_\alpha})$ functional calculus, is B similar to an operator G such that

$$\|f(G)\| \leq \|f\|_{\mathcal{A}(H_\alpha)}, \ \ \forall f \in H^\infty(H_\alpha) \cap C(\overline{H_\alpha})?$$

This, and Theorem 2.4, suggest many open questions.

Open Questions 2.6. Consider the following, for $\omega > 0$, $\mathcal{D}(B)$ dense.

1. The operator iB generates a strongly continuous group such that $\|e^{itB}\| = O(e^{\omega|t|})$.
2. The operator iB generates a strongly continuous group, and there exists $S \in B(H)$ such that $0 \in \rho(S)$ and
$$\|Se^{itB}S^{-1}\| \leq e^{\omega|t|}, \ \forall t \in \mathbf{R}.$$
3. The operator B has an $H^\infty(H_\omega) \cap C(\overline{H_\omega})$ functional calculus.
4. The operator B has an $H^\infty(H_\omega) \cap C(\overline{H_\omega})$ functional calculus, and there exists $S \in B(H)$ such that $0 \in \rho(S)$, and
$$\|Sf(B)S^{-1}\| \leq \|f\|_{H^\infty(H_\omega)} \ \forall f \in H^\infty(H_\omega) \cap C(\overline{H_\omega}).$$

What are the relationships between (1)–(4)? It is straightforward to show that (3) $\to$ (1) and (4) $\to$ (2).

Does (1) imply (2)? By Theorem 2.4, (1) implies that, for any α *greater* than ω, there exists S_α such that
$$\|S_\alpha e^{itB}S_\alpha^{-1}\| \leq e^{\alpha|t|} \ \forall t \in \mathbf{R}.$$
But now we are asking for sharp boundary behaviour.

As I commented before, (3) $\to$ (4) is equivalent to Halmos's conjecture about polynomially bounded operators being similar to a contraction.

For $\omega \leq \frac{\pi}{2}$, assertion (4) is equivalent to e^B being sectorial; thus (2) $\to$ (4) would be a numerical range mapping theorem; that is, it would be stating that, if the numerical range of B is contained in H_ω, then the numerical range of e^B is contained in $e^{H_\omega} = S_\omega$. Note in particular that (4), with $\omega = \frac{\pi}{2}$, is equivalent to e^B having numerical range and spectrum contained in $S_{\frac{\pi}{2}}$, which is equivalent to $-e^B$ generating a strongly continuous semigroup of contractions.

Does (2) imply (3), or, better yet, (4)? Note that, for $\omega \leq \frac{\pi}{2}$, the sectorial analogue of (2) does imply the sectorial analogue of (4); by "sectorial analogue" I mean, in (4), replace H_ω with S_ω, and replace (2) with

($2'$) $-A$ generates a strongly continuous analytic semigroup $\{e^{-zA}\}_{z \in S_{(\frac{\pi}{2}-\omega)}}$ such that
$$\|Se^{-zA}S^{-1}\| \leq 1 \ \forall z \in S_{(\frac{\pi}{2}-\omega)},$$
(for $\omega < \frac{\pi}{2}$); or

(2′) $-A$ generates a strongly continuous semigroup of contractions (for $\omega = \frac{\pi}{2}$).

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