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AN EVALUATION OF EFFICIENT POINTS FOR VECTOR OPTIMIZATION

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Abstract. In this paper, to decide the best point of many efficient points in vector optimization, we consider an evaluate method of efficient points for solutions in vector optimization problem. We introduce an evaluate function of efficient points, and show properties of the evaluate function.

1. Introduction

The following efficiencies in vector optimization have been studied by many researchers, see [9].

Definition 1. Let A be a subset of a topological vector space E and K a convex cone of E which is not the whole space. We say that

(1) $x \in A$ is an ideal efficient point of A with respect to K if $y \in x + K$ for all $y \in A$;

(2) $x \in A$ is an efficient point of A with respect to K if $y \in x+K$ for all $y \in A$ with $x \in y + K$;

(3) $x \in A$ is a (global) properly efficient point of A with respect to K if there exists a convex cone L of E, which is not the whole space, such that $K \setminus l(K)$ is contained in int$L$ and x is an efficient point of A with respect to L, where $l(K) = K \cap (-K)$;

(4) Supposing that int$K$ is nonempty, $x \in A$ is a weakly efficient point of A with respect to K if x is an efficient point of A with respect to int$K \cup \{\theta\}$;

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The set of all efficient points (resp. properly efficient points, weakly efficient points, ideal efficient points) of A with respect to K is denoted by $\mathrm{Min}(A|K)$ (resp. $\mathrm{PrMin}(A|K)$, $\mathrm{WMin}(A|K)$, $\mathrm{IMin}(A|K)$).

Note that

$$\mathrm{IMin}(A|K) \subset \mathrm{PrMin}(A|K) \subset \mathrm{Min}(A|K) \subset \mathrm{WMin}(A|K)$$

hold.

We have many notions of proper efficiencies. For instance, one is Benson's proper efficiency, and one is Borwein's proper efficiency, see [1, 2]. Global proper efficiency corresponds to Benson's proper efficiency, see [4]. On the other hand, the following definition corresponds to Borwein's proper efficiency.

Definition 2. Let A be a subset of a metric space E and K a convex cone of E which is not the whole space. We say that x is a local properly efficient point with respect to K if, for each $r > 0$, there exists a convex cone L of E, which is not the whole space, such that $K \setminus l(K)$ is contained in int$L$ and x is an efficient point of $A \cap B(x, r)$ with respect to L. The set of all local properly efficient points of A with respect to K is denoted by $\mathrm{LPrMin}(A|K)$.

In many cases, the number of elements of $\mathrm{Min}(A \mid K)$ is more than two. For instance, let $E = \mathbb{R}^2$, $A = \bar{B}(\theta, 1)$ and $K = \mathbb{R}^2_+$ where $\bar{B}(\theta, 1)$ is the closed ball of radius 1 centered at the null vector in E and $\mathbb{R}_+$ is the set of all non-negative numbers, then $\mathrm{Min}(A \mid K) = \{x \in \mathbb{R}^2_- \mid \|x\| = 1\}$ holds. Thus, it is interested which efficient point is the most suitable for the solution of vector optimization problem.

In this paper, we consider an evaluation of efficient points for vector optimization problem. We introduce an evaluate function, which is a function from E to $[-\infty, \infty]$, as ways of an evaluation of efficient points.

2. Flutter Cones

First, we state a family of flutter cones, which is needed to define an evaluate function. The family consists of expansion and reduction cones made from the original cone. In previous researches, we considered certain families of enlarged cones, see [3, 5, 6, 7] and [8].

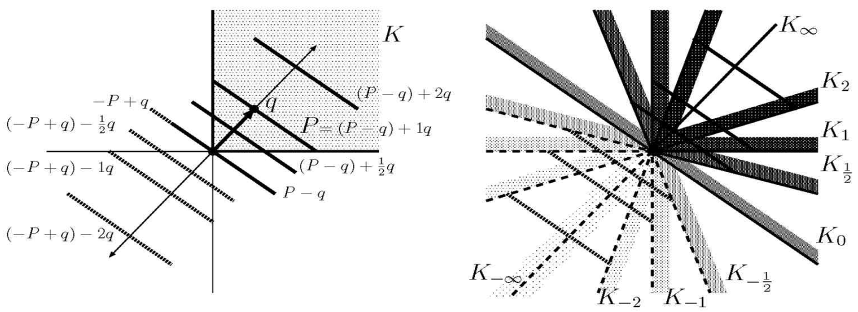
Let E be a topological vector space over the real field $\mathbb{R}$, E^* the dual space of E, K a convex cone which is acute, that is, cl$K \cap (-$cl$K)$ consists of only the null vector θ of E, K^+ the positive polar cone of K, that is $K^+ = \{\xi \in E^* \mid \langle \xi, k\rangle \geq 0$ for each $k \in K\}$, ξ in K^+ which is not the null vector θ^* of E^*, and $P = \{x \in K \mid \langle \xi, x\rangle = 1\}$. We assume that int$K$ is nonempty, and cl$P$ is compact.

Definition 3. Let $q \in \mathrm{int}K \cap P$, and define

$$K_\lambda^q = \begin{cases} \mathrm{cone}(\{q\}) & \text{if } \lambda = \infty, \\ \mathrm{cone}((P-q)+\lambda q) & \text{if } \lambda \in (0,\infty), \\ \{x \in E \mid \langle \xi, x\rangle \geq 0\} & \text{if } \lambda = 0, \\ \mathrm{cone}((-P+q)+\lambda q)^c \cup \{\theta\} & \text{if } \lambda \in (-\infty, 0), \\ \mathrm{cone}(\{-q\})^c \cup \{\theta\} & \text{if } \lambda = -\infty. \end{cases}$$

for each $\lambda \in [-\infty, \infty]$. We say that K_λ^q is flutter cone of K at λ with respect to q.

Forms of the flutter cone depend on positions of the base of the original cone K, see the following figure.



Other expression of the flutter cone is as follows:

$$K_\lambda^q = \begin{cases} \{\alpha q | \alpha \geq 0\} & \text{if } \lambda = \infty, \\ \{\alpha(p-(1-\lambda)q) | \alpha \geq 0, p \in P\} & \text{if } \lambda \in (0,\infty), \\ \displaystyle\bigcup_{\mu \in [0,\infty)} \{\alpha(p-(1-\lambda)q) | \alpha \geq 0, p \in P\} & \text{if } \lambda = 0, \\ \{-\alpha(p-(1+\lambda)q) | \alpha > 0, p \in P\}^c & \text{if } \lambda \in (-\infty, 0), \\ \{-\alpha q | \alpha > 0\}^c & \text{if } \lambda = -\infty. \end{cases}$$

Proof of the assertion on $\lambda = 0$, see Lemma 1.

Example 1. Let $E = \mathbb{R}^n$, $q = \xi = (1, 0, 0, \ldots, 0)$, $P = \{p = (1, p_2, p_3, \ldots, p_n) \mid \|p - q\| \leq 1\}$ and $K = \{\alpha p \mid \alpha \geq 0, p \in P\}$. Then,

$$K_\lambda^q = \begin{cases} \{(\alpha, 0, 0, \ldots, 0) \in \mathbb{R}^n \mid \alpha \geq 0\} & \text{if } \lambda = \infty, \\ \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 & \\ \qquad \geq \lambda\sqrt{x_2^2 + x_3^2 + \cdots + x_n^2}\} & \text{if } \lambda \in [0, \infty), \\ \{x \mid x_1 > \lambda\sqrt{x_2^2 + x_3^2 + \cdots + x_n^2}\} \cup \{\theta\} & \text{if } \lambda \in (-\infty, 0), \\ \{(\alpha, 0, 0, \ldots, 0) \in \mathbb{R}^n \mid \alpha < 0\}^c & \text{if } \lambda = -\infty. \end{cases}$$

Example 2. Let e_i is in $\mathbb{R}^n$ such that ith component of e_i is 1 and other components of e_i are 0 $(i = 1, 2, \ldots, n)$, $K = \mathbb{R}_+^n$, $q = \frac{1}{n}(1, 1, \ldots, 1)$, $\xi = h = (1, 1, \ldots, 1)$ and $P = \{p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}_+^n \mid p_1 + p_2 + \cdots + p_n = 1\}$, where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$. Then,

$$K_\lambda^q = \begin{cases} \{(\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^n \mid \alpha \geq 0\} & \text{if } \lambda = \infty, \\ \bigcap_{i=1}^n \{x \mid \langle a_i(\lambda), x\rangle \geq 0\} & \text{if } \lambda \in [0, \infty), \\ \bigcup_{i=1}^n \{x \mid \langle a_i(\lambda), x\rangle > 0\} \cup \{\theta\} & \text{if } \lambda \in (-\infty, 0), \\ \{(\alpha, \alpha, \ldots, \alpha) \in \mathbb{R}^n \mid \alpha < 0\}^c & \text{if } \lambda = -\infty. \end{cases}$$

where $a_i(\lambda) = (1 - |\lambda|)h + |\lambda|ne_i$. Next, we state properties of the flutter cones.

Lemma 1. *Let $q \in \mathrm{int}K \cap P$. Then,*

(1) $K_1^q = K$, $K_0^q = \{x \in E \mid \langle \xi, x\rangle \geq 0\}$ *and* $K_{-1}^q = (-K)^c \cup \{\theta\}$*;*

(2) $\forall\lambda \in [0, \infty]$, K_λ^q *is a convex cone;*

(3) $\forall\lambda \in [-\infty, 0]$, K_λ^q *is a cone,* $(K_\lambda^q)^c \cup \{\theta\}$ *is a convex cone;*

(4) $\forall\lambda, \lambda' \in [-\infty, \infty]$ *with* $\lambda < \lambda'$, $K_{\lambda'}^q \subset K_\lambda^q$.

Proof. We show only (1) and (4) because it is easy to show (2) and (3) hold. (1) To show $K_1^q = K$ and $K_{-1}^q = (-K)^c \cup \{\theta\}$ is easy, and we prove $K_0^q = \{x \in E \mid \langle \xi, x\rangle \geq 0\}$ holds. It is clear $K_0^q \subset \{x \in E \mid \langle \xi, x\rangle \geq 0\}$ holds. On the other hand, let x be in E satisfying $\langle \xi, x\rangle \geq 0$. Since $q \in \mathrm{int}K$, we can choose a natural number n_0 such that $\frac{1}{n_0}x + (1 - \frac{1}{n_0}\langle \xi, x\rangle)q \in K$. Let $\alpha_0 = n_0$ and $p_0 = \frac{1}{n_0}x + (1 - \frac{1}{n_0}\langle \xi, x\rangle)q$, then we have $\alpha_0 \geq 0$, $p_0 \in P$, and $x = \alpha_0(p_0 - (1 - \frac{1}{n_0}\langle \xi, x\rangle)q)$, and consequently $x \in K_0^q$. Proof of (1) is completed. (4) Assume that $\lambda, \lambda' \in [-\infty, \infty]$ with $\lambda < \lambda'$. We show $K_{\lambda'}^q \subset K_\lambda^q$ in every situation.

Case 1. $\lambda \in (0, \infty)$ and $\lambda' = \infty$. Let $x \in K_{\lambda'}^q$, then there exists $\alpha \geq 0$ such that $x = \alpha q$. Since $\alpha q = \frac{1}{\lambda}\alpha(q - (1 - \lambda)q)$, we have $x \in K_\lambda^q$.

Case 2. $\lambda, \lambda' \in (0, \infty)$. Let $x \in K^q_{\lambda'}$, then there exist $\alpha \geq 0$ and $p \in P$ such that $x = \alpha(p - (1 - \lambda')q)$. Since $\alpha\frac{\lambda'}{\lambda} \geq 0$ and $\frac{\lambda}{\lambda'}p + (1 - \frac{\lambda}{\lambda'})q \in P$, we have $x = \alpha\frac{\lambda'}{\lambda}(\frac{\lambda}{\lambda'}p + (1 - \frac{\lambda}{\lambda'})q - (1 - \lambda)q) \in K^q_\lambda$.

Case 3. $\lambda = 0$ and $\lambda' \in (0, \infty)$. It is clear in this case.

Case 4. $\lambda \in (-\infty, 0)$ and $\lambda' = 0$. Suppose that there exist $x \in K^q_{\lambda'}$ such that $x \in (K^q_\lambda)^c$, then there exist $\mu \in [0, \infty)$, $\alpha_1 \geq 0$, $\alpha_2 > 0$ and $p_1, p_2 \in P$ such that $x = \alpha_1(p_1 - (1 - \mu)q) = -\alpha_2(p_2 - (1 - \lambda)q)$.

$$0 = \langle \xi, \alpha_1(p_1 - (1 - \mu)q) - (-\alpha_2(p_2 - (1 - \lambda)q)) \rangle = \alpha_1\mu + \alpha_2(-\lambda),$$

and this is a contradiction.

Case 5. $\lambda, \lambda' \in (-\infty, 0)$. We can obtain $(K^q_\lambda)^c \subset (K^q_{\lambda'})^c$ as the same as Case 2.

Case 6. $\lambda = -\infty$ and $\lambda' \in (-\infty, 0)$. We can prove that $(K^q_\lambda)^c \subset (K^q_{\lambda'})^c$ holds as the same as Case 1.

Case 7. Otherwise. By using the above results, the proof of $K^q_{\lambda'} \subset K^q_\lambda$ is given. ∎

Lemma 2 and Lemma 3 play important roles to consider the interior and the closure of the flutter cones.

Lemma 2. *For each $q \in \mathrm{int}K \cup P$ and $\mu \in (0, \infty)$, we define $t^q_\mu : E \to E$ as follows:*

$$t^q_\mu(x) = x - \langle \xi, x \rangle (1 - \mu)q \textit{ for each } x \in E.$$

Then,

(1) *t^q_μ is a homeomorphism on E, and $(t^q_\mu)^{-1} = t^q_{\frac{1}{\mu}}$.*

(2) *$t^q_\mu(K^q_\lambda) = K^q_{\lambda\mu}$ for all $\lambda \in (0, \infty)$.*

Proof. (1) It is clear that t^q_μ is continuous on E. We show that t^q_μ is a injection. Assume that $t^q_\mu(x) = t^q_\mu(y)$. Then, we have $x - y = \langle \xi, x - y \rangle (1 - \mu)q$, and

$$\langle \xi, x - y \rangle = \langle \xi, \langle \xi, x - y \rangle (1 - \mu)q \rangle = \langle \xi, x - y \rangle (1 - \mu),$$

therefore

$$0 = \langle \xi, x - y \rangle \mu.$$

Since $\mu \neq 0$, we have $\langle \xi, x - y \rangle = 0$ and also $x - y = \theta$. Next, we show that t^q_μ is a surjection. Let y be in E, then

$$t^q_\mu(y - \langle \xi, y \rangle (1 - \frac{1}{\mu})q) = y.$$

This equation directly shows $(t^q_\mu)^{-1} = t^q_{\frac{1}{\mu}}$. (2) Let x be in $t^q_\mu(K^q_\lambda)$, then there exist

$\alpha \geq 0$ and $p \in P$ such that $x = t^q_\mu(\alpha(p-(1-\lambda)q))$. We have

$$t^q_\mu(\alpha(p-(1-\lambda)q)) = \alpha(p-(1-\lambda\mu)q) \in K^q_{\lambda\mu},$$

therefore $t^q_\mu(K^q_\lambda) \subset K^q_{\lambda\mu}$ holds. By applying the result, we see $t^q_{\frac{1}{\mu}}(K^q_{\lambda\mu}) \subset K^q_\lambda$. Since $(t^q_{\frac{1}{\mu}})^{-1} = t^q_\mu$, we conclude $t^q_\mu(K^q_\lambda) = K^q_{\lambda\mu}$. ■

Lemma 3. *The following properties are satisfied:*
(1) $\mathrm{int}K = \{\alpha p \mid \alpha > 0, p \in \mathrm{int}K \cap P\}$;
(2) $\mathrm{cl}K = \{\alpha p \mid \alpha \geq 0, p \in \mathrm{cl}P\}$.

Proof. It is easily confirmed the above Lemma holds. ■

Lemma 4. *Let $q \in \mathrm{int}K \cap P$. Then,*
(1) $\forall \lambda \in [-\infty, \infty]$,

$$\mathrm{int}K^q_\lambda = \begin{cases} \emptyset & \text{if } \lambda = \infty, \\ \{\alpha(p-(1-\lambda)q) \mid \alpha > 0, p \in \mathrm{int}K \cap P\} & \text{if } \lambda \in (0,\infty), \\ \bigcup_{\mu \in (0,\infty)} \{\alpha(p-(1-\mu)q) \mid \alpha > 0, p \in \mathrm{int}K \cap P\} & \text{if } \lambda = 0, \\ \{-\alpha(p-(1+\lambda)q) \mid \alpha \geq 0, p \in \mathrm{cl}P\}^c & \text{if } \lambda \in (-\infty, 0), \\ \{-\alpha q \mid \alpha \geq 0\}^c & \text{if } \lambda = -\infty; \end{cases}$$

(2) $\forall \lambda, \lambda' \in [-\infty, \infty]$ *with* $\lambda < \lambda'$, $K^q_{\lambda'} \subset \mathrm{int}K^q_\lambda \cup \{\theta\}$.

Proof. At first, we show the proof of (1) without $\lambda = 0$. If $\lambda \in \{-\infty, \infty\}$, this proof is easy and omitted. Assume that $\lambda \in (0, \infty)$. By using Lemma 2 and Lemma 3, we have

$$\begin{aligned} \mathrm{int}K^q_\lambda &= \mathrm{int}t^q_\lambda(K) \\ &= t^q_\lambda(\mathrm{int}K) \\ &= t^q_\lambda(\{\alpha p \mid \alpha > 0, p \in \mathrm{int}K \cap P\}) \\ &= \{\alpha(p-(1-\lambda)q) \mid \alpha > 0, p \in \mathrm{int}K \cap P\}. \end{aligned}$$

Assume that $\lambda \in (-\infty, 0)$. To show that $\mathrm{int}K^q_\lambda = \{-\alpha(p-(1+\lambda)q) \mid \alpha \geq 0, p \in \mathrm{cl}P\}^c$ holds, it is satisfied that $\mathrm{cl}((K^q_\lambda)^c) = \{-\alpha(p-(1+\lambda)q) \mid \alpha \geq 0, p \in \mathrm{cl}P\}$. In the same way as $\lambda \in (0,\infty)$, we obtain $\mathrm{cl}(\{\alpha(p-(1-(-\lambda))q) \mid \alpha > 0, p \in P\}) = \{\alpha(p-(1+\lambda)q) \mid \alpha \geq 0, p \in \mathrm{cl}P\}$. Then, we have

$$(*) \qquad \mathrm{cl}((K^q_\lambda)^c) = \{-\alpha(p-(1+\lambda)q) \mid \alpha \geq 0, p \in \mathrm{cl}P\}.$$

Then, proof of (1) without $\lambda = 0$ is finished. Next, we show the proof of (2) in condition that $\lambda, \lambda' \in (0, \infty]$ or $\lambda, \lambda' \in [-\infty, 0)$. If $\lambda' = \infty$ or $\lambda = -\infty$, the proof is easy and omitted. Assume that $\lambda, \lambda' \in (0, \infty)$. We show that $K \subset \text{int}K_{\frac{\lambda}{\lambda'}} \cup \{\theta\}$. Let x be in K, then there exist a nonnegative number α and $p \in P$ such that $x = \alpha p$. We have $\frac{\alpha\lambda'}{\lambda} \geq 0$, $\frac{\lambda}{\lambda'}p + (1 - \frac{\lambda}{\lambda'})q \in \text{int}K \cap P$ and $x = \frac{\alpha\lambda'}{\lambda}((\frac{\lambda}{\lambda'}p + (1-\frac{\lambda}{\lambda'})q) - (1-\frac{\lambda}{\lambda'})q)$. By using Lemma 4(1) when $\lambda \in (0, \infty)$, $\frac{\alpha\lambda'}{\lambda}((\frac{\lambda}{\lambda'}p + (1 - \frac{\lambda}{\lambda'})q) - (1 - \frac{\lambda}{\lambda'})q)$ is in $\text{int}K^q_{\frac{\lambda}{\lambda'}} \cup \{\theta\}$, and then we have $K \subset \text{int}K_{\frac{\lambda}{\lambda'}} \cup \{\theta\}$. By using Lemma 2, we obtain

$$K^q_{\lambda'} = t^q_{\lambda'}(K) \subset t^q_{\lambda'}(\text{int}K^q_{\frac{\lambda}{\lambda'}} \cup \{\theta\}) = \text{int}K^q_{\lambda} \cup \{\theta\}.$$

Assume that $\lambda, \lambda' \in (-\infty, 0)$. To show that $K^q_\lambda \subset K^q_{\lambda'}$ holds, it is enough if $(K^q_{\lambda'})^c \subset \text{cl}((K^q_\lambda)^c) \setminus \{\theta\}$ is shown. The inclusion is obtained in the same way as $\lambda, \lambda' \in (0, \infty)$. Next, we show the proof of (1) in condition that $\lambda = 0$. It is clear that $\bigcup_{\mu \in (0,\infty)} \{\alpha(p - (1 - \mu)q) \mid \alpha > 0, p \in \text{int}K \cap P\} \subset K^q_0$ holds. Since $\{\alpha(p - (1 - \mu)q) \mid \alpha > 0, p \in \text{int}K \cap P\}$ is open for each $\mu \in (0, \infty)$, $\bigcup_{\mu \in (0,\infty)} \{\alpha(p - (1 - \mu)q) \mid \alpha > 0, p \in \text{int}K \cap P\}$ is open. Let O be a open subset of E with $O \subset K^q_0$ and x in $O \setminus \{\theta\}$, then there exists $\mu_0 \in [0, \infty)$ such that $x \in \{\alpha(p - (1 - \mu_0)q) \mid \alpha \geq 0, p \in P\} = K^q_{\mu_0}$. we show now that $\mu_0 \neq 0$ holds. Suppose that $\mu_0 = 0$. There exists a neighborhood U of θ such that $x + U \subset O$, and there exist $\alpha \geq 0$ and $p \in P$ such that $x = \alpha(p - (1 - \mu_0)q)$. We can choose $n_0 \in \mathbb{N}$ such that $-\frac{\alpha}{n_0}q \in U \subset K^q_0$ since $\alpha(p - (1 + \frac{1}{n})q) \to x$, therefore we have $-q \in K^q_0$. Since $q \in \text{int}K$, a radial circled neighborhood V of θ such that $q + V \subset K^q_0$. Then $V \subset K^q_0$ holds, this means that $E = K^q_0$. The equation is a contradiction, and we have $\mu_0 \neq 0$. By $x \neq \theta$ and using Lemma 4(2) in condition that $\lambda, \lambda' \in (0, \infty)$, we have

$$x \in \text{int}K^q_{\mu_0+1} \subset \bigcup_{\mu \in (0,\infty)} \{\alpha(p - (1 - (\mu_0 + 1))q) \mid \alpha > 0, p \in \text{int}K \cap P\}.$$

Finally, we show the proof of (2) in other conditions. Assume that $\lambda = 0$ and $\lambda' \in (0, \infty)$. Because of (1),

$$\text{int}K^q_\lambda \cup \{\theta\} \supset \text{int}K^q_{\frac{1}{2}\lambda'} \supset K^q_{\lambda'}.$$

If $\lambda \in (-\infty, 0)$ and $\lambda' = 0$, we can prove the same as $\lambda = 0$ and $\lambda' \in (0, \infty)$. In other case, we can prove by using the above results. ■

Lemma 5. *Let $q \in \text{int}K \cap P$. Then, the following properties are satisfied:*

(1) $\forall\lambda \in [-\infty, \infty]$,

$$\mathrm{cl}K_\lambda^q = \begin{cases} \{\alpha q \mid \alpha \geq 0\} & \text{if } \lambda = \infty, \\ \{\alpha(p-(1-\lambda)q) \mid \alpha \geq 0, p \in \mathrm{cl}P\} & \text{if } \lambda \in (0,\infty), \\ \bigcup_{\mu\in[0,\infty)} \{\alpha(p-(1-\mu)q) \mid \alpha \geq 0, p \in P\} & \text{if } \lambda = 0, \\ \{-\alpha(p-(1+\lambda)q) \mid \alpha > 0, p \in \mathrm{int}K \cap P\}^c & \text{if } \lambda \in (-\infty, 0), \\ E & \text{if } \lambda = -\infty; \end{cases}$$

(2) $\forall\lambda, \lambda' \in [-\infty, \infty]$ *with* $\lambda < \lambda'$, $\mathrm{cl}K_{\lambda'}^q \subset K_\lambda^q$.

Proof. Since $-P = \{x \in -K \mid \langle -\xi, x\rangle = 1\}$ is a base of $-K$, and $-q \in -P$, we can consider the flutter cones of $-K$ with respect to $-q$. Then we have

$$\mathrm{cl}\left(K_\lambda^q\right) = \left(\mathrm{int}(-K)_{-\lambda}^{-q}\right)^c$$

for each $\lambda \in [-\infty, \infty]$. Indeed, when $\lambda \neq 0$, we can check easily

$$(-K)_{-\lambda}^{-q} \cap K_\lambda^q = \{\theta\} \quad \text{and} \quad (-K)_{-\lambda}^{-q} \cup K_\lambda^q = E$$

by the definition of the flutter cones, and also

$$\left((-K)_{-\lambda}^{-q}\right)^c = K_\lambda^q \setminus \{\theta\}.$$

Therefore we have

$$\mathrm{cl}\left(K_\lambda^q\right) = \mathrm{cl}\left(K_\lambda^q \setminus \{\theta\}\right) = \mathrm{cl}\left(((-K)_{-\lambda}^{-q})^c\right) = \left(\mathrm{int}(-K)_{-\lambda}^{-q}\right)^c.$$

On the other hand, when $\lambda = 0$,

$$K_0^q = \{x \in E \mid \langle \xi, x\rangle \geq 0\} \quad \text{and} \quad (-K)_0^{-q} = \{x \in E \mid \langle -\xi, x\rangle \geq 0\}$$

by using Lemma 1, and it is easy to show that

$$\mathrm{cl}\left(K_0^q\right) = \left(\mathrm{int}(-K)_0^{-q}\right)^c.$$

Hence we obtain $\mathrm{cl}\left(K_\lambda^q\right) = \left(\mathrm{int}(-K)_{-\lambda}^{-q}\right)^c$ for each $\lambda \in [-\infty, \infty]$. From this and Lemma 4, we complete the proof. ∎

Now we have the following Lemma by using Lemma 4(2) and Lemma 5.

Lemma 6. *Let* $q \in \mathrm{int}K \cap P$. *Then,*

$$\mathrm{cl}K_{\lambda'}^q \subset \mathrm{int}K_\lambda^q \cup \{\theta\}$$

for each $\lambda, \lambda' \in [-\infty, \infty]$ *with* $\lambda < \lambda'$.

Lemma 7. *The following properties are satisfied:*

(1) $\bigcup_{\mu\in(1,\infty)} K_\mu^q = \mathrm{int}K \cup \{\theta\}$;

(2) $\bigcap_{\mu\in(0,1)} K_\mu^q = \mathrm{cl}K$.

Proof. (1) By using Lemma 4, we can show that $\bigcup_{\mu\in(1,\infty)} K_\mu^q \subset \mathrm{int}K \cup \{\theta\}$ holds. Let x be an element of $\mathrm{int}K\cup\{\theta\}$. If $x=\theta$, it is clear that $x \in \bigcup_{\mu\in(1,\infty)} K_\mu^q$. Assume that $x \neq \theta$, then there exist a positive number α and $p \in \mathrm{int}K \cap P$ such that $x = \alpha p$. We can choose a radial open neighborhood U of θ such that $p + U \subset K$, and then there exists a positive number r such that $r(p-q) \in U$. Therefore, we obtain

$$x - \frac{\alpha r}{1+r}q = \frac{\alpha}{1+r}(p + r(p-q)) \in \frac{\alpha}{1+r}(p+U) \subset K.$$

$\left\langle \xi, x - \frac{\alpha r}{1+r}\right\rangle \neq 0$ holds now, so we denote that

$$\mu = \frac{\frac{\alpha r}{1+r}}{\left\langle \xi, x - \frac{\alpha r}{1+r}q\right\rangle} + 1.$$

Then, we have

$$\begin{aligned} x &= \left\langle \xi, x - \frac{\alpha r}{1+r}q\right\rangle \left(\frac{x - \frac{\alpha r}{1+r}q}{\left\langle \xi, x - \frac{\alpha r}{1+r}q\right\rangle} + \frac{\frac{\alpha r}{1+r}q}{\left\langle \xi, x - \frac{\alpha r}{1+r}q\right\rangle} \right) \\ &= \left\langle \xi, x - \frac{\alpha r}{1+r}q\right\rangle \left(\frac{x - \frac{\alpha r}{1+r}q}{\left\langle \xi, x - \frac{\alpha r}{1+r}q\right\rangle} - (1-\mu)\, q \right) \in K_\mu^q \subset \bigcup_{\mu\in(1,\infty)} K_\mu^q. \end{aligned}$$

(2) By using Lemma 5, we can show that $\mathrm{cl}K \subset \bigcap_{\mu\in(0,1)} K_\mu^q$ holds. Let x be an element of $\bigcap_{\mu\in(0,1)} K_\mu^q$. If $x=\theta$, it is clear that $x \in \mathrm{cl}K$. Assume that $x \neq \theta$ and let $n \in \mathbb{N}$, then there exist a positive number α_n and $p_n \in P$ such that $x = \alpha_n(p_n - (1-(1-\frac{1}{n+1}))q)$. Calculating the value of $\langle \xi, x\rangle$, we have

$$\alpha_n = \frac{\langle \xi, x\rangle}{1 - \frac{1}{n+1}}.$$

From this and $x = \alpha_n(p_n - (1-(1-\frac{1}{n+1}))q)$, we have

$$p_n = \frac{1-\frac{1}{n+1}}{\langle \xi, x\rangle}x + \frac{1}{n+1}q.$$

Therefore $\{p_n\}_{n\in\mathbb{N}}$ converges to $\dfrac{1}{\langle \xi, x\rangle}x$, and we have $\dfrac{1}{\langle \xi, x\rangle}x \in \mathrm{cl}P$. ■

Let $q \in \mathrm{int}K \cap P$ and $\mu \in (0,\infty)$, then we can consider flutter cones of K^q_μ with respect to μq since K^q_μ is also an acute convex cone of E, $P-(1-\mu)q$ is equal to $\{x \in E \mid \left\langle \frac{1}{\mu}\xi, x\right\rangle = 1\}$ and is a base of K^q_μ. Then we have the following lemma, the proof is easy and omitted.

Lemma 8. *For each $q \in \mathrm{int}K \cap P$, $\lambda \in \mathbb{R}$ and $\mu \in (0,\infty)$,*

$$(K^q_\mu)^{\mu q}_\lambda = K^q_{\lambda\mu}.$$

Lemma 9. *The following properties are satisfied:*

(1) $\lambda \in [0,\infty) \Longrightarrow \mathrm{cl}K^q_\lambda = (\mathrm{cl}K)^q_\lambda$;
(2) $\lambda \in (0,\infty) \Longrightarrow \mathrm{int}K^q_\lambda \cup \{\theta\} = (\mathrm{int}K \cup \{\theta\})^q_\lambda$;
(3) $\lambda \in (-\infty,0] \Longrightarrow \mathrm{cl}K^q_\lambda = (\mathrm{int}K \cup \{\theta\})^q_\lambda$;
(4) $\lambda \in (-\infty,0) \Longrightarrow \mathrm{int}K^q_\lambda \cup \{\theta\} = (\mathrm{cl}K)^q_\lambda$.

Proof. If $\lambda \neq 0$, we obtain the properties by using Lemma 4 and Lemma 5. If $\lambda = 0$, we obtain $\mathrm{cl}K^q_0 = (\mathrm{cl}K)^q_0 = (\mathrm{int}K \cup \{\theta\})^q_0 = \{x \in E \mid \langle \xi, x\rangle \geq 0\}$ by using Lemma 1. ■

We state representations of efficiencies by using flutter cones.

Theorem 1. *The following properties are satisfied:*

(1) $\mathrm{Min}(A|K) = \mathrm{Min}(A|K^q_1)$;
(2) *K is closed* $\Longrightarrow \mathrm{PrMin}(A|K) = \displaystyle\bigcup_{\mu\in(0,1)} \mathrm{Min}(A|K^q_\mu)$;
(3) $\mathrm{WMin}(A|K) = \displaystyle\bigcap_{\mu\in(1,\infty)} \mathrm{Min}(A|K^q_\mu)$;

Proof. It is clear that (1) holds. (2) Assume that K is closed, and let x be an element of $\mathrm{PrMin}(A|K)$. Then there exists a convex cone L of E, which is not the whole space, such that $\mathrm{cl}K \setminus \{\theta\} \subset \mathrm{int}L$ and $x \in \mathrm{Min}(A|L)$. We show that we can choose $\mu_0 \in (0,1)$ such that $\mathrm{cl}K^q_\mu \subset L$. Assume that $\mathrm{cl}K^q_\mu \not\subset L$ for each $\mu \in (0,1)$, and let n be a natural number. Then there exists $x_n \in \mathrm{cl}K^q_{1-\frac{1}{2n}}$ such that $x_n \notin L$, and there exist a positive number α_n and $p_n \in \mathrm{cl}P$ such that

$x_n = \alpha_n(p_n-(1-(1-\frac{1}{2n}))q)$. We denote $y_n = p_n-(1-(1-\frac{1}{2n}))q$, and then $y_n \in \mathrm{cl}K^q_{1-\frac{1}{2n}} \cap (\mathrm{int}L)^c$. Since cl$P$ is compact, we can choose a subsequence $\{p_{n_i}\}_{i\in\mathbb{N}}$ of $\{p_n\}_{n\in\mathbb{N}}$ and $p \in \mathrm{cl}P$ such that $\{p_{n_i}\}_{i\in\mathbb{N}}$ converges to p. Then $\{y_{n_i}\}_{i\in\mathbb{N}}$ converges to p, and we have $p \in (\mathrm{int}L)^c \subset (\mathrm{cl}K)^c$. This is a contradiction. By using Proposition 2.4 of Chapter 2 in [9],

$$x \in \mathrm{Min}(A|L) \subset \mathrm{Min}(A|K^q_{\mu_0}) \subset \bigcup_{\mu\in(0,1)} \mathrm{Min}(A|K^q_\mu).$$

(3) It is easy to show that $\mathrm{WMin}(A|K) \subset \bigcap_{\mu\in(1,\infty)} \mathrm{Min}(A|K^q_\mu)$ holds. Let x be an element of $\bigcap_{\mu\in(1,\infty)} \mathrm{Min}(A|K^q_\mu)$. By using proposition 2.3 of Chapter 2 in [9], we obtain

$$\{x\} = \bigcup_{\mu\in(1,\infty)} (A\cap(x-K^q_\mu)) = A \cap \left(x - \bigcup_{\mu\in(1,\infty)} K^q_\mu\right).$$

By using Lemma 7, we have $A\cap(x-(\mathrm{int}K\cup\{\theta\})) = \{x\}$. This means that $x \in \mathrm{WMin}(A|K)$. ■

Before we state representation of ideal efficiency, we introduce new notions of ideal efficiency, which are proper ideal efficiency and weakly ideal efficiency.

Definition 4. Let A be a subset of a topological vector space E and K a convex cone of E which is not the whole space. We say that

(1) $x \in A$ is a proper ideal efficient point of A with respect to K if there exists a cone L of E, which does not consist only the null vector, such that cl$L$ is contained in K and x is a ideal efficient point of A with respect to L;

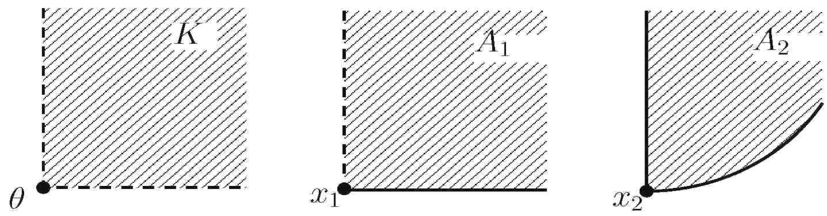
(2) $x \in A$ is a weakly ideal efficient point of A with respect to K if x is a ideal efficient point of A with respect to cl$K$;

The set of all proper ideal efficient points (resp. weakly ideal efficient points) of A with respect to K is denoted by $\mathrm{PrIMin}(A|K)$ (resp. $\mathrm{WIMin}(A|K)$). Note that

$$\begin{aligned}&\mathrm{PrIMin}(A|K) \subset \mathrm{IMin}(A|K) \subset \mathrm{WIMin}(A|K)\\ &\subset \mathrm{PrMin}(A|K) \subset \mathrm{Min}(A|K) \subset \mathrm{WMin}(A|K)\end{aligned}$$

hold. In the following figure, x_1 is a component of $\mathrm{WIMin}(A_1|K)$, but is not a component of $\mathrm{IMin}(A_1|K)$. On the other hand, x_2 is not a component of PrIMin $(A_2|K)$, but is a component of $\mathrm{IMin}(A_2|K)$.

We obtain results concerned with ideal efficiency, which is similar to Theorem 1.



Theorem 2. *The following properties are satisfied:*

(1) $\mathrm{IMin}(A|K) = \mathrm{IMin}(A|K_1^q)$;

(2) $K \setminus \{\theta\}$ *is open* $\Longrightarrow \mathrm{PrIMin}(A|K) = \displaystyle\bigcup_{\mu \in (1,\infty)} \mathrm{IMin}(A|K_\mu^q)$;

(3) $\mathrm{WIMin}(A|K) = \displaystyle\bigcap_{\mu \in (0,1)} \mathrm{IMin}(A|K_\mu^q)$.

Proof. It is clear that (1) holds. (2) Let x be an element of $\mathrm{PrIMin}(A|K)$, there exists a convex cone L of E, which does not consist only the null vector, such that $\mathrm{cl}L \subset K$ and $x \in \mathrm{IMin}(A|L)$. We show that there exists $\mu_0 \in (1, \infty)$ such that $\mathrm{cl}L \subset K_{\mu_0}^q$. Assume that $\mathrm{cl}L \not\subset K_\mu^q$ for all $\mu \in (1, \infty)$. Let n be a natural number, then there exists $x_n \in \mathrm{cl}L$ such that $x_n \notin K_{1+\frac{1}{n}}^q$. Because of the condition of K, there exist $\alpha_n > 0$ and $p_n \in \mathrm{int}K \cap P$ such that $x_n = \alpha_n p_n$. There exist a subsequence $\{p_{n_i}\}_{i \in \mathbb{N}}$ of $\{p_n\}_{n \in \mathbb{N}}$ and $p \in \mathrm{cl}P$ such that $\{p_{n_i}\}_{i \in \mathbb{N}}$ converges to p, since $\mathrm{cl}P$ is compact, and then we obtain $p \in \mathrm{cl}L \subset K$. On the other hand, we also obtain

$$p \in \bigcap_{n \in \mathbb{N}} (K_{1+\frac{1}{n}}^q)^c = K^c$$

by $K = \mathrm{int}K \cup \{\theta\}$ and using Lemma 4 and Lemma 7. This is a contradiction. Therefore,

$$x \in \mathrm{IMin}(A|L) \subset \mathrm{IMin}(A|K_{\mu_0}^q) \subset \bigcup_{\mu \in (1,\infty)} \mathrm{IMin}(A|K_\mu^q).$$

By using Lemma 5, we obtain $\displaystyle\bigcup_{\mu \in (1,\infty)} \mathrm{IMin}(A|K_\mu^q) \subset \mathrm{IMin}(A|K)$. (3) By using Lemma 6 and Lemma 7, we can show that $\mathrm{WIMin}(A|K) = \displaystyle\bigcap_{\mu \in (0,1)} \mathrm{IMin}(A|K_\mu^q)$ holds. ■

Equivalent definitions of efficiency and weakly efficiency are as follows:

- $x \in \mathrm{Min}(A|K) \iff A \cap (x - K) = \{x\}$;
- $x \in \mathrm{WMin}(A|K) \iff A \cap (x - \mathrm{int}K) = \emptyset$;

see [9]. By using the flutter cones, we obtain equivalent definitions of ideal efficiency which are the same as the above.

Theorem 3. *An equivalent definition of ideal efficiency:*

(1) $x \in \mathrm{IMin}(A|K) \iff A \cap (x - K^q_{-1}) = \{x\}$;

(2) $x \in \mathrm{WIMin}(A|K) \iff A \cap (x - \mathrm{int}K^q_{-1}) = \emptyset$.

Proof. (1)

$$\begin{aligned} x \in \mathrm{IMin}(A|K) &\iff A \subset x + K \\ &\iff A \cap (x+K)^c = \emptyset \\ &\iff A \cap (x + K^c \cup \{\theta\}) = \{x\} \\ &\iff A \cap (x - K^q_{-1}) = \{x\}. \end{aligned}$$

The proof of (2) is the same as (1). ■

Next, we redefine the notions of ideal efficiency without convexity of the cone, and we consider $\mathrm{IMin}(A|K^q_\mu)$ when $\mu < 0$. Then by using the flutter cones, we also have representations of notions of efficiency by ideal efficient points.

Theorem 4. *The following properties are satisfied:*

(1) $\mathrm{Min}(A|K) = \mathrm{IMin}(A|K^q_{-1})$;

(2) K *is closed* $\Longrightarrow \mathrm{PrMin}(A|K) = \displaystyle\bigcup_{\mu \in (-1,0)} \mathrm{IMin}(A|K^q_\mu)$;

(3) $\mathrm{WMin}(A|K) = \displaystyle\bigcap_{\mu \in (-\infty,-1)} \mathrm{IMin}(A|K^q_\mu)$.

Proof. (1) Let x be an element of $\mathrm{Min}(A|K)$, and we show that $A \subset x + K^q_{-1}$. Assume that there exists $a_0 \in A$ such that $a_0 \notin x + K^q_{-1}$, then we have $x - a_0 \in K \setminus \{\theta\}$. Since $x \in \mathrm{Min}(A|K)$ and K is acute, $A \cap (x - K) = \{x\}$. This means that $a_0 = x$, and this is a contradiction. Therefore, we have $x \in \mathrm{IMin}(A|K^q_{-1}$. Let $x \in \mathrm{IMin}(A|K^q_{-1})$ and $y \in A$ with $x \in y + K$. Since $x \in \mathrm{IMin}(A|K^q_{-1})$, we have $x - y \in K^c \cup \{\theta\}$. And then, we obtain $x = y$ since $x - y \in K$. The proofs of (2) and (3) are given by using Lemma 8, Theorem 1 and Theorem 4(1).

3. Evaluate Function of Efficient Points

By representations of efficiencies, we know the following relations between flutter cones and efficiencies.

- $x \in \mathrm{WMin}(A \mid K) \iff A \subset x + K^q_\lambda$ for each $\lambda \in (-\infty, -1)$;
- $x \in \mathrm{Min}(A \mid K) \iff A \subset x + K^q_{-1}$;

- Supposing that K is closed, $x \in \mathrm{PrMin}(A \mid K) \iff$ there exists $\lambda \in (-1, 0)$ such that $A \subset x + K_\lambda^q$;
- $x \in \mathrm{IMin}(A|K) \iff A \subset x + K_1^q$.

Since $\mathrm{IMin}(A|K) \subset \mathrm{PrMin}(A|K) \subset \mathrm{Min}(A|K) \subset \mathrm{WMin}(A|K)$, as the condition of efficiency which has $x \in A$ is stronger, a real number λ which keeps $A \subset x + K_\lambda^q$ is higher. Moreover, as λ which keeps $A \subset x + K_\lambda^q$ is higher, the condition of efficiency which has $x \in A$ is stronger. So, we can consider that the supremum of all real number λ which keeps $A \subset x + K_\lambda^q$ determines value of x as efficiency in vector optimization.

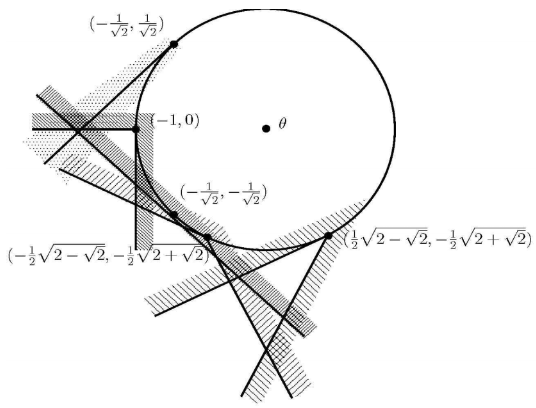
Under this consideration, we define the following evaluate function of efficient points:

Definition 5. Let EV_A^q be a function from E to $[-\infty, \infty]$ defined by

$$EV_A^q(x) := \sup\{\lambda \in \mathbb{R} \mid A \subset x + K_\lambda^q\} \text{ for each } x \in E.$$

Example 3. Let $E = \mathbb{R}^2$, $A = \bar{B}(\theta, 1)$, $K = \mathbb{R}_+^2$, $P = \{(1-\alpha, \alpha) \mid \alpha \in [0, 1]\}$ and $q = \xi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Then, we have

- $EV_A^q(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = EV_A^q(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -\infty$,
- $EV_A^q(-\frac{1}{2}\sqrt{2+\sqrt{2}}, \frac{1}{2}\sqrt{2-\sqrt{2}}) = EV_A^q(\frac{1}{2}\sqrt{2-\sqrt{2}}, -\frac{1}{2}\sqrt{2+\sqrt{2}}) = -2$,
- $EV_A^q(-1, 0) = EV_A^q(0, -1) = -1$,
- $EV_A^q(-\frac{1}{2}\sqrt{2+\sqrt{2}}, -\frac{1}{2}\sqrt{2-\sqrt{2}}) = EV_A^q(-\frac{1}{2}\sqrt{2-\sqrt{2}}, -\frac{1}{2}\sqrt{2+\sqrt{2}}) = -\frac{1}{2}$,
- $EV_A^q(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 0$.



We state a fundamental relation between the evaluate function and efficiencies.

Theorem 5. *Let x be in A. The following properties are satisfied:*

(1) $EV_A^q(x) \geq -1 \iff x \in \mathrm{WMin}(A \mid K)$*;*

(2) K *is closed* $\Longrightarrow (EV_A^q(x) > -1 \iff x \in \mathrm{PrMin}(A \mid K))$*;*

(3) $EV_A^q(x) \geq 1 \iff x \in \mathrm{WIMin}(A \mid K)$*;*

(4) $K \setminus \{\theta\}$ *is open* $\Longrightarrow (EV_A^q(x) > 1 \iff x \in \mathrm{PrIMin}(A \mid K))$.

Proof. (1) Assume that $EV_A^q(x) \geq -1$ holds. Because of the definition of EV_A^q, we have $A \subset x + K\ q_\mu = x + (K^q_{-\mu})^{-\mu q}_{-1}$. By using Theorem 4 and Theorem 1, we obtain $x \in \bigcap_{\mu>1} \mathrm{Min}(A \mid K^q_\mu) = \mathrm{WMin}(A \mid K)$. Next, assume that $x \in \mathrm{WMin}(A \mid K)$ holds. Let $\mu < -1$. By using Theorem 1 and Theorem 4, we have $A \subset x+(K^q_{-\mu})^{-\mu q}_{-1} = x+K^q_\mu$. Therefore, we obtain $EV_A^q(x) \geq \mu$. (2) Assume that $EV_A^q(x) > -1$. Then, there exists $\lambda \in (-1, 0)$ such that $A \subset x + (K^q_{-\lambda})^{-\lambda q}_{-1}$. By using Theorem 1, we have $x \in \bigcup_{\lambda \in (0,1)} \mathrm{Min}(A \mid K^q_\lambda) = \mathrm{PrMin}(A \mid K)$. Next, assume that $x \in \mathrm{PrMin}(A \mid K)$. By using Theorem 1 and Theorem 4, there exists $A \subset x + (K^q_{-\lambda_0})^{-\lambda q}_{-1} = x + K^q_{\lambda_0}$. Therefore we obtain $EV_A^q(x) \geq \lambda_0 > -1$. (3) Assume that $EV_A^q(x) \geq 1$. Let λ be in $(0, 1)$. Then, there exists $\mu_\lambda > -\lambda$ such that $A \subset x + K^q_{\mu_\lambda}$. By using Lemma 1 and Theorem 2, we have $x \in \bigcap_{\lambda \in (0,1)} \mathrm{IMin}(A \mid K^q_\lambda) = \mathrm{WIMin}(A \mid K)$. Next, assume that $\mathrm{WIMin}(A \mid K)$. Let λ be in $(0, 1)$. By using Theorem 2, we have $x \in \mathrm{IMin}(A \mid K^q_\lambda)$. Therefore, we obtain $EV_A^q(x) \geq \lambda$. (4) Assume that $EV_A^q(x) > 1$. Then, there exists $\mu > 1$ such that $A \subset x + K^q_\mu$. Since $\mathrm{cl}K^q_\mu$, we obtain $x \in \mathrm{PrIMin}(A \mid K)$. Next, assume that $x \in \mathrm{PrIMin}(A \mid K)$. By using Theorem 2, we have $\bigcup_{\lambda \in (1,\infty)} \mathrm{IMin}(A \mid K^q_\mu)$. Therefore, there exists $\mu \in (1, \infty)$ such that $EV_A^q(x) \geq \mu$. ■

We could show that points, which has superior values with respect to the evaluate function, is strongly efficient points. Next, we state other merits which are had their points.

Theorem 6. *Let x_0 be in A and $EV_A^q(x_0) \in (-\infty, 0]$. Assume that*

(1) $L \subset \mathrm{int}(K^q_{-EV_A^q(x_0)}) \cup \{\theta\}$ *and L is a convex cone;*

(2) $A \subset B \subset A + \mathrm{int}(K^q_{-EV_A^q(x_0)}) \cup \{\theta\}$.

Then, $x_0 \in \mathrm{Min}(B \mid L)$ holds.

Proof. At first, suppose that $EV_A^q(x_0) < 0$. Since $A \subset x_0 + K^q_{\mu EV_A^q(x_0)}$, we have

$$\begin{aligned} A &\subset x_0 + \bigcap_{\mu \in (-\infty,1)} (K^q_{-EV_A^q(x_0)})_\mu^{-EV_A^q(x_0)q} \\ &= x_0 + \mathrm{cl}(K^q_{EV_A^q(x_0)}) \end{aligned}$$

by using Lemma 8. Since $(\mathrm{int}K^q_{-EV^q_A(x_0)} \cup \{\theta\})^{-EV^q_A(x_0)}_{-1} \subset -L^c \cup \{\theta\}$, we obtain

$$\begin{aligned} B &\subset x_0 + (\mathrm{int}K^q_{-EV^q_A(x_0)} \cup \{\theta\})^{-EV^q_A(x_0)}_{-1} + (\mathrm{int}K^q_{-EV^q_A(x_0)} \cup \{\theta\}) \\ &= x_0 + (\mathrm{int}K^q_{-EV^q_A(x_0)} \cup \{\theta\}) \\ &= x_0 + (-L)^c \cup \{\theta\}. \end{aligned}$$

This means that $x_0 \in \mathrm{Min}(B \mid L)$ holds. Next, suppose that $EV^q_A(x_0) = 0$. Since $A \subset x_0 + K^q_{-\mu}$ for $\mu \in (0,1)$, we have $A \subset x_0 + \bigcap_{\mu\in(0,1)} K^q_{-\mu} = x_0 + K^q_0$ by using Lemma 5. Since $K^q_0 \subset (-L)^c \cup \{\theta\}$, we obtain $B \subset x_0 + K^q_0 + \mathrm{int}K^q_0 \cup \{\theta\} \subset x_0 + (-L)^c \cup \{\theta\}$. ■

Theorem 7. *Let x_0 be in A, $EV^q_A(x_0) \in (0,\infty)$, η be in E^*, $H := \{x \in E \mid \langle\eta, x\rangle \geq 0\}$ and $H \subset \mathrm{int}(K^q_{-EV^q_A(x_0)}) \cup \{\theta\}$. Assume that*

(1) *$L \subset K^q_{-EV^q_A(x_0)}$ and L is a pointed convex cone;*

(2) *$A \subset B \subset A + N$, where $N = \bigcup\{M \subset E \mid M$ is a half-space, $M \subset (\mathrm{int}(K^q_{-EV^q_A(x_0)}) \cup \{\theta\}) \cap ((-L)^c \cup \{\theta\})\}$.*

Then, $x_0 \in \mathrm{Min}(B \mid L)$ holds.

Proof. Since $A \subset x_0 + K^q_{\mu EV^q_A(x_0)}$ for each $\mu \in (0,1)$, we have

$$A \subset x_0 + \bigcap_{\mu\in(0,1)} K^q_{\mu EV^q_A(x_0)} = x_0 + \mathrm{cl}(K^q_{EV^q_A(x_0)})$$

by using Lemma 9 and Lemma 8. Therefore, we obtain

$$B \subset A + K^q_{EV^q_A(x_0)} \subset x_0 + \mathrm{cl}(K^q_{EV^q_A(x_0)}0) \subset x_0 + (-N)^c \cup \{\theta\} + N.$$

Since $K^q_0 \subset (\mathrm{int}(K^q_{-EV^q_A(x_0)}) \cup \{\theta\}) \cap ((-L)^c \cup \{\theta\})$, we have N is nonempty, and $(-N)^q \cup \{\theta\}$ is a pointed convex cone. Therefore we obtain $(-N)^c \subset \{\theta\} + N = (-N)^c \subset \{\theta\}$, also $B \subset x_0 + (-L)^c \cup \{\theta\}$. This means that $x_0 \in \mathrm{Min}(B|L)$ holds. ■

4. Properties of the Evaluate Function and Existence for Maxima of the Evaluate Function

In this section, we state some properties of the evaluate function which was defined in the previous section. And, by using given properties, we obtain existence theorems for the maxima of the evaluate function.

Proposition 1. *Let e be in E, x in A and α a positive number. The following properties are satisfied:*

(1) $EV^q_{A+\mathbb{R}_+q}(x+\alpha q) = -\infty$;
(2) $EV^q_{A+e}(x+e) = EV^q_A(x)$;
(3) $EV^q_{\alpha A}(\alpha x) = EV^q_A(x)$.

Proof.

(1) Since $-q \notin K^q_\lambda$ for each λ, x is not belong to $(x+\alpha q)+K^q_\lambda$ for each $\lambda \in \mathbb{R}$. Therefore, we have $EV^q_{A+\mathbb{R}_+q}(x+\alpha q) = -\infty$.
(2) This proof is easy and omitted.
(3) Since K^q_λ is a cone, we have that $\alpha A \subset \alpha x + K^q_\lambda$ holds if and only if $A \subset x + K^q_\lambda$ holds. ■

Theorem 8. *The evaluate function EV^q_A is upper semi-continuous on A.*

Proof. Assume that there exist $x \in A$ and a net $\{x_\alpha\}_{\alpha\in I}$ of A such that $\{x_\alpha\}_{\alpha\in I}$ converges at x and $EV^q_A(x) < \limsup_{\alpha\in I} EV^q_A(x_\alpha)$. When $EV^q_A(x) = \infty$ holds, it is clear that $EV^q_A(x) < \limsup_{\alpha\in I} EV^q_A(x_\alpha)$ is a contradiction. Next, suppose that $EV^q_A(x) \neq \infty$, and let a real number λ be $EV^q_A(x) < \lambda < \limsup_{\alpha\in I} EV^q_A(x_\alpha)$. Since $EV^q_A(x) < \lambda$, there exists $a \in A$ such that $a \notin x + \mathrm{cl}(K^q_\lambda)$. On the other hand, there exists $\alpha_0 \in I$ such that $\lambda < \sup_{\alpha\geq\beta} EV^q_A(x_\beta)$ for each $\alpha \geq \alpha_0$ since $\lambda < \limsup_{\alpha\in I} EV^q_A(x_\alpha)$. For each $\alpha \geq \alpha_0$, we can choose $\beta_\alpha \geq \alpha$ such that $a \in x_{\beta_\alpha} + \mathrm{cl}(K^q_\lambda)$. It is clear that $\{x_{\beta_\alpha}\}_{\alpha\in I}$ converges at x, then we obtain $a - x \in \mathrm{cl}(K^q_\lambda)$. This is a contradiction. ■

If A is convex, $x \in A$ is important for value of evaluate function

Theorem 9. *Let E be a normed space, x in A, r positive number and $y \in A \cap B(x,r)$, where $B(x,r) := \{z \in E \mid \|z - x\| < r\}$. If A is convex, then $EV^q_A(y) = EV^q_{A\cap\bar{B}(x,r)}(y)$ holds, where $\bar{B}(x,r)(y)$ is the closure of $B(x,r)$ in E.*

Proof. At first, we show $EV^q_A(x) = EV^q_{A\cap\bar{B}(x,r)}(x)$. Since $A \cap \bar{B}(x,r) \subset A$, we have $EV^q_A(x) \leq EV^q_{A\cap\bar{B}(x,r)}(x)$. It is clear that $EV^q_A(x) \geq EV^q_{A\cap\bar{B}(x,r)}(x)$ when $EV^q_A(x) = \infty$, so we suppose that $EV^q_A(x) \neq \infty$. Assume that $EV^q_A(x) < \lambda < EV^q_{A\cap\bar{B}(x,r)}(x)$. Since $EV^q_A(x) < \lambda$, there exists $a \in A$ such that $a \in x + K^q_\lambda$. Let $\alpha = \max\{r, \|a - x\|\}$, then we have $\frac{r}{2\alpha} \in (0,1)$. Since A is convex and $\lambda < EV^q_{A\cap\bar{B}(x,r)}(x)$, we obtain $x + \frac{r}{2\alpha}(a - x) \in A \cap \bar{B}(x,r) \subset x + K^q_\lambda$. Therefore we have $a - x \in K^q_\lambda$, and this is a contradiction. Next, we show $EV^q_A(y) = EV^q_{A\cap\bar{B}(x,r)}(y)$. Since $y \in B(x,r)$, we have there exists $r' > 0$ such that $A \cap \bar{B}(y,r') \subset A \cap \bar{B}(x,r) \subset A$. Therefore we have $EV^q_{A\cap\bar{B}(y,r')}(y) \leq EV^q_{A\cap\bar{B}(x,r)} \leq EV^q_A(y)$. Moreover we have $EV^q_{A\cap\bar{B}(y,r')}(y) = EV^q_A(y)$ by the above proof, and the proof is finished. ■

Lemma 10. *Let x be in A. Then, we have*

$$EV_A^q(x) = EV_{A+\mathbb{R}_+q}^q(x)$$

Proof. This proof is easy, and omitted. ■

Lemma 11. *Let E be a normed space, A a nonempty convex subset of E, x in A, $r > 0$ and y in $A \cap B(x,r)$. Then, we have*

$$EV_A^q(y) = EV_{A\cap\bar{B}(x,r)}^q(y).$$

Proof. There exists $r' > 0$ such that $\bar{B}(y,r') \subset B(x,r)$. By Lemma 9, we have $EV_{A\cap B(y,r')}^q(y) = EV_A^q(y)$. Since $A\cap\bar{B}(y,r') \subset A\cap B(x,r) \subset A$, we obtain

$$EV_A^q(y) \leq EV_{A\cap B(x,r)}^q(y) \leq EV_{A\cap\bar{B}(y,r')}^q.$$ ■

In the rest of this section, we state existence theorems for maxima of the evaluate function.

Proposition 2. *Let x_0 be in A. If $EV_A^q(x_0) > 0$, then $EV_A^q(x_0) = \max EV_A^q(A)$ holds.*

Proof. It is clear that $EV_A^q(x_0) = \max EV_A^q(A)$ when $EV_A^q(x_0) = \infty$, so we suppose that $EV_A^q(x_0) \neq \infty$. Assume that there exists $y_0 \in A$ such that $EV_A^q(x_0) < EV_A^q(y_0)$. Since $EV_A^q(x_0) > 0$, we have $x_0 \in \mathrm{IMin}(A \mid K_{\lambda_0}^q)$. Also, we have $y_0 \in \mathrm{IMin}(A \mid K_{\lambda_0}^q)$ since $\lambda_0 < EV_A^q(y)$. By using Proposition 2.2 of Chapter 2 in [9], we obtain $\{x_0\} = \mathrm{IMin}(A \mid K_{\lambda_0}^q) = \{y_0\}$. This is a contradiction. ■

Proposition 3. *If A is compact, then there exists $x_0 \in A$ such that $EV_A^q(x_0) = \max EV_A^q(A)$.*

Theorem 10. *Let $E := \mathbb{R}^n$, $0 < r$ and y in A. Assume that $A \subset B(y,r) + \mathbb{R}q$ and A is $\mathbb{R}_+q$-closed and $\mathbb{R}_+q$-convex. Then, there exists $x_0 \in A$ such that $EV_A^q(x_0) = \max EV_A^q(A)$.*

Proof. Since $(A + \mathbb{R}_+q) \cap \bar{B}(y,r)$ is compact, there exists $x_0 \in (A + \mathbb{R}_+q) \cap \bar{B}(y,r)$ such that

$$EV_{(A+\mathbb{R}_+q)\cap\bar{B}(y,r)}^q(x_0) = \max EV_{(A+\mathbb{R}_+q)\cap\bar{B}(y,r)}^q((A + \mathbb{R}_+q) \cap \bar{B}(y,r)).$$

Let $r < s$. By using Lemma 9 and Lemma 10, we have

$$EV_A^q(x_0) = \max EV_A^q((A + \mathbb{R}_+q) \cap B(y,s)).$$

We show that $EV_A^q(x) \le EV_A^q(x_0)$ for each $x \in A$. Let $x \in A$. When $x \in B(y,r)$, it is clear that $EV_A^q(x) \le EV_A^q(x_0)$ holds. Assume that $x \notin B(y,r)$. Since $A \subset B(y,r) + \mathbb{R}q$, there exists $z \in B(y,r)$ and $\alpha > 0$ such that $x = z + \alpha q$. For each $\lambda \in \mathbb{R}$, z is not in $x + K$. Therefore, we obtain $EV_A^q(x) = -\infty$. ■

5. A Evaluate Function for Local Properly Efficient Points

In the rest of paper, let E be a metric space. We do not obtain a relation between evaluate function EV_A^q and local proper efficiency. In this section, we introduce another evaluate function and consider a relation between this evaluate function and local proper efficiency.

Definition 6. Let $EV_A'^q$ be a function from E to $[-\infty, \infty]$ defined by

$$EV_A'^q(x) := \lim_{r \to \infty} EV^q_{A \cap B(x,r)}(x) \text{ for each } x \in E.$$

Since $EV^q_{A\cap B(x,r)}$ is non-increasing with respect to r when r is sufficiently large, $EV_A'^q(x)$ is well-define.

Theorem 11. *Let x be in A. The following properties are satisfied:*

(1) $EV_A'^q(x) > -1 \Rightarrow x \in \mathrm{LPrMin}(A|K)$;

(2) $x \in \mathrm{LPrMin}(A|K) \Rightarrow EV_A'^q(x) \ge -1$.

Proof. (1) Let r be a positive number. By the definition of $EV_A'^q$, there exists $\varepsilon_0 > 0$ such that $-1 < EV^q_{A\cap B(x,\varepsilon)}(x)$ for each $\varepsilon > \varepsilon_0$. Assume that $\varepsilon_0 \le r$. Let $-\lambda$ be in $(-1, EV^q_{A\cap B(x,r)})$, then we have $A \cap B(x,r) \subset x + K^q_{-\lambda}$. By using Theorem 4, we have $x \in \mathrm{Min}(A \cap B(x,r)|K^q_\lambda)$. By using Lemma 4, we obtain $K \subset \mathrm{int}K^q_\lambda \cup \{\theta\}$. On the other hand, assume that $r < \varepsilon_0$. Let $-\lambda$ be in $(-1, EV^q_{A\cap B(x,\varepsilon_0)})$, then we have $A \cap B(x,r) \subset x + K^q_{-\lambda}$. By the same as $\varepsilon_0 \le r$, we obtain $x \in \mathrm{Min}(A \cap B(x,r)|K^q_\lambda)$ and $K \subset \mathrm{int}K^q_\lambda \cup \{\theta\}$. (2) Let $\varepsilon > 0$. Since $x \in \mathrm{PrMin}(A \cap B(x,\varepsilon)|K)$, there exists $\lambda \in (0,1)$ such that $x \in \mathrm{Min}(A|K^q_\lambda)$ by using 1. By using Theorem 4, we have $A \cap B(x,\varepsilon) \subset x + K^q_{-\lambda}$. Therefore, we obtain $EV^q_{A\cap B(x,\varepsilon)}(x) > -1$. ■

We state relations between EV_A^q and $EV_A'^q$. When A is a convex set, the evaluate function for local proper efficiency is equivalent the previous evaluate function.

Theorem 12. *Let x be in A.*

(1) $EV_A^q(x) \le EV_A'^q(x)$;

(2) *If A is convex, then we have*

$$EV_A^q(x) = EV_A'^q(x).$$

Proof. (1) It is easy and omitted. (2) By Lemma 9, it is clear. ■

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