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A SCHUR-NEWTON ALGORITHM FOR ROBUST POLE ASSIGNMENT OF DESCRIPTOR SYSTEMS

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Abstract. We propose an algorithm for the pole assignment problem for descriptor systems with proportional and derivative state feedback. The algorithm is the first of its kind, making use of the Schur form and minimizing the departure from normality of the closed-loop poles by Newton's method. Three illustrative examples are given.

1. Introduction

We consider the robust pole assignment of the descriptor system (RPAP_DS)

$$E\lambda x(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, A, B and E are real matrices of appropriate dimensions with E possibly singular and $\lambda x(t) = \dot{x}(t)$ (or $x(t+1)$) for a continuous- (or discrete-) time system. We assume that the pencil (A, E) is *regular* or $\det(\alpha A - \beta E) \not\equiv 0$, and that the system (A, E, B) is *controllable* [24], i.e. $\text{rank}(\alpha A - \beta E, B) = n = \text{rank}(E, B)$, for arbitrary $\alpha, \beta \in \mathbb{C}$. Also consult [2, 4, 11] on the issue of controllability and the related problem of regularization.

By *robust pole assignment* (RPAP), we mean to seek feedback matrices F and G so that the closed-loop pencil $(A+BF, E+BG)$ possesses a prescribed desirable spectrum. It is equivalent to modifying the system in (1) with proportional and derivative state feedback $u = Fx - G\dot{x}$. It is well-known [24] that our problem is solvable when the control system (1) is controllable.

There have been many previous attempts in tackling RPAPs. For ordinary systems, please consult [3, 6, 8, 16, 17, 21] and the references therein. Second-order systems have been considered in [7, 9] and descriptor systems in [2, 5, 10, 23, 24] (some with

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only proportional feedback). Robustness is optimized directly in [6-10, 15-17, 23, 24] and indirectly in [3, 5]. For ordinary systems, a method in [16] has been implemented in the MATLAB command `place` [18]. The Schur form [12] has been utilized in pole assignment problems in [3, 5, 8, 17, 21, 22]. However, robustness is not directly optimized in [3, 5] and the Schur form is not directly computed in [21, 22]. A Schur form optimizing the robustness measure of the departure from normality for ordinary systems is computed directly in [8] in a non-iterative manner. Only suboptimality is achieved in [8], due to the freedom in the first Schur vector, but full optimality is possible after a Newton refinement step [17].

In this paper, our methods for descriptor systems is a generalization of the ones in [8, 17] for ordinary systems. This Schur-Newton algorithm represents the only known method which minimizes the departure from normality of the closed-loop system by computing directly the closed-loop Schur form.

Lastly, there are many concepts of controllability for descriptor systems and many different possibilities in measuring robustness. Various optimization techniques can be applied to these robustness measures, as in [23, 24]. Comparing methods under different assumptions or optimizing different robustness measures is difficult, if at all possible. Optimizing robustness measures blindly usually runs into slow convergence, the lack of a good feasible starting value or other related problems. The main contribution of this paper is the availability of a good feasible starting value, from the Schur algorithm in Section 3, which can be refined efficiently by the Schur-Newton refinement in Section 4. The numerical examples in Section 5 show much promise for the Schur-Newton algorithm but more thorough testing will have to be done.

2. Departure From Normality

We shall quote the departure from normality measure for generalized eigenvalue problems [19], generalizing the similar measure for ordinary systems [1, 12, 14, 20].

Definition 2.1. Let $\{A, B\}$ be a regular matrix pair and $\mathcal{U}_{\{A,B\}}$ be the set of all pairs of transformations $\{Z, U\}$ which satisfy the following conditions:

(i) $Z, U \in \mathbb{C}^{n\times n}$, Z is nonsingular and U is unitary;

(ii) $Z^{-1}AU$ and $Z^{-1}BU$ are both upper triangular; and

(iii) $|(Z^{-1}AU)_{ii}|^2 + |(Z^{-1}BU)_{ii}|^2 = 1$ $(i = 1, \cdots, n)$ where $(A)_{ij}$ is the (i, j) element of A.

Let $(Z, U) \in \mathcal{U}_{\{\mathcal{A},\mathcal{B}\}}$ and $\operatorname{diag}(A) \in \mathbb{C}^n$ denote the diagonal matrix sharing the diagonal of A. Denote

$$\mu(Z, U) \equiv \|(Z^{-1}AU - \operatorname{diag}(Z^{-1}AU), Z^{-1}BU - \operatorname{diag}(Z^{-1}BU)\|_2$$

and

$$\Delta_2(A,B) \equiv \inf_{\{Z,U\}\in\mathcal{U}_{\{\mathcal{A},\mathcal{B}\}}} \mu(Z,U).$$

Then $\Delta_2(A,B)$ is called the *departure from normality measure* of $\{A,B\}$.

Definition 2.2. Let $\sigma(A,B) = \{(\alpha_i,\beta_i)\}$ denote the spectrum of the pencil $\alpha A - \beta B$ and let $(\alpha,\beta) \in \sigma(C,D)$. The *spectral variation* of (C,D) from (A,B) equals

$$s_{(A,B)}(C,D) \equiv \max_{(\alpha,\beta)}\{s_{(\alpha,\beta)}\}, \quad s_{(\alpha,\beta)} \equiv \min_i\{|\alpha\beta_i - \beta\alpha_i|\}$$

Theorem 2.1. (Henrici Theorem [19]). *Let $\{A,B\}$ and $\{C,D\}$ be regular pairs of the same dimension, $\Delta_2(A,B)$ is the departure from normality measure of $\{A,B\}$, and suppose $\Delta_2(A,B) \neq 0$. Let $W = (A,B)$ and $\widetilde{W} = (C,D)$, then*

$$s_{(A,B)}(C,D) \leq \frac{\eta}{g(\eta)}[1+\Delta_2(A,B)]\, d_2(W,\widetilde{W})$$

where $d_2(W,\widetilde{W}) = \|\sin\Theta(W,\widetilde{W})\|_2$ denotes the distance between W and $\widetilde{W}$,

$$\eta = \frac{\Delta_2(A,B)}{[1+\Delta_2(A,B)]\, d_2(W,\widetilde{W})},$$

and $g(\eta)$ is the unique nonnegative root of $g + g^2 + \cdots + g^n = \eta$ $(\eta > 0)$.

Based on Theorem 2.1, we shall minimize the departure from normality of the closed-loop matrix pencil in the effort to control the robustness of the closed-loop spectrum or system.

2. Schur Algorithm

Multiplying the nonsingular matrix Z^{-1} and orthogonal matrix X on the both sides of $A+BF$ and $E+BG$, we get

$$Z^{-1}(A+BF)X = D_\alpha + N_\alpha, \quad Z^{-1}(E+BG)X = D_\beta + N_\beta,$$

or

$$(A+BF)X = Z(D_\alpha + N_\alpha), \quad (E+BG)X = Z(D_\beta + N_\beta), \tag{2}$$

where D_α, D_β are diagonal, and N_α, N_β are straightly upper triangular.

Assuming without loss of generality that the feedback matrix B has full rank and possesses the QR decomposition

$$B = [Q_1, Q_2] \begin{bmatrix} R_B \\ 0 \end{bmatrix} = Q_1 R_B,$$

then $Q_2^\top B = 0$ and $B^\dagger = R_B^{-1} Q_1^\top$.

Pre-multiplying the equations in (2), respectively, by $Q_2^\top$ and $B^\dagger$, we obtain

$$Q_2^\top AX - Q_2^\top Z D_\alpha - Q_2^\top Z N_\alpha = 0, \quad Q_2^\top EX - Q_2^\top Z D_\beta - Q_2^\top Z N_\beta = 0, \tag{3}$$

and

$$F = R_B^{-1} Q_1^\top [Z(D_\alpha + N_\alpha) X^\top - A], \quad G = R_B^{-1} Q_1^\top [Z(D_\beta + N_\beta) X^\top - E]. \tag{4}$$

For a given eigenvalue pairs $\{D_\alpha,\ D_\beta\}$, we can select $Z,\ X$ from (3) then obtain the solution to the pole assignment problem using (4).

In this paper, we denote $C \oplus D = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ (where $C,\ D$ need not to be square), $X = [x_1, x_2, \cdots, x_n] \in \mathbb{R_B}^{n\times n}$, $v(X) = [x_1^\top, x_2^\top, \cdots, x_n^\top]^\top$, $v(AXB) = (B^\top \otimes A)\, v(X)$, $\mathrm{Vec}(I) = [1|0,1|0,0,1|\cdots|0,\cdots,0,1]^\top \in \mathbb{R}^{n(n+1)/2 \equiv q}$. Note that both $v(\cdot)$ and $\mathrm{Vec}(\cdot)$ stack columns of matrices but the latter discards zeroes for strictly upper triangular matrices.

3.1. Real Eigenvalues

Let us first consider the case when all the closed-loop eigenvalues are real, with the closed-loop system matrix pair $(A_c, E_c) = (A + BF, E + BG) = (Z\Lambda_\alpha X^\top, Z \Lambda_\beta X^\top)$ in Schur form. Here we have $(\Lambda_\alpha, \Lambda_\beta) = (D_\alpha + N_\alpha, D_\beta + N_\beta)$, with $D_\alpha = \mathrm{diag}\{\alpha_1, \cdots, \alpha_n\}$, $D_\beta = \mathrm{diag}\{\beta_1, \cdots, \beta_n\}$ being real, $N_\alpha = [\hat\eta_1, \hat\eta_2, \cdots, \hat\eta_n]$, $N_\beta = \left[\hat\zeta_1, \hat\zeta_2, \cdots, \hat\zeta_n\right]$ being straightly upper triangular and nilpotent, and $\eta_j = [\eta_{1,j}, \cdots, \eta_{j-1,j}]^\top$, $\zeta_j = [\zeta_{1,j}, \cdots, \zeta_{j-1,j}]^\top$ are the vectors constructed from $\hat\eta_j$ and $\hat\zeta_j$ with the zeroes at the bottom deleted (thus η_1, ζ_1 are degenerate and $\eta_j, \zeta_j \in \Re^{j-1}$). The Schur vector matrix X is orthogonal. The case when some of the eigenvalues of Λ are complex will be considered later. From (3), for $j = 1, 2, \cdots, n$, we then have

$$Q_2^\top A x_j - \alpha_j Q_2^\top z_j - Q_2^\top \sum_{k=1}^{j-1} \eta_{kj} z_k = 0 = Q_2^\top E x_j - \beta_j Q_2^\top z_j - Q_2^\top \sum_{k=1}^{j-1} \zeta_{kj} z_k.$$

With $X_{-j} \equiv [x_1, \cdots, x_{j-1}]$, $Z_{-j} \equiv [z_1, \cdots, z_{j-1}]$ $(j \geq 2)$, we can select $x_j,\ z_j,\ \eta_j$ and ζ_j from

$$Q_2^\top A x_j - \alpha_j Q_2^\top z_j - Q_2^\top Z_{-j} \eta_j = 0 = Q_2^\top E x_j - \beta_j Q_2^\top z_j - Q_2^\top Z_{-j} \zeta_j.$$

In order to obtain the Schur decomposition $(A_c, E_c) = (A+BF, E+BG) = (Z\Lambda_\alpha X^\top, Z\Lambda_\beta X^\top)$ with X being unitary and Z being nonsingular, we select $X_{-j}^\top x_j = 0 = Z_{-j}^\top z_j$ and $\|x_j\| = 1$. Consequently, we select $x_j,\ z_j,\ \eta_j$ and ζ_j from

$$
(5) \qquad M_j \begin{bmatrix} x_j \\ z_j \\ \eta_j \\ \zeta_j \end{bmatrix} = 0 \ , \quad M_j \equiv \begin{bmatrix} Q_2^\top A & -\alpha_j Q_2^\top & -Q_2^\top Z_{-j} & 0 \\ Q_2^\top E & -\beta_j Q_2^\top & 0 & -Q_2^\top Z_{-j} \\ X_{-j}^\top & 0 & 0 & 0 \\ 0 & Z_{-j}^\top & 0 & 0 \end{bmatrix} .
$$

Notice that the null space $\mathcal{N}(M_j)$ above is non-empty and the algorithm is feasible, as its dimension lies between $2m$ and $2(m+j-1)$.

From Theorem 2.1, we can minimize the size of (η_j, ζ_j), leading to the optimal departure from normality measure and robustness of the closed-loop spectrum. From here on, $\|N_\alpha\|^2 + \|N_\beta\|^2$, the departure from normality, is the robustness measure in the RPAP_DS. We thus arrive to the subproblem from which $x_j,\ z_j,\ \eta_j$ and ζ_j are chosen: (for $j > 1$)

$$
(6) \qquad \min_{\|x_j\|=1} \|\eta_j\|_F^2 + \|\zeta_j\|_F^2 \quad \text{subject to} \quad M_j \begin{bmatrix} x_j \\ z_j \\ \eta_j \\ \zeta_j \end{bmatrix} = 0.
$$

Let $[S_{j1}^\top,\ S_{j2}^\top,\ S_{j3}^\top,\ S_{j4}^\top]^\top$ be a unitary basis of the null space of M_j, with $x_j = S_{j1}u_j$, $z_j = S_{j2}u_j$, $\eta_j = S_{j3}u_j$ and $\zeta_j = S_{j4}u_j$. It is easy to see that $x_j,\ z_j,\ \eta_j$ and ζ_j can be chosen, for a given value of j, by finding the smallest generalized singular value (GSV) [12] and its associated singular vector for $\left\{(S_{j1}^\top,\ S_{j2}^\top)^\top,\ (S_{j3}^\top,\ S_{j4}^\top)^\top\right\}$.

For a given ordering of the close-loop poles, $x_1,\ z_1$ are constrained in (6) but cannot be chosen uniquely, as $\eta_1,\ \zeta_1$ are degenerate. Similar comments apply in the complex case below.

3.2. Complex Eigenvalues

When some of the closed-loop eigenvalues are complex, we can modify our algorithm using real arithmetic so that a real feedback matrix $F,\ G$ can be obtained. Using the following modified real Schur form, the real vectors $x_j,\ x_{j+1},\ z_j,\ z_{j+1},\ \eta_j,\ \eta_{j+1}$ and $\zeta_j,\ \zeta_{j+1}$ are chosen via

$$
Q_2^\top A\,[x_j, x_{j+1}] - Q_2^\top\,[z_j, z_{j+1}]\,D_{\alpha j} - Q_2^\top Z_{-j}\,[\eta_j, \eta_{j+1}] = 0 \ ,
$$

$$
Q_2^\top E\,[x_j, x_{j+1}] - Q_2^\top\,[z_j, z_{j+1}]\,D_{\beta j} - Q_2^\top Z_{-j}\,[\zeta_j, \zeta_{j+1}] = 0 \ , \quad X_{-j}^\top\,[x_j, x_{j+1}] = 0
$$

where

$$D_{\alpha j}=\left[\begin{array}{cc}\mu_j & \nu_j\\ -\nu_j & \mu_j\end{array}\right],\quad D_{\beta j}=\left[\begin{array}{cc}\kappa_j & \tau_j\\ -\tau_j & \kappa_j\end{array}\right].$$

Equivalently, we have

$$M_j\left[\begin{array}{c}x_j\\ x_{j+1}\\ z_j\\ z_{j+1}\\ \hline \eta_j\\ \eta_{j+1}\\ \zeta_j\\ \zeta_{j+1}\end{array}\right]=0,\quad M_j \tag{7}$$

$$\equiv\left[\begin{array}{cccc}I_2\otimes(Q_2^\top A) & D_{\alpha j}^\top\otimes Q_2^\top & -I_2\otimes Q_2^\top Z_{-j} & 0\\ I_2\otimes(Q_2^\top E) & D_{\beta j}^\top\otimes Q_2^\top & 0 & -I_2\otimes Q_2^\top Z_{-j}\\ I_2\otimes X_{-j}^\top & 0 & 0 & 0\\ 0 & I_2\otimes Z_{-j}^\top & 0 & 0\end{array}\right]$$

while minimizing $\|[\eta_j,\ \eta_{j+1}]\|_F^2+\|[\zeta_j,\ \zeta_{j+1}]\|_F^2$ with $\|[x_j,x_{j+1}]\|_F=1$. Notice that the null space $\mathcal{N}(M_j)$ in (7) is non-empty and the algorithm is feasible, as its dimension lies between $4m$ and $4(m+j-1)$. Also, we have not imposed the quadratic constraint that $x_j\perp x_{j+1}$. The conditioning of the pseudo-Schur vectors $[x_j,x_{j+1}]$ is then controlled by $\|[\eta_j,\ \eta_{j+1}]\|_F^2+\|[\zeta_j,\ \zeta_{j+1}]\|_F^2$, or the size of the upper triangular parts of $\Lambda_\alpha,\ \Lambda_\beta$ corresponding to the complex conjugate pair of eigenvalues in $(D_{\alpha j},D_{\beta j})$. Note, in the real Schur form, that $(D_{\alpha j},D_{\beta j})$ can be replaced by any matrix pair with the same eigenvalues, with $x_j^\top x_{j+1}=0$. This orthogonality condition is a difficult quadratic constraint and is abandoned for simplicity. To pay for this simplicity, the pseudo-Schur vector matrix X is no longer orthogonal. However, it is still nearly orthogonal, with $X^\top X$ being block-diagonal and 2×2 diagonal blocks for individual complex conjugate pairs of closed-loop eigenvalues. The conditioning of the eigenvalues are then partly controlled by the sizes of $\eta_j,\ \eta_{j+1}$ in N_α and $\zeta_j,\ \zeta_{j+1}$ in N_β.

Let $\left[S_{j1}^\top,S_{j2}^\top,S_{j3}^\top,S_{j4}^\top,S_{j5}^\top,S_{j6}^\top,S_{j7}^\top,S_{j8}^\top\right]^\top$ be a unitary basis for the null space defined in (7). We are looking for the vectors

$$x_j=S_{j1}u_j,\qquad x_{j+1}=S_{j2}u_j,\qquad z_j=S_{j3}u_j,\qquad z_{j+1}=S_{j4}u_j\ ,$$
$$\eta_j=S_{j5}u_j,\qquad \eta_{j+1}=S_{j6}u_j,\qquad \zeta_j=S_{j7}u_j,\qquad \zeta_{j+1}=S_{j8}u_j$$

which satisfy $\min_{u_j} \|[\eta_j, \eta_{j+1}]\|_F^2 + \|[\zeta_j, \zeta_{j+1}]\|_F^2$ s.t. $\|[x_j, x_{j+1}]\|_F^2 = 1$, or

$$\min_{u_j} u_j^\top (S_{j5}^\top S_{j5} + S_{j6}^\top S_{j6}) u_j + u_j^\top (S_{j7}^\top S_{j7} + S_{j8}^\top S_{j8}) u_j \quad \text{s.t.} \quad u_j^\top (S_{j1}^\top S_{j1} + S_{j2}^\top S_{j2}) u_j = 1$$

Similar to the real case earlier, the minimization can be achieved via the GSVs of the matrix pair $\left\{ \left(S_{j1}^\top, S_{j2}^\top, S_{j3}^\top, S_{j4}^\top \right)^\top, \left(S_{j5}^\top, S_{j6}^\top, S_{j7}^\top, S_{j8}^\top \right)^\top \right\}$.

Remark. In Theorem 2.1, Z is only required to be nonsingular, but this will be difficult to achieve in practice. If it is unconstrained, an ill-conditioned Z may cause problems in the Schur-Newton refinement in the next Section. Consequently, we require in the calulcations in (5) and (7) that Z has orthogonal columns.

4. Schur-newton Optimization Algorithm

From the Schur algorithm in Section 3, we obtain the starting value for the Newton refinement technique in this Section.

4.1. Real Eigenvalues

We seek feedback matrices $F, G \in \mathbb{R}^{m\times n}$ such that

$$\lambda(A + BF, E + BG) = \lambda \left\{ \left[\begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right] \right\} = \lambda(D_\alpha, D_\beta)$$

for given $D_\alpha \equiv \mathrm{diag}\{\alpha_1, \cdots, \alpha_n\}$, $D_\beta \equiv \{\beta_1, \cdots, \beta_n\}$. We have $(A + BF)Y D_\beta = (E + BG)Y D_\alpha$, where the columns of Y are the eigenvectors. With Z is nonsingular, X is orthogonal and $\alpha_j^2 + \beta_j^2 = 1$ ($j = 1, \cdots, n$), we then have

$$Z^{-1}(A + BF)X = D_\alpha + N_\alpha, \quad Z^{-1}(E + BG)X = D_\beta + N_\beta.$$

Let Q denote Q_2, then we arrive at:

Optimization Problem 1.
Given $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, (A, B) is regular, $Q \in \mathbb{R}^{n\times(n-m)}$ is unitary and $Q^\top B = 0$;

$$\min \|[N_\alpha, N_\beta]\|_F^2$$

$$\textbf{s.t.} \begin{cases} Q^\top AX - Q^\top ZD_\alpha - Q^\top ZN_\alpha = 0 & N_\alpha, N_\beta \text{ are } n\times n \text{ strictly upper triangular,} \\ Q^\top EX - Q^\top ZD_\beta - Q^\top ZN_\beta = 0, & X \text{ is } n\times n \text{ orthogonal, } Z \text{ is nonsingular.} \\ X^\top X - I = 0 & \end{cases}$$

Optimization Problem 1 is equivalent to:

$$\min \mathrm{Vec}(N_\alpha)^\top \mathrm{Vec}(N_\alpha) + \mathrm{Vec}(N_\beta)^\top \mathrm{Vec}(N_\beta)$$

$$\textbf{s.t.} \begin{cases} (I \otimes Q^\top A)v(X) - (D_\alpha^\top \otimes Q^\top)v(Z) - (N_\alpha^\top \otimes Q^\top)v(Z) = 0 \\ (I \otimes Q^\top E)v(X) - (D_\beta^\top \otimes Q^\top)v(Z) - (N_\beta^\top \otimes Q^\top)v(Z) = 0 \\ d_0(X)^\top v(X) - \mathrm{Vec}(I) = 0 \end{cases}$$

where

$$N_\alpha = \begin{bmatrix} 0 & \eta_{12} & \eta_{13} & \cdots & \eta_{1n} \\ 0 & 0 & \eta_{23} & \cdots & \eta_{2n} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \eta_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \ \mathrm{Vec}(N_\alpha) = \begin{bmatrix} \underline{\eta_{12}} \\ \eta_{13} \\ \underline{\eta_{23}} \\ \vdots \\ \overline{\eta_{1n}} \\ \vdots \\ \eta_{n-1,n} \end{bmatrix} \in \mathbb{R}^{n(n-1)/2 \equiv p},$$

$$N_\beta = \begin{bmatrix} 0 & \zeta_{12} & \zeta_{13} & \cdots & \zeta_{1n} \\ 0 & 0 & \zeta_{23} & \cdots & \zeta_{2n} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \zeta_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \ \mathrm{Vec}(N_\beta) = \begin{bmatrix} \underline{\zeta_{12}} \\ \zeta_{13} \\ \underline{\zeta_{23}} \\ \vdots \\ \overline{\zeta_{1n}} \\ \vdots \\ \zeta_{n-1,n} \end{bmatrix} \in \mathbb{R}^{p}.$$

Here, we write $C^{k\times n} = [c_1, \cdots, c_n]$, so

$$d_0(C) = [c_1 \oplus [c_1, c_2] \oplus \cdots \oplus [c_1, \cdots, c_n]] \in \mathbb{R}^{kn\times q}.$$

We then consider the Lagrangian function of Optimization Problem 1:

$$\begin{aligned} &L\left(\gamma, \varepsilon, \delta, v(X), v(Z), \mathrm{Vec}(N_\alpha), \mathrm{Vec}(N_\beta)\right) = \mathrm{Vec}(N_\alpha)^\top \mathrm{Vec}(N_\alpha) \\ &\quad + \mathrm{Vec}(N_\beta)^\top \mathrm{Vec}(N_\beta) + \gamma^\top[(I \otimes Q^\top A)v(X) - (D_\alpha^\top \otimes Q^\top)v(Z) \\ &\quad - (N_\alpha^\top \otimes Q^\top)v(Z)] + \varepsilon^\top[(I \otimes Q^\top E)v(X) \\ &\quad - (D_\beta^\top \otimes Q^\top)v(Z) - (N_\beta^\top \otimes Q^\top)v(Z)] + \delta^\top[d_0(X)^\top v(X) - \mathrm{Vec}(I)] \end{aligned}$$

where

$$\gamma = [\underbrace{\gamma_1^\top}_{l} | \underbrace{\gamma_2^\top}_{l} | \cdots | \underbrace{\gamma_n^\top}_{l}]^\top \in \mathbb{R}^{ln},$$
$$R = \left[\gamma_1, \gamma_2, \cdots, \gamma_n\right],$$
$$\varepsilon = [\underbrace{\varepsilon_1^\top}_{l} | \underbrace{\varepsilon_2^\top}_{l} | \cdots | \underbrace{\varepsilon_n^\top}_{l}]^\top \in \mathbb{R}^{ln},$$
$$W = \left[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\right],$$
$$\delta = [\underbrace{\delta_1^\top}_{1} | \underbrace{\delta_2^\top}_{2} | \cdots | \underbrace{\delta_n^\top}_{n}]^\top = [\delta_{11} \big| \delta_{21}, \delta_{22} \big| \cdots \big| \delta_{n1}, \delta_{n2}, \cdots, \delta_{nn}]^\top \in \mathbb{R}^q.$$

The derivatives of L satisfy

(8) $$f_1 \equiv \frac{\partial L}{\partial \gamma} = (I \otimes Q^\top A)v(X) - (D_\alpha^\top \otimes Q^\top)v(Z) - (N_\alpha^\top \otimes Q^\top)v(Z) = 0,$$

(9) $$f_2 \equiv \frac{\partial L}{\partial \varepsilon} = (I \otimes Q^\top E)v(X) - (D_\beta^\top \otimes Q^\top)v(Z) - (N_\beta^\top \otimes Q^\top)v(Z) = 0,$$

(10) $$f_3 \equiv \frac{\partial L}{\partial \delta} = d_0(X)^\top v(X) - \mathrm{Vec}(I) = 0,$$

(11) $$f_4 \equiv \frac{\partial L}{\partial v(X)} = (I \otimes A^\top Q)\gamma + (I \otimes E^\top Q)\varepsilon + v(X\Delta) = 0,$$

(12) $$f_5 \equiv \frac{\partial L}{\partial v(Z)} = -[(D_\alpha \otimes Q) + (N_\alpha \otimes Q)]\gamma - [(D_\beta \otimes Q) + (N_\beta \otimes Q)]\varepsilon = 0,$$

(13) $$f_6 \equiv \frac{\partial L}{\partial \mathrm{Vec}(N_\alpha)} = 2\,\mathrm{Vec}(N_\alpha) - d_1(Q^\top Z)^\top \gamma = 0,$$

(14) $$f_7 \equiv \frac{\partial L}{\partial \mathrm{Vec}(N_\beta)} = 2\,\mathrm{Vec}(N_\beta) - d_1(Q^\top Z)^\top \varepsilon = 0$$

where

$$\Delta = \begin{bmatrix} 2\delta_{11} & \delta_{21} & \delta_{31} & \cdots & \delta_{n1} \\ \delta_{21} & 2\delta_{22} & \delta_{32} & \cdots & \delta_{n2} \\ \vdots & \vdots & 2\delta_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \delta_{n,n-1} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} & \cdots & 2\delta_{nn} \end{bmatrix}.$$

We can apply Newton's method to $f \equiv (f_1^\top, f_2^\top, \cdots, f_7^\top)^\top$, which can be formulated as

$$\begin{bmatrix} \gamma \\ \varepsilon \\ \delta \\ v(X) \\ v(Z) \\ \mathrm{Vec}(N_\alpha) \\ \mathrm{Vec}(N_\beta) \end{bmatrix}_{\texttt{new}} = \begin{bmatrix} \gamma \\ \varepsilon \\ \delta \\ v(X) \\ v(Z) \\ \mathrm{Vec}(N_\alpha) \\ \mathrm{Vec}(N_\beta) \end{bmatrix} - J_f^{-1} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} \tag{15}$$

where the symmetric

$$J_f = \begin{bmatrix} \frac{\partial f_1}{\partial \gamma} & \frac{\partial f_1}{\partial \varepsilon} & \frac{\partial f_1}{\partial \delta} & \frac{\partial f_1}{\partial v(X)} & \frac{\partial f_1}{\partial v(Z)} & \frac{\partial f_1}{\partial \mathrm{Vec}(N_\alpha)} & \frac{\partial f_1}{\partial \mathrm{Vec}(N_\beta)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_7}{\partial \gamma} & \frac{\partial f_7}{\partial \varepsilon} & \frac{\partial f_7}{\partial \delta} & \frac{\partial f_7}{\partial v(X)} & \frac{\partial f_7}{\partial v(Z)} & \frac{\partial f_7}{\partial \mathrm{Vec}(N_\alpha)} & \frac{\partial f_7}{\partial \mathrm{Vec}(N_\beta)} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & I \otimes Q^\top A & -D_\alpha^\top \otimes Q^\top - N_\alpha^\top \otimes Q^\top & -d_1(Q^\top Z) & 0 \\ 0 & 0 & 0 & I \otimes Q^\top E & -D_\beta^\top \otimes Q^\top - N_\beta^\top \otimes Q^\top & 0 & -d_1(Q^\top Z) \\ * & 0 & 0 & d_0(X)^\top + d_2(X^\top) & 0 & 0 & 0 \\ * & * & * & \Delta \otimes I & 0 & 0 & 0 \\ * & * & * & * & 0 & -d_3\left(Q[\gamma_2, \cdots, \gamma_n]\right) & -d_3\left(Q[\varepsilon_2, \cdots, \varepsilon_n]\right) \\ * & * & * & * & * & 2\,I_p & 0 \\ * & * & * & * & * & 0 & 2\,I_p \end{bmatrix}$$

and

$$d_1(C) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \multicolumn{6}{c}{c_1 \oplus [c_1, c_2] \oplus \cdots \oplus [c_1, \ldots, c_{n-1}]} \end{bmatrix} \in \mathbb{R}^{kn \times p},$$

$$d_2(C^\top) = \begin{bmatrix} c_1^\top \\ c_2^\top \oplus c_2^\top \\ c_3^\top \oplus c_3^\top \oplus c_3^\top \\ \vdots \\ c_n^\top \oplus \cdots \oplus c_n^\top \end{bmatrix} \in \mathbb{R}^{q \times kn},$$

$$d_3(C) = \begin{matrix} \\ k\{ \end{matrix} \begin{bmatrix} c_2 & c_3 \oplus c_3 & c_4 \oplus c_4 \oplus c_4 & \cdots & c_n \oplus \cdots \oplus c_n \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{kn \times p}.$$

Applying Newton's method to Optimization Problem 1, we obtain Z, X then by using (4), the feedback matrices F, G. Now, we can write down the Schur-Newton Algorithm for the RPAP_DS with real eigenvalues:

Algorithm 1 (Real Schur-Newton).

(1) Use the Schur algorithm in Section 3 to find an initial X_0, Z_0 and $N_{\alpha 0}$, $N_{\beta 0}$.

(2) Substitute X_0, Z_0, $N_{\alpha 0}$, $N_{\beta 0}$ into (11)-(14), producing an over-determined linear system for $(\gamma_0^\top, \varepsilon_0^\top, \delta_0^\top)^\top$:

$$
\begin{array}{ll} n^2 & \{ \\ n^2 & \{ \\ p & \{ \\ p & \{ \end{array}
\begin{bmatrix} \overbrace{I \otimes A^\top Q}^{ln} & \overbrace{I \otimes E^\top Q}^{ln} & \overbrace{d_0(X) + d_2(X^\top)^\top}^{q} \\ -(D_\alpha \otimes Q + N_\alpha \otimes Q) & -(D_\beta \otimes Q + N_\beta \otimes Q) & 0 \\ d_1(Q^\top Z)^\top & 0 & 0 \\ 0 & d_1(Q^\top Z)^\top & 0 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \varepsilon_0 \\ \delta_0 \end{bmatrix} \tag{16}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ 2\,\mathrm{Vec}(N_\alpha) \\ 2\,\mathrm{Vec}(N_\beta) \end{bmatrix}
$$

where $l = n - m$, p, q are defined as before and $n^2 + p \geq ln + q$. Use the least squares method to solve the over-determined system in (**??**) for γ_0, ε_0 and δ_0.

(3) Choose $\{\gamma_0, \varepsilon_0, \delta_0, v(X_0), v(Z_0), \mathrm{Vec}(N_{\alpha 0}), \mathrm{Vec}(N_{\beta 0})\}$ to be the starting values, run Newton's iteration (15) until convergence to X, Z and N_α, N_β.

(4) Substitute the X, Z and N_α, N_β into (**??**) to obtain the feedback matrices F, G.

4.2. Complex Eigenvalues

Let $\{(\alpha_1, \beta_1), \cdots, (\alpha_{n-2s}, \beta_{n-2s}); (\mu_1 \pm i\nu_1, \kappa_1 \pm i\tau_1), \cdots, (\mu_s \pm i\nu_s, \kappa_s \pm i\tau_s)\}$ be the prescribed eigenvalues, where s is the number of complex eigenvalue pairs. As in the real eigenvalue case, we seek feedback matrices $F, G \in \mathbb{R}^{n\times n}$ such that

$$
\lambda(A + BF, E + BG) = \lambda \left\{ \left[\begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right] \right\} = \lambda(D_\alpha, D_\beta)
$$

where $\alpha_j, \beta_j, \mu_l, \nu_l, \kappa_l, \tau_l \in \mathbb{R}$, $\alpha_j^2 + \beta_j^2 = 1 = \mu_l^2 + \nu_l^2 + \kappa_l^2 + \tau_l^2$ ($j = 1, \cdots, n - 2s$; $l = 1, \cdots, s$), and $D_\alpha \equiv \mathrm{diag}\{\alpha_1, \cdots, \alpha_{n-2s}; \mu_1 \pm i\nu_1, \cdots, \mu_s \pm i\nu_s\}, D_\beta \equiv \mathrm{diag}\{\beta_1, \cdots, \beta_{n-2s}; \kappa_1 \pm i\tau_1, \cdots, \kappa_s \pm i\tau_s\}$. With a nonsingular Z and orthogonal X, we require

$$
Z^{-1}(A + BF)X = D_\alpha + N_\alpha, \quad Z^{-1}(E + BG)X = D_\beta + N_\beta,
$$

where

$$
D_\alpha = \left[\alpha_1 \oplus \cdots \oplus \alpha_{n-2s} \oplus \begin{bmatrix} a_1 & b_2 \\ b_1 & a_2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} a_{2s-1} & b_{2s} \\ b_{2s-1} & a_{2s} \end{bmatrix} \right]
$$

with

$$\begin{cases} a_1, \cdots, a_{2s}, b_1, \cdots, b_{2s} \in \mathbb{R}, \\ a_{2j-1} + a_{2j} = 2\mu_j, \ j = 1, \cdots, s; \\ a_{2j-1}a_{2j} - b_{2j-1}b_{2j} = \mu_j^2 + \nu_j^2, \ j = 1, \cdots, s. \end{cases},$$

$$D_\beta = \left[\beta_1 \oplus \cdots \oplus \beta_{n-2s} \oplus \begin{bmatrix} c_1 & d_2 \\ d_1 & c_2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} c_{2s-1} & d_{2s} \\ d_{2s-1} & c_{2s} \end{bmatrix} \right],$$

$$\begin{cases} c_1, \cdots, c_{2s}, d_1, \cdots, d_{2s} \in \mathbb{R}, \\ c_{2j-1} + c_{2j} = 2\kappa_j, \ j = 1, \cdots, s; \\ c_{2j-1}c_{2j} - d_{2j-1}d_{2j} = \kappa_j^2 + \tau_j^2, \ j = 1, \cdots, s. \end{cases},$$

$$N_\alpha = \begin{array}{c} \begin{matrix} & \overset{\eta_2}{\downarrow} & \overset{\eta_3}{\downarrow} & \cdots & \overset{\eta_{n-2s}}{\downarrow} & \overset{\eta_{n-2s+1}}{\downarrow} & \overset{\eta_{n-2s+2}}{\downarrow} & \cdots & \overset{\eta_{n-1}}{\downarrow} & \overset{\eta_n}{\downarrow} \end{matrix} \\ \begin{bmatrix} 0 & \eta_{1,2} & \eta_{1,3} & \cdots & \eta_{1,n-2s} & \eta_{1,n-2s+1} & \eta_{1,n-2s+2} & \cdots & \eta_{1,n-1} & \eta_{1,n} \\ & 0 & \eta_{2,3} & & \eta_{2,n-2s} & \eta_{2,n-2s+1} & \eta_{2,n-2s+2} & \cdots & \eta_{2,n-1} & \eta_{2,n} \\ & & 0 & \cdots & \vdots & \vdots & \vdots & & & \vdots \\ & & & & \eta_{n-2s-1,n-2s} & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & 0 & \eta_{n-2s,n-2s+1} & \eta_{n-2s,n-2s+2} & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & \vdots & \vdots & \vdots \\ & & & & & & & & \eta_{n-2,n-1} & \eta_{n-2,n} \\ & & & & & & & & 0 & 0 \\ & & & & & & & & 0 & 0 \end{bmatrix} \end{array},$$

$$N_\beta = \begin{array}{c} \begin{matrix} & \overset{\zeta_2}{\downarrow} & \overset{\zeta_3}{\downarrow} & \cdots & \overset{\zeta_{n-2s}}{\downarrow} & \overset{\zeta_{n-2s+1}}{\downarrow} & \overset{\zeta_{n-2s+2}}{\downarrow} & \cdots & \overset{\zeta_{n-1}}{\downarrow} & \overset{\zeta_n}{\downarrow} \end{matrix} \\ \begin{bmatrix} 0 & \zeta_{1,2} & \zeta_{1,3} & \cdots & \zeta_{1,n-2s} & \zeta_{1,n-2s+1} & \zeta_{1,n-2s+2} & \cdots & \zeta_{1,n-1} & \zeta_{1,n} \\ & 0 & \zeta_{2,3} & & \zeta_{2,n-2s} & \zeta_{2,n-2s+1} & \zeta_{2,n-2s+2} & \cdots & \zeta_{2,n-1} & \zeta_{2,n} \\ & & 0 & \cdots & \vdots & \vdots & \vdots & & & \vdots \\ & & & & \zeta_{n-2s-1,n-2s} & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & 0 & \zeta_{n-2s,n-2s+1} & \zeta_{n-2s,n-2s+2} & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & \vdots & \vdots & \vdots \\ & & & & & & & & \zeta_{n-2,n-1} & \zeta_{n-2,n} \\ & & & & & & & & 0 & 0 \\ & & & & & & & & 0 & 0 \end{bmatrix} \end{array}.$$

We arrive at the optimization problem for complex eigenvalues:

Optimization Problem 2.
Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, (A, B) regular, $Q \in \mathbb{R}^{n\times(n-m)}$ is orthogonal and

$Q^\top B = 0;$

$$\min \|[N_\alpha, N_\beta]\|_F^2$$

$$\textbf{s.t.} \begin{cases} Q^\top AX - Q^\top Z(D_\alpha + N_\alpha) = 0 & D_\alpha, D_\beta, N_\alpha, N_\beta \text{ are as defined before,} \\ Q^\top EX - Q^\top Z(D_\beta + N_\beta) = 0 & X \text{ is } n\times n \text{ orthogonal, } Z \text{ is } n\times n \text{ nonsingular.} \\ X^\top X - I = 0 & \end{cases}$$

Optimization Problem 2 is equivalent to:

$$\min \mathrm{Vec}(N_\alpha)^\top \mathrm{Vec}(N_\alpha) + \mathrm{Vec}(N_\beta)^\top \mathrm{Vec}(N_\beta) \quad \text{s.t.}$$

$$\begin{aligned} (I\otimes Q^\top A)v(X) - (D_\alpha^\top \otimes Q^\top)v(Z) - (N_\alpha^\top \otimes Q^\top)v(Z) = 0 \\ (I\otimes Q^\top E)v(X) - (D_\beta^\top \otimes Q^\top)v(Z) - (N_\beta^\top \otimes Q^\top)v(Z) = 0 \\ d_0(X)^\top v(X) - \mathrm{Vec}(I) = 0 \end{aligned}$$

$$\begin{bmatrix} a_1 + a_2 - 2\mu_1 = 0 \\ \vdots \\ a_{2s-1} + a_{2s} - 2\mu_s = 0 \end{bmatrix}$$
$$\begin{bmatrix} c_1 + c_2 - 2\kappa_1 = 0 \\ \vdots \\ c_{2s-1} + c_{2s} - 2\kappa_s = 0 \end{bmatrix}$$
$$\begin{bmatrix} a_1a_2 - b_1b_2 - (\mu_1^2 + \nu_1^2) = 0 \\ \vdots \\ a_{2s-1}a_{2s} - b_{2s-1}b_{2s} - (\mu_s^2 + \nu_s^2) = 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1c_2 - d_1d_2 - (\kappa_1^2 + \tau_1^2) = 0 \\ \vdots \\ c_{2s-1}c_{2s} - d_{2s-1}d_{2s} - (\kappa_s^2 + \tau_s^2) = 0 \end{bmatrix}$$

for which the Lagrangian function equals

$$\begin{aligned} & L(\gamma, \varepsilon, \delta, \omega, \theta, \xi, \sigma, v(X), v(Z), a, b, c, d, \mathrm{Vec}(N_\alpha), \mathrm{Vec}(N_\beta)) \\ &= \mathrm{Vec}(N_\alpha)^\top \mathrm{Vec}(N_\alpha) + \mathrm{Vec}(N_\beta)^\top \mathrm{Vec}(N_\beta) \\ &\quad + \gamma^\top[(I\otimes Q^\top A)v(X) - (D_\alpha^\top \otimes Q^\top)v(Z) - (N_\alpha^\top \otimes Q^\top)v(Z)] \\ &\quad + \varepsilon^\top[(I\otimes Q^\top E)v(X) - (D_\beta^\top \otimes Q^\top)v(Z) - (N_\beta^\top \otimes Q^\top)v(Z)] \\ &\quad + \delta^\top[d_0(X)^\top v(X) - \mathrm{Vec}(I)] + \sum_{j=1}^{s} \omega_j(a_{2j-1} \end{aligned}$$

$$
\begin{aligned}
&+ a_{2j} - 2\mu_j) + \textstyle\sum_{j=1}^{s} \theta_j (c_{2j-1} + c_{2j} - 2\kappa_j) \\
&+ \sum_{j=1}^{s} \xi_j [a_{2j-1}a_{2j} - b_{2j-1}b_{2j} - (\mu_j^2 + \nu_j^2)] \\
&+ \sum_{j=1}^{s} \sigma_j [c_{2j-1}c_{2j} - d_{2j-1}d_{2j} - (\kappa_j^2 + \tau_j^2)]
\end{aligned}
$$

where

$$
\begin{aligned}
\omega &= (\omega_1, \omega_2, \cdots, \omega_s)^\top, & \theta &= (\theta_1, \theta_2, \cdots, \theta_s)^\top, \\
\xi &= (\xi_1, \xi_2, \cdots, \xi_s)^\top, & \sigma &= (\sigma_1, \sigma_2, \cdots, \sigma_s)^\top, \\
a &= (a_1, a_2, \cdots, a_{2s})^\top, & b &= (b_1, b_2, \cdots, b_{2s})^\top, \\
c &= (c_1, c_2, \cdots, c_{2s})^\top, & d &= (d_1, d_2, \cdots, d_{2s})^\top.
\end{aligned}
$$

The derivatives of L are

$$
f_1 \equiv \frac{\partial L}{\partial \gamma} = (I \otimes Q^\top A)v(X) - (D_\alpha^\top \otimes Q^\top)v(Z) - (N_\alpha^\top \otimes Q^\top)v(Z) = 0, \tag{17}
$$

$$
f_2 \equiv \frac{\partial L}{\partial \varepsilon} = (I \otimes Q^\top E)v(X) - (D_\beta^\top \otimes Q^\top)v(Z) - (N_\beta^\top \otimes Q^\top)v(Z) = 0, \tag{18}
$$

$$
f_3 \equiv \frac{\partial L}{\partial \delta} = d_0(X)^\top v(X) - \mathrm{Vec}(I) = 0, \tag{19}
$$

$$
f_4 \equiv \frac{\partial L}{\partial \omega} = \begin{bmatrix} a_1 + a_2 - 2\mu_1 \\ \vdots \\ a_{2s-1} + a_{2s} - 2\mu_s \end{bmatrix}, \tag{20}
$$

$$
f_5 \equiv \frac{\partial L}{\partial \theta} = \begin{bmatrix} c_1 + c_2 - 2\kappa_1 \\ \vdots \\ c_{2s-1} + c_{2s} - 2\kappa_s \end{bmatrix}, \tag{21}
$$

$$
f_6 \equiv \frac{\partial L}{\partial \xi} = \begin{bmatrix} a_1a_2 - b_1b_2 - (\mu_1^2 + \nu_1^2) = 0 \\ \vdots \\ a_{2s-1}a_{2s} - b_{2s-1}b_{2s} - (\mu_s^2 + \nu_s^2) = 0 \end{bmatrix}, \tag{22}
$$

$$
f_7 \equiv \frac{\partial L}{\partial \sigma} = \begin{bmatrix} c_1c_2 - d_1d_2 - (\kappa_1^2 + \tau_1^2) = 0 \\ \vdots \\ c_{2s-1}c_{2s} - d_{2s-1}d_{2s} - (\kappa_s^2 + \tau_s^2) = 0 \end{bmatrix}, \tag{23}
$$

$$
f_8 \equiv \frac{\partial L}{\partial v(X)} = (I \otimes A^\top Q)\gamma + (I \otimes E^\top Q)\varepsilon + v(X\Delta) = 0, \tag{24}
$$

$$
f_9 \equiv \frac{\partial L}{\partial v(Z)} = -[(D_\alpha \otimes Q) + (N_\alpha \otimes Q)]\gamma - [(D_\beta \otimes Q) + (N_\beta \otimes Q)]\varepsilon = 0, \tag{25}
$$

(26)

$$f_{10} \equiv \frac{\partial L}{\partial a} = \begin{bmatrix} \omega_1 \\ \omega_1 \\ \vdots \\ \omega_s \\ \omega_s \end{bmatrix} + \begin{bmatrix} \xi_1 a_2 \\ \xi_1 a_1 \\ \vdots \\ \xi_s a_{2s} \\ \xi_s a_{2s-1} \end{bmatrix} - \begin{bmatrix} \gamma_{n-2s+1}^\top & & \\ & \ddots & \\ & & \gamma_n^\top \end{bmatrix} v(Q^\top [z_{n-2s+1}, \cdots, z_n]) = 0,$$

(27)

$$f_{11} \equiv \frac{\partial L}{\partial b} = - \begin{bmatrix} \xi_1 b_2 \\ \xi_1 b_1 \\ \vdots \\ \xi_s b_{2s} \\ \xi_s b_{2s-1} \end{bmatrix} - \begin{bmatrix} \gamma_{n-2s+1}^\top & & \\ & \ddots & \\ & & \gamma_n^\top \end{bmatrix} \Pi_s\, v(Q^\top [z_{n-2s+1}, \cdots, z_n]) = 0,$$

(28)

$$f_{12} \equiv \frac{\partial L}{\partial c} = \begin{bmatrix} \theta_1 \\ \theta_1 \\ \vdots \\ \theta_s \\ \theta_s \end{bmatrix} + \begin{bmatrix} \sigma_1 c_2 \\ \sigma_1 c_1 \\ \vdots \\ \sigma_s c_{2s} \\ \sigma_s c_{2s-1} \end{bmatrix} - \begin{bmatrix} \varepsilon_{n-2s+1}^\top & & \\ & \ddots & \\ & & \varepsilon_n^\top \end{bmatrix} v(Q^\top [z_{n-2s+1}, \cdots, z_n]) = 0,$$

(29)

$$f_{13} \equiv \frac{\partial L}{\partial d} = - \begin{bmatrix} \sigma_1 d_2 \\ \sigma_1 d_1 \\ \vdots \\ \sigma_s d_{2s} \\ \sigma_s d_{2s-1} \end{bmatrix} - \begin{bmatrix} \varepsilon_{n-2s+1}^\top & & \\ & \ddots & \\ & & \varepsilon_n^\top \end{bmatrix} \Pi_s\, v(Q^\top [z_{n-2s+1}, \cdots, z_n]) = 0,$$

(30) $$f_{14} \equiv \frac{\partial L}{\partial \mathrm{Vec}(N_\alpha)} = 2\,\mathrm{Vec}(N_\alpha) - \widehat{d_1}(Q^\top Z, s)^\top \gamma = 0,$$

(31) $$f_{15} \equiv \frac{\partial L}{\partial \mathrm{Vec}(N_\beta)} = 2\,\mathrm{Vec}(N_\beta) - \widehat{d_1}(Q^\top Z, s)^\top \varepsilon = 0$$

where $\widehat{d_1}(C, s) =$

$$\begin{bmatrix} 0 & \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & 0 \\ c_1 \oplus \cdots \oplus & [c_1, \cdots, c_{n-2s-1}] \oplus \begin{bmatrix} c_1 \cdots c_{n-2s} & 0 \\ 0 & c_1 \cdots c_{n-2s} \end{bmatrix} \oplus \cdots \oplus & \begin{bmatrix} c_1 \cdots c_{n-2} & 0 \\ 0 & c_1 \cdots c_{n-2} \end{bmatrix} \end{bmatrix} \}k,$$

$$\Pi_s = \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{s}.$$

We can then obtain the symmetric gradient matrix of $f \equiv [f_1^\top, f_2^\top, \cdots, f_{15}^\top]^\top$:

$$J_f = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}$$

where $J_1 = 0,\ J_3 = J_2^\top$,

$$J_2 = \begin{bmatrix} \Omega_1 & \Phi_1 & -\widehat{d_4}(Q^\top Z, s) & -d_4(Q^\top Z, s) & 0 & 0 & -\widehat{d_1}(Q^\top Z, s) & 0 \\ \Omega_2 & \Phi_2 & 0 & 0 & -\widehat{d_4}(Q^\top Z, s) & -d_4(Q^\top Z, s) & 0 & -\widehat{d_1}(Q^\top Z, s) \\ \Xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_5(e) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5(e) & 0 & 0 & 0 \\ 0 & 0 & d_5(a) & -d_5(b) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5(c) & -d_5(d) & 0 & 0 \end{bmatrix},$$

$$J_4 = \begin{bmatrix} \Delta\otimes I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & -\widehat{d_4}(QR, s) & -d_6(QR, s) & -\widehat{d_4}(QW, s) & -d_6(QW, s) & -d_8(QR, s)^\top & -d_8(QW, s)^\top \\ & & d_7(\xi) & 0 & 0 & 0 & 0 & 0 \\ & & & -d_7(\xi) & 0 & 0 & 0 & 0 \\ & & & & d_7(\sigma) & 0 & 0 & 0 \\ & & & & & -d_7(\sigma) & 0 & 0 \\ & & * & & & & 2\,I_{\widehat{p}} & 0 \\ & & & & & & & 2\,I_{\widehat{p}} \end{bmatrix}$$

with

$$\begin{gathered} \Omega_1 = I \otimes Q^\top A, \quad \Omega_2 = I \otimes Q^\top E, \\ \Phi_1 = -D_\alpha^\top \otimes Q^\top - N_\alpha^\top \otimes Q^\top, \quad \Phi_2 = -D_\beta^\top \otimes Q^\top - N_\beta^\top \otimes Q^\top, \\ \widehat{p} = (n-2s)(n-2s-1)/2 + 2s(n-s-1), \\ e = [\underbrace{1, 1, \cdots, 1}_{2s}]^\top, \quad \Xi = d_0(X)^\top + d_2(X^\top), \end{gathered}$$

$$d_4(C, s) = \left[\begin{array}{ccccc} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ \begin{bmatrix} c_{n-2s+2} & \\ & c_{n-2s+1} \end{bmatrix} & \oplus\cdots\oplus & & \begin{bmatrix} c_n & \\ & c_{n-1} \end{bmatrix} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} k(n-2s) \\ \left.\vphantom{\begin{bmatrix} c \\ c \end{bmatrix}}\right\} k(2s) \end{array},$$

$$\widehat{d_4}(C, s) = \left[\begin{array}{ccccccc} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ c_{n-2s+1} & \oplus & c_{n-2s+2} & \oplus\cdots\oplus & c_{n-1} & \oplus & c_n \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} k(n-2s) \\ \left.\vphantom{c}\right\} k(2s) \end{array},$$

$$d_5(\psi) \equiv d_5([\psi_1, \cdots, \psi_{2s}]^\top)) = [\psi_2, \psi_1] \oplus [\psi_4, \psi_3] \oplus \cdots \oplus [\psi_{2s}, \psi_{2s-1}],$$

$$d_6(C,s) = \left[\begin{array}{ccc} 0 & \cdots\ \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\ \cdots & 0 \\ \begin{bmatrix} 0 & c_{n-2s+2} \\ c_{n-2s+1} & 0 \end{bmatrix} & \oplus\cdots\oplus & \begin{bmatrix} 0 & c_n \\ c_{n-1} & 0 \end{bmatrix} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\ \vdots\\ 0\end{array}}\right\} k(k-2s) \\ \left.\vphantom{\begin{bmatrix}0\\ 0\end{bmatrix}}\right\} k(2s) \end{array},$$

$$d_7(\psi) \equiv d_7([\psi_1, \cdots, \psi_{2s}]^\top)) = \begin{bmatrix} 0 & \psi_1 \\ \psi_1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \psi_2 \\ \psi_2 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \psi_s \\ \psi_s & 0 \end{bmatrix},$$

$$d_8(C,s) = \begin{bmatrix} (c_2)^\top & 0 \\ (c_3)^\top \oplus (c_3)^\top & 0 \\ (c_4)^\top \oplus (c_4)^\top \oplus (c_4)^\top & 0 \\ \vdots & \\ (c_{n-2s})^\top \oplus \cdots \oplus \quad c_{n-2s})^\top & 0 \\ (c_{n-2s+1})^\top \oplus \cdots \oplus (c_{n-2s+1})^\top & 0 \\ (c_{n-2s+2})^\top \oplus \cdots \oplus (c_{n-2s+2})^\top & 0 \\ \vdots & \\ (c_{n-1})^\top \oplus \cdots \oplus (c_{n-1})^\top & 0 \\ (c_n)^\top \oplus \cdots \oplus \quad (c_n)^\top & 0 \end{bmatrix} \begin{array}{l} \}\ \ 1 \\ \}\ \ 2 \\ \}\ \ 3 \\ \\ \\ \}\ \ n-2s-1 \\ \}\ \ n-2s \\ \}\ \ n-2s \\ \\ \}\ \ n-2 \\ \}\ \ n-2 \end{array}.$$

Similar to Algorithm 1, we solve Optimization Problem 2 by Newton's iteration for the real F, G:

Algorithm 2 (Complex Schur-Newton).

(1) Use the Schur method in Section 3 to find the initial X_0, Z_0 and $N_{\alpha 0}$, $N_{\beta 0}$.

(2) Substitute X_0, Z_0, $N_{\alpha 0}$, $N_{\beta 0}$ into (24)-(31), we obtain $\{\gamma_0, \varepsilon_0, \delta_0, \omega_0, \theta_0, \xi_0, \sigma_0, a_0, b_0, c_0, d_0\}$.

(3) With $\{\gamma_0, \varepsilon_0, \delta_0, \omega_0, \theta_0, \xi_0, \sigma_0, a_0, b_0, c_0, d_0, v(X_0), v(Z_0), \mathrm{Vec}(N_{\alpha 0}), \mathrm{Vec}(N_{\beta 0})\}$ as starting values, run Newton's iteration until convergence to X, Z and N_α, N_β.

(4) Substitute X, Z and N_α, N_β into (4) to obtain the feedback matrices F, G.

Remark.

- At Step 2, we set a, b, c, d being the same as the given eigenvalues, obtaining γ_0 by substituting $Z_0, N_{\alpha 0}$ into (30) and ε_0 by substituting $Z_0, N_{\beta 0}$ into (31). Then from (24)-(29), we obtain $\delta_0, \omega_0, \theta_0, \xi_0$ and σ_0.

- The starting point $\{\gamma_0, \varepsilon_0, \delta_0, \omega_0, \theta_0, \xi_0, \sigma_0, a_0, b_0, c_0, d_0, v(X_0), v(Z_0), \mathrm{Vec}(N_{\alpha 0}), \mathrm{Vec}(N_{\beta 0})\}$ is often far away from being optimal. In such an event, we apply the GBB Gradient method [13] to decrease the objective function sufficiently, before Newton's iteration is applied.
- At Step 4, since the matrix X is orthogonal, we can use $X^\top$ in place of X^{-1}.

These remarks also hold for Algorithm 1.

5. Numerical Examples

Algorithms 1 and 2 are applied to three examples, all with singular Es. The convergence tolerance is 10^{-8}. The numerical computations were carried out on a MATLAB 7.01 [18] with machine accuracy equals 2.22×10^{-16}. We use $\mathtt{Obj}_{\mathtt{Schur}}$ and $\mathtt{Obj}_{\mathtt{Newton}}$ respectively to denote the values of the departure from normality measure from the Schur and Schur-Newton algorithms (before and after the Newton refinement in Section 4).

Ex1. $n = 4$, $m = 2$, $\lambda_\alpha = \{1, 1, 1, 1.0e - 8\}$, $\lambda_\beta = \{-20, -1, -3, -4\}$;

$$A = \begin{bmatrix} -65 & 65 & -19.5 & 19.5 \\ 0.1 & -0.1 & 0 & 0 \\ 1 & 0 & -0.5 & -1 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}, \; E = \begin{bmatrix} 1000 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 90 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 65 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.4 \end{bmatrix};$$

$$\mathtt{Obj}_{\mathtt{SCHUR}} = 239, \; \mathtt{Obj}_{\mathtt{Newton}} = 14;$$

$$F = \begin{bmatrix} 0.925 & -0.906 & 0.581 & -0.455 \\ 11.075 & -12.4 & 17.8 & -7.25 \end{bmatrix}, G = \begin{bmatrix} -15.3 & -0.005 & -0.860 & 0.440 \\ 6.39 & 17.4 & -54.9 & 22.5 \end{bmatrix}.$$

Ex2. $n = 4$, $m = 2$, $\lambda_\alpha = \{1, 1, 1, 1\}$, $\lambda_\beta = \{-1, -2, -3, -4\}$;

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 0 & 1 & 10 & 100 \\ 0 & 0 & 1 & 10 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$\mathtt{Obj}_{\mathtt{SCHUR}} = 30789, \quad \mathtt{Obj}_{\mathtt{Newton}} = 42;$$

$$F = \begin{bmatrix} 0.196 & -0.576 & -0.125 & -3.99 \\ -1.24 & -10.4 & -100 & -997 \end{bmatrix}, \; G = \begin{bmatrix} -19.0 & -0.115 & 3.78 & -8.02 \\ 17.8 & -0.036 & 4.73 & 1.40 \end{bmatrix}.$$

Ex3. $n = 5$, $m = 2$, $\lambda_\alpha = \{1, 1, 1, 1, 1\}$, $\lambda_\beta = \{-0.2, -0.5, -1, -1+i, -1-i\}$;

$$A = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.0023 & 0 & 0.1636 & -0.1625 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0.0638 & 0 \\ 0.0838 & -0.1396 \\ 0.1004 & -0.206 \\ 0.0063 & -0.0128 \end{bmatrix};$$

$$\texttt{Obj}_{\texttt{SCHUR}} = 60, \ \texttt{Obj}_{\texttt{Newton}} = 54;$$

$$F = \begin{bmatrix} -46.4 & 50.9 & -55.0 & 6.12 & 29.5 \\ -23.3 & 22.1 & -25.1 & 5.53 & 21.4 \end{bmatrix}, \ G = \begin{bmatrix} -22.5 & -2.46 & 17.5 & -216 & 21.2 \\ -18.6 & -2.05 & 21.0 & -150 & 21.8 \end{bmatrix}.$$

Comments.

(1) For Ex1 and Ex2 with real eigenvalues, the starting vectors from the Schur algorithm in Section 3 fall within the domain of convergence for the Schur-Newton algorithm. This coincides with our experience with other RPAP_DS with real eigenvalues. The subsequent Newton refinement produces a local minimum which improves the robustness measure substantially.

(2) For the RPAP_DS with complex eigenvalues like Ex3, the starting vectors from Schur are often infeasible. Preliminary correction by Newton's iteration can be applied to the constraints in Optimization Problem 2, with gradient J_2. This produces a feasible starting vector for the Schur-Newton algorithm. However, the improvement in the robustness measure can be limited, as shown in Ex3. Apart from having an infeasible starting vector far from a local minimum, the main difficulty lies in the choice of finding accurate starting values for the Lagrange multipliers. However, improvements are still possible theoretically and achieved in practice, as seen in Ex3.

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