TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 6, pp. 1561-1580, September 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

RECURRENT DIMENSIONS AND EXTENDED COMMON MULTIPLES OF QUASI-PERIODIC ORBITS GIVEN BY SOLUTIONS OF SECOND ORDER NONLINEAR EVOLUTION EQUATIONS

Koichiro Naito

Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. In our previous paper we introduced the sequence of Extended Common Multiples (abr. ECM) and the ECM condition for a pair of rationally independent irrational numbers to estimate the recurrent dimensions of quasi-periodic orbits. In this paper first we define the quasi-periodic properties for the functions, which have recurrent properties related with some ECM sequences. Next we study a second order nonlinear evolution equation with a quasi-periodic perturbation term, which gives an ECM sequence, and we show the existence of solutions, which have the quasi-periodic properties induced by this ECM sequence. Furthermore, we estimate the recurrent dimensions of the discrete orbits given by this solution for the case where the irrational frequencies of the q.p. pertubations satisfy the ECM condition.

1. Introduction

Recurrent properties of quasi-periodic orbits depend on rationally approximable properties of their irrational frequencies. In our previous paper [7] we introduced the sequence of Extended Common Multiples (abr. ECM) and the ECM condition for a pair of rationally independent irrational numbers to estimate the recurrent dimensions of quasi-periodic orbits. In this paper we define the quasi-periodic properties for the functions, which have recurrent properties related with some ECM sequences. Next we study a second order nonlinear evolution equation with a quasi-periodic perturbation term, which gives an ECM sequence, and we show the existence of solutions, which have the quasi-periodic properties induced by this ECM sequence. Furthermore, we estimate the recurrent dimensions of discrete orbits given by the

Received April 8, 2008.
2000 *Mathematics Subject Classification*: 58F27, 11K60, 28A80, 35B15.
Key words and phrases: Diophantine approximation, Quasi-periodic solutions, Fractal dimension.

solutions of the second order nonlinear evolution equation with the quasi-periodic perturbation term, the irrational frequencies of which satisfy the ECM condition.

In this section we introduce the notations and definitions given in [7].

First we introduce the definitions of recurrent dimensions.

Let T be a nonlinear operator on a Banach space X. For an element $x \in X$ we consider a discrete dynamical system given by

$$x_n = T^n x, \quad n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}$$

and its orbit is denoted by

$$\Sigma_x = \{T^n x : \quad n \in \mathbf{N}_0\}.$$

For a small $\varepsilon > 0$, define upper and lower first ε-recurrent times by

$$\overline{M}_\varepsilon = \sup_{n\in\mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x), \quad m \in \mathbf{N}\},$$
$$\underline{M}_\varepsilon = \inf_{n\in\mathbf{N}_0} \min\{m : T^{m+n}x \in V_\varepsilon(T^n x), \quad m \in \mathbf{N}\},$$

respectively, where $V_\varepsilon(z) = \{y \in X : \|y - z\| < \varepsilon\}$. Then upper and lower recurrent dimensions are defined as follows:

$$\overline{D}_r(\Sigma_x) = \limsup_{\varepsilon\to 0} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon}, \quad \overline{d}_r(\Sigma_x) = \limsup_{\varepsilon\to 0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon},$$
$$\underline{d}_r(\Sigma_x) = \liminf_{\varepsilon\to 0} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon}, \quad \underline{D}_r(\Sigma_x) = \liminf_{\varepsilon\to 0} \frac{\log \underline{M}_\varepsilon}{-\log \varepsilon}.$$

Instead of considering the whole orbit we can treat a local point in the orbit, say, $T^{n_0}x,\ n_0 \in \mathbf{N}$. Define the first ε-recurrent time by

$$M_\varepsilon(n_0) = \min\{m \in \mathbf{N} : T^{m+n_0}x \in V_\varepsilon(T^{n_0}x)\}$$

and the upper and lower recurrent dimensions by

$$\overline{D}_r(n_0) = \limsup_{\varepsilon\to 0} \frac{\log M_\varepsilon(n_0)}{-\log \varepsilon},$$
$$\underline{D}_r(n_0) = \liminf_{\varepsilon\to 0} \frac{\log M_\varepsilon(n_0)}{-\log \varepsilon}.$$

It follows from the definitions that we have

$$\overline{D}_r(\Sigma_x) \geq \overline{D}_r(n_0) \geq \overline{d}_r(\Sigma_x),$$

$$\underline{d}_r(\Sigma_x) \geq \underline{D}_r(n_0) \geq \underline{D}_r(\Sigma_x).$$

Here we treat discrete 2-frequencies quasi-periodic orbits. Let a function g from $[0,+\infty)\times[0,+\infty)$ to X be 1-periodic with respect to each variable:

$$g(t,s)=g(t+1,s)=g(t,s+1),\qquad \forall t,s\in[0,+\infty)$$

and consider the following Hölder condition.

(H1) There exist constants $\sigma_1,\sigma_1' : 0<\sigma_1,\sigma_1'\le 1$ and $L_1,L_1'>0$, which satisfy

$$\|g(t,s)-g(t',s')\| \;\le\; L_1|t-t'|^{\sigma_1}+L_1'|s-s'|^{\sigma_1'},$$
$$t,t',s,s'\ge 0 : |t-t'|<\varepsilon_0, |s-s'|<\varepsilon_0$$

for a small constant $\varepsilon_0>0$.

Let τ_1,τ_2 be irrational numbers, which have the Diophantine approximations $\{n_j/m_j\},\{r_j/l_j\}$, respectively. Then we consider the quasi-periodic orbit defined by

$$\varphi(n)=g(\tau_1 n,\tau_2 n),\quad \Sigma=\{\varphi(n)\in X:\ n\in\mathbf{N}_0\}.$$

Hereafter we consider the case where τ_1,τ_2 are **(KL)** class numbers;

$$C_1^k\le m_k\le C_2^k,\quad \forall k\in\mathbf{N}:k\ge k_0, \tag{1.1}$$

$$D_1^s\le l_s\le D_2^s,\quad \forall s\in\mathbf{N}:s\ge s_0 \tag{1.2}$$

for some large $k_0,s_0\in\mathbf{N}$ and $C_2>C_1>1,\ D_2>D_1>1$.

Remark 1.1. **(KL)** class number, the abbreviation of Khinchin-Lévy number, was introduced in [7]. In [2] Khinchin proved that the denominators $\{m_k\}$ of the Diophantine approximations of almost all irrational numbers satisfy (1.1) and furthermore, he had shown that there exists a constant γ_0, which satisfies

$$\lim_{k\to\infty}(m_k)^{\frac{1}{k}}=\gamma_0$$

for almost all irrational numbers. By Lévy this constant was estimated:

$$\gamma_0=e^{\frac{\pi^2}{12\log 2}}\sim 3.27582...$$

To simplify our argument we introduce the following constants.

$$E_1=\min\{C_1,D_1\},\quad E_2=\max\{C_2,D_2\},$$
$$\bar\sigma_1=\min\{\sigma_1,\sigma_1'\},\quad \bar L_1=\max\{L_1,L_1'\}.$$

We define the following sets of positive integers by using $\{m_j\}, \{l_j\}$ as the bases. Let $0 \le \alpha, \beta < 1$ and $k, s \in \mathbf{N}$, then we put

$$\begin{aligned}
[M]_k^\alpha := \{ & m \in \mathbf{N} : m = p_k m_k + p_{k-1} m_{k-1} + \cdots + p_u m_u, \\
& k \ge u \ge 1 : \quad \frac{k-u}{k} = \alpha, \quad p_i \in \mathbf{N}_0, i = u, u+1, ..., k : \\
& p_k, p_u \ge 1, \quad p_j < \frac{m_{j+1}}{m_j}, \quad j = u, u+1, ..., k\}, \\
[L]_s^\beta := \{ & l \in \mathbf{N} : l = q_s l_s + q_{s-1} l_{s-1} + \cdots + q_t l_t, \\
& s \ge t \ge 1 : \quad \frac{s-t}{s} = \beta, \quad q_i \in \mathbf{N}_0, i = t, t+1, ..., s : \\
& q_s, q_t \ge 1, \quad q_j < \frac{l_{j+1}}{l_j}, \quad j = t, t+1, ..., s\}
\end{aligned}$$

and define

$$[M]^\alpha := \bigcup_{k=1}^{\infty} [M]_k^\alpha, \qquad [L]^\beta := \bigcup_{s=1}^{\infty} [L]_s^\beta.$$

Furthermore, we put

$$\begin{aligned}
[M]_k^{(d)} := \{ & m \in \mathbf{N} : m = p_k m_k + p_{k-1} m_{k-1} + \cdots + p_d m_d, \\
& p_i \in \mathbf{N}, i = d, d+1, ..., k : \\
& \quad 1 \le p_k < \frac{m_{k+1}}{m_k}, \quad 0 \le p_j < \frac{m_{j+1}}{m_j}, \quad j = d, d+1, ..., k-1\}, \\
[L]_s^{(d)} := \{ & l \in \mathbf{N} : l = q_s l_s + q_{s-1} l_{s-1} + \cdots + q_d l_d, \\
& q_i \in \mathbf{N}, i = d, d+1, ..., s : \\
& 1 \le q_s < \frac{l_{s+1}}{l_s}, \quad 0 \le q_j < \frac{l_{j+1}}{l_j}, \quad j = d, d+1, ..., s-1\}
\end{aligned}$$

and define

$$[M]^{(d)} := \bigcup_{k=1}^{\infty} [M]_k^{(d)}, \qquad [L]^{(d)} := \bigcup_{s=1}^{\infty} [L]_s^{(d)}$$

for $d = 0, 1, 2, ...$ Since $m_0 = l_0 = 1$, we note that

$$\mathbf{N} = [M]^{(0)} = [L]^{(0)}.$$

Since

$$\begin{aligned}
& m_1 \ge 2 \text{ if } 0 < \tau_1 < \frac{1}{2} \text{ and, } l_1 \ge 2 \text{ if } 0 < \tau_2 < \frac{1}{2}, \\
& m_1 = 1, \ m_2 \ge 2 \text{ if } \frac{1}{2} < \tau_1 < 1 \text{ and, } l_1 = 1, \ l_2 \ge 2 \text{ if } \frac{1}{2} < \tau_2 < 1,
\end{aligned}$$

we define $[M]^{(d_1)} \cap [L]^{(d_2)}$ as follows:

$$(1.3) \qquad d_i = 1 \text{ if } 0 < \tau_i < \frac{1}{2} \text{ and, } d_i = 2 \text{ if } \frac{1}{2} < \tau_i < 1, \quad i = 1, 2.$$

For our purpose, to estimate recurrent dimensions, we choose a suitable subsequence in $[M]^{(d_1)} \cap [L]^{(d_2)}$ by the following construction method.

(T) For positive integers m, l:

$$m = p_k m_k + \cdots + p_{u+1} m_{u+1} + p_u m_u,$$
$$l = q_s l_s + \cdots + q_{t+1} l_{t+1} + q_t l_t,$$

define $\zeta_1, \zeta_2 : \mathbf{N} \to \mathbf{N}$ by

$$\zeta_1(m) = u, \qquad \zeta_2(l) = t.$$

Define a sequence of positive integers $T_j \in [M]^{(d_1)} \cap [L]^{(d_2)}$ as follows. Let

$$T_1 = \min\{m : m \in [M]^{(d_1)} \cap [L]^{(d_2)}\}$$

and

$$T_2 = \min\{m \in [M]^{(d_1)} \cap [L]^{(d_2)} : \min\{\zeta_1(m), \zeta_2(m)\} > \min\{\zeta_1(T_1), \zeta_2(T_1)\}\}.$$

Iteratively, let

$$T_{j+1} = \min\{m \in [M]^{(d_1)} \cap [L]^{(d_2)} : \min\{\zeta_1(m), \zeta_2(m)\} > \min\{\zeta_1(T_j), \zeta_2(T_j)\}\}.$$

Let $\zeta_1(T_j) = t_j, \zeta_2(T_j) = u_j$, then we note that the sequence $\{\min\{u_j, t_j\}\}$ is strictly increasing.

$[T_j]$ denotes the sequence $\{T_j\}$ in $[M]^{(d_1)} \cap [L]^{(d_2)}$, which is constructed by the method **(T)** and then we call $[T_j]$ the sequence of extended common multiples (abr. ECM).

Now we can estimate the upper bounds of the recurrent dimensions for 2-frequencies discrete quasi-periodic orbits. (For details, see [7].) We give its proof, since we will use the notations and the argument in the following sections.

Theorem 1.2. ([7]). *Under Hypothesis* **(H1)***, let* τ_1, τ_2 *be* **(KL)** *class numbers, which satisfy* (1.1) *and* (1.2) *and for the sequence* $[T_j]$ *of ECM, constructed by the method* **(T)***, such that*

$$T_j = M_{k_j} = L_{s_j} : \quad M_{k_j} \in [M]_{k_j}^{\alpha_j}, \quad L_{s_j} \in [L]_{s_j}^{\beta_j}, \quad j = 1, 2, ...,$$

assume that the pair of sequences of real numbers $\alpha_j, \beta_j : 0 \le \alpha_j, \beta_j < 1, \quad j = 1, 2, \ldots$, *satisfies*

$$\delta_0 := \liminf_j \max\{\alpha_j, \beta_j\} < 1. \tag{1.4}$$

Then we have

$$\underline{D}_r(\Sigma) \le \underline{d}_r(\Sigma) \le \frac{\log E_2}{(1-\delta_0)\bar{\sigma}_1 \log E_1}. \tag{1.5}$$

Proof. Put

$$\begin{aligned}
M_{k_j} &= p_{k_j} m_{k_j} + p_{k_j-1} m_{k_j-1} + \cdots + p_{u_j} m_{u_j}, \\
N_{k_j} &= p_{k_j} n_{k_j} + p_{k_j-1} n_{k_j-1} + \cdots + p_{u_j} n_{u_j} : \\
\frac{k_j - u_j}{k_j} &= \alpha_j, \ p_{k_j}, p_{u_j} \ge 1, \quad p_k < \frac{m_{k+1}}{m_k}, \quad k = u_j, \ldots, k_j, \\
L_{s_j} &= q_{s_j} l_{s_j} + q_{s_j-1} l_{s_j-1} + \cdots + q_{t_j} l_{t_j}, \\
R_{s_j} &= q_{s_j} r_{s_j} + q_{s_j-1} r_{s_j-1} + \cdots + q_{t_j} r_{t_j} : \\
\frac{s_j - t_j}{s_j} &= \beta_j, \quad q_{s_j}, q_{t_j} \ge 1, \quad 0 \le q_s < \frac{l_{s+1}}{l_s}, \quad s = t_j, \ldots, s_j.
\end{aligned}$$

Then, since $M_{k_j} = L_{s_j}$, it follows from Hypothesis **(H1)** that we have

$$\begin{aligned}
&\|\varphi(M_{k_j} + n) - \varphi(n)\| \\
=\ & \|g(\tau_1(M_{k_j} + n), \tau_2(L_{s_j} + n)) - g(\tau_1 n + N_{k_j}, \tau_2 n + R_{s_j}))\| \\
\le\ & L_1 |\tau_1 M_{k_j} - N_{k_j}|^{\sigma_1} + L_1' |\tau_2 L_{s_j} - R_{s_j}|^{\sigma_1'} \\
\le\ & L_1 \{p_{k_j} |\tau_1 m_{k_j} - n_{k_j}| + \cdots + p_{u_j} |\tau_1 m_{u_j} - n_{u_j}|\}^{\sigma_1} \\
& + L_1' \{q_{s_j} |\tau_2 l_{s_j} - r_{s_j}| + \cdots + q_{t_j} |\tau_2 l_{t_j} - r_{t_j}|\}^{\sigma_1'}.
\end{aligned}$$

Using the above estimate and the conditions of **(KL)**, we can estimate

$$\begin{aligned}
&\|\varphi(M_{k_j} + n) - \varphi(n)\| \\
\le\ & L_1 \left\{ \frac{m_{k_j+1}}{m_{k_j}} \cdot \frac{1}{m_{k_j+1}} + \cdots + \frac{m_{u_j+1}}{m_{u_j}} \cdot \frac{1}{m_{u_j+1}} \right\}^{\sigma_1} \\
& + L_1' \left\{ \frac{l_{s_j+1}}{l_{s_j}} \cdot \frac{1}{l_{s_j+1}} + \cdots + \frac{l_{t_j+1}}{l_{t_j}} \cdot \frac{1}{l_{t_j+1}} \right\}^{\sigma_1'}
\end{aligned}$$

$$
\begin{aligned}
&\leq L_1 \left\{ \frac{1}{C_1^{k_j}} \left(\frac{C_1^{k_j - u_j + 1} - 1}{C_1 - 1} \right) \right\}^{\sigma_1} + L_1' \left\{ \frac{1}{D_1^{s_j}} \left(\frac{D_1^{s_j - t_j + 1} - 1}{D_1 - 1} \right) \right\}^{\sigma_1'} \\
&\leq c E_1^{-\min\{u_j, t_j\} \bar{\sigma}_1} := \varepsilon_j, \quad \forall j \geq j_1
\end{aligned}
$$

for a sufficiently large j_1 such that $k_0 \leq u_{j_1},\ s_0 \leq t_{j_1}$. For the ε_j-recurrent time we can estimate

$$
\begin{aligned}
M_{k_j} &= p_{k_j} m_{k_j} + p_{k_j - 1} m_{k_j - 1} + \cdots + p_{u_j} m_{u_j} \\
&\leq \left(\frac{m_{k_j+1}}{m_{k_j}}\right) m_{k_j} + \cdots + \left(\frac{m_{u_j+1}}{m_{u_j}}\right) m_{u_j} \\
&\leq C_2^{u_j+1} \cdot \frac{C_2^{k_j - u_j + 1} - 1}{C_2 - 1} \leq c' C_2^{k_j}.
\end{aligned}
$$

It follows from Hypothesis that we have

$$
\begin{aligned}
\inf_{0 < \varepsilon \leq \varepsilon_j} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon} &= \frac{\log \overline{M}_{\varepsilon_j}}{-\log \varepsilon_j} \leq \frac{\log M_{k_j}}{-\log \varepsilon_j} \\
&= \frac{\log c' + k_j \log E_2}{\log c^{-1} + \min\{k_j, s_j\}(1 - \max\{\alpha_j, \beta_j\}) \bar{\sigma}_1 \log E_1}.
\end{aligned}
$$

Similarly, we have

$$
\inf_{0 < \varepsilon \leq \varepsilon_j} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon} \leq \frac{\log c'' + s_j \log E_2}{\log c^{-1} + \min\{k_j, s_j\}(1 - \max\{\alpha_j, \beta_j\}) \bar{\sigma}_1 \log E_1}.
$$

It follows that

$$
\inf_{0 < \varepsilon \leq \varepsilon_j} \frac{\log \overline{M}_\varepsilon}{-\log \varepsilon} \leq \frac{\log \max\{c', c''\} + \min\{k_j, s_j\} \log E_2}{\log c^{-1} + \min\{k_j, s_j\}(1 - \max\{\alpha_j, \beta_j\}) \bar{\sigma}_1 \log E_1}.
$$

Taking the limit infimum as $j \to \infty$ of the both sides above, we obtain the conclusion.

Remark 1.3. If a pair of irrational numbers $\{\tau_1, \tau_2\}$ satisfies the condition (1.4), we say that the pair $\{\tau_1, \tau_2\}$ satisfies δ_0-ECM condition. In [7] we have shown some equivalence relations between δ_0-ECM condition and d_0-(D) condition, which was also defined as the parametrized Diophantine condition for a pair of irrational numbers and we also have shown that $\delta_0 = 1/2$ for almost all irrational pairs.

2. Nonlinear Evolution Equation

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and we consider the class of flexible systems that can be described by the following second-order damped evolution equation in $Y = L^2(\Omega)$ with a nonlinear forcing term under a quasi-periodic perturbation:

$$\frac{d^2u(t)}{dt^2} + 2\alpha A\frac{du(t)}{dt} + Au(t) = F(u(t)) + w(t), \quad t > 0, \tag{2.1}$$

$$u(0) = u_0, u_t(0) = u_1 \tag{2.2}$$

where $w(t)$ is a quasi-periodic function from $\mathbf{R}$ to Y and we assume that A is a self-adjoint positive definite operator with dense domain $D(A)$ in Y, and A^{-1} exists and compact. It is well known that there exist eigenvalues λ_i and corresponding eigenfunctions $\varphi_{i,j}(x)$ of the operator A satisfying the following conditions:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots, \quad \lim_{i\to\infty} \lambda_i = \infty,$$

$$A\varphi_{i,j} = \lambda_i\varphi_{i,j}, \quad j = 1, ..., m_i, \quad i = 1, 2, ...,$$

$$\{\varphi_{i,j}(\cdot)\} \text{ forms a complete orthnormal system in } Y.$$

For each constant $0 \le \sigma \le 1$, the domain $D(A^\sigma)$ of the fractional power A^σ, denoted by Y_σ, is topologized by the norm

$$|x|_\sigma^2 := |A^\sigma x|_0^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \lambda_i^{2\sigma}|(x, \varphi_{i,j})|^2, \quad x \in Y_\sigma$$

where $|\cdot|_0$ denotes the norm of Y. We also assume that the perturbation function w is continuous and uniformly bounded and we denote its usual supremum norm by

$$|w|_\infty = \sup\{|w(t)|_0 : \ t \in \mathbf{R}\}.$$

We consider the following condition on the nonlinear function F for a given fixed constant $\beta : 0 < \beta < 1$. F is locally Lipschitz continuous from Y_β to Y: there exits a constant $k(c)$ such that

$$|F(x) - F(y)|_0 \le k(c)|x - y|_\beta \quad \text{for } |x|_\beta, |y|_\beta \le c \tag{2.3}$$

and has the linear growth rate: there exists a positive constant K_0 such that

$$|F(x)|_0 \le K_0(1 + |x|_\beta), \quad x \in Y_\beta. \tag{2.4}$$

The formulation (2.1) includes vibrations in mechanichally flexible systems such as flexible arms or antennas of industrial machinery structures; robots, space aircrafts,... (cf. [9] for linear systems: $f \equiv 0$). Here we treat the case with nonlinear forcing, which is determined not only by the displacement $u(x,t)$, but also by the bending force $u_{xx}(x,t)$. In our previous paper [3], giving some inequality relations between system parameters, such as the eigenvalues of the linear term and the growth rate or the (locally) Lipschitz constant of the nonlinear term, we have studied the periodic stability of this system. If the values $\lambda_1, \lambda_{h+1} - (1/\alpha^2), (1/\alpha^2) - \lambda_h$ are sufficiently large where

$$0 < \lambda_1 < \cdots < \lambda_h < \frac{1}{\alpha^2} < \lambda_{h+1} < \cdots ,$$

we have shown periodicity and asymptotic periodicity of solutions under periodic perturbations.

Here under the quasi-periodic perturbations and the same (inequality) Hypotheses as those of the periodic case we show quasi-periodic properties of solutions and estimate recurrent dimensions of the discrete orbits given by these solutions.

Following the formulation in [3], we assume that

$$\alpha^2\lambda_i^2 - \lambda_i \neq 0, \quad i = 1, 2, ...$$

and that $\alpha > 0$ is so small that

$$\alpha\lambda_1 < \frac{1}{2\alpha}. \tag{2.5}$$

Define a complex-valued function b by

$$b(\lambda) = \sqrt{\alpha^2\lambda^2 - \lambda}.$$

Then, since A is self-adjoint, we can define an operator $b(A)$ by

$$b(A)u = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} b(\lambda_i)(u_i, \varphi_{i,j})\varphi_{i,j},$$

$$D(b(A)) = \{u \in Y : \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} |b(\lambda_i)(u_i, \varphi_{i,j})|^2 < \infty\}.$$

Note that $D(b(A)) = D(A)$ and define the following two operators by

$$A^+ := \alpha A - b(A), \quad A^- := \alpha A + b(A).$$

Then for each $u \in D(A)$,

$$A^{\pm}u = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}(\alpha\lambda_i \mp b(\lambda_i))(u_i, \varphi_{i,j})\varphi_{i,j}$$

and the eigenvalues and the eigenfunctions of $A^{\pm}$ are given by

$$\nu_i = \alpha\lambda_i - b(\lambda_i), \quad \mu_i = \alpha\lambda_i + b(\lambda_i),$$
$$A^{+}u = \nu_i\varphi_{i,j}, \quad A^{-}u = \mu_i\varphi_{i,j}, \quad j = 1, ..., m_i, \ i = 1, 2, ...$$

It follows from (2.5) that there is an integer $h \geq 1$ such that

$$\alpha^2\lambda_h^2 - \lambda_h < 0, \quad \alpha^2\lambda_{h+1}^2 - \lambda_{h+1} > 0.$$

Since the operators $-A^{+}, -A^{-}$ generate analytic semigroups $S_1(t), S_2(t)$, respectively, and since A^{+} is a bounded operator, we can consider the following systems of the semilinear equations:

$$\dot{\xi} + A^{+}\xi(t) = b^{-1}(A)[F(\frac{\xi+\eta}{2}) + w(t)], \tag{2.6}$$

$$\dot{\eta} + A^{-}\eta(t) = -b^{-1}(A)[F(\frac{\xi+\eta}{2}) + w(t)] \tag{2.7}$$

where $b^{-1}(A)$ is the inverse operator of $b(A)$, that is,

$$b^{-1}(A)u = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}(b(\lambda_i))^{-1}(u_i, \varphi_{i,j})\varphi_{i,j}.$$

Also, their mild forms are described as follows:

$$\xi(t) = S_1(t)\xi_0 + \int_0^t S_1(t-s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s)) + w(s)]ds, \tag{2.8}$$

$$\eta(t) = S_2(t)\eta_0 - \int_0^t S_2(t-s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s)) + w(s)]ds. \tag{2.9}$$

Under the conditions (2.3), (2.4) for a fixed constant $0 < \beta < 1$, we can admit the (classical) solution

$$\begin{aligned} &[\xi(t), \eta(t)] \in C([0, +\infty) : D(A^{\beta}) \times D(A^{\beta})) \\ &\cap C^1([0, +\infty) : L^2(\Omega) \times L^2(\Omega)) \end{aligned} \tag{2.10}$$

for each initial data $[\xi_0, \eta_0] \in D(A^{\beta}) \times D(A^{\beta})$.

Furthermore, we can estimate the regularity of the solutions as follows: if $[\xi_0, \eta_0] \in D(A) \times D(A^{1+s})$ for some constant $s : 0 < \beta \leq s < 1$, then by multiplying λ_i^2 and $\lambda_i^{2(1+s)}$ to the spectral expansions of (2.8), (2.9), respectively, and applying the direct estimation, we have

$$\xi \in C([0, +\infty) : D(A)), \quad \eta \in C([0, +\infty) : D(A^{1+s})). \tag{2.11}$$

Then, it follows from (2.6), (2.7) that

$$\dot{\xi} \in C([0,+\infty) : D(A)), \quad \dot{\eta} \in C([0,+\infty) : D(A^s)). \tag{2.12}$$

Now, define the functions u, v by

$$u = \frac{\xi + \eta}{2}, \quad v = \frac{\xi - \eta}{2}. \tag{2.13}$$

Then from (2.11), (2.12) it follows that

$$u, v \in C([0,+\infty) : D(A)) \cap C^1([0,+\infty) : D(A^s)).$$

Hereafter, we consider the case $s = \beta$. Thus we have

$$[u, \dot{u}] \in C([0,+\infty) : D(A)) \times C([0,+\infty) : D(A^\beta)).$$

We need some notations:

$$\lambda(\beta) := \min\{\sqrt{\lambda_1}, \lambda_1^{1-\beta}\} \tag{2.14}$$

$$M_h := \max\{\frac{1}{\sqrt{1-\alpha^2\lambda_h}}, \sqrt{\frac{\alpha^2\lambda_{h+1}}{\alpha^2\lambda_{h+1}-1}} + 1\}, \tag{2.15}$$

$$C_h := \max\{\sqrt{\frac{\lambda_h}{1-\alpha^2\lambda_h}}, \sqrt{\frac{\lambda_{h+1}}{\alpha^2\lambda_{h+1}-1}}\}. \tag{2.16}$$

Furthermore, for a given constant $\delta : 0 < \delta < \alpha\lambda_1$, define

$$M_\beta := M_h(\lambda_1^\beta + \frac{1}{\alpha})(\frac{\beta}{\alpha\lambda_1 - \delta})^\beta e^{-\beta}. \tag{2.17}$$

By applying the proof of Theorem 2.1 in [3] we obtain the unique existence and boundednes of solutions.

Theorem 2.1. *Under Hypotheses* (2.3), (2.4)*, let* $[\xi_0, \eta_0] \in D(A) \times D(A^{1+\beta})$ *and assume that system parameters* $\delta, \alpha, \beta, \lambda_1, \lambda_h, \lambda_{h+1}, K_0$*, satisfy the following inequality conditions:* $0 < \delta < \alpha\lambda_1,\ 0 < \beta \leq 1/2$*, and*

$$\delta > \vartheta := (\frac{M_\beta K_0 \Gamma(\bar{\beta})}{\lambda(\beta)})^{1/\bar{\beta}}, \tag{2.18}$$

where $\bar{\beta} = 1 - \beta$*. Then the estimate*

$$|A\xi(t)|_0 + |A^-\eta(t)|_\beta \leq K_1(t)(|A\xi_0|_0 + |A^-\eta_0|_\beta) + K_2|w|_\infty + K_3 \tag{2.19}$$

holds for some constants K_2, K_3, *and*

$$K_1(t) = \frac{e^{-(\delta-\vartheta)t}}{\bar{\beta}} + e^{-\alpha\lambda_1 t}\Gamma(\bar{\beta}). \tag{2.20}$$

Consequently, the solution $[u(t), \dot{u}(t)]$, *given by* $u = (\xi + \eta)/2$, *has a global attractor in* $Y_1 \times Y_\beta : \{[x, y] \in Y_1 \times Y_\beta : \ |x|_1 + |y|_\beta \le K_p(K_2(|w|_\infty + K_3)\}$ *for some* $K_p > 0$.

Remark 2.2. In case $1/2 < \beta < 1$ the assertion of Theorem 2.1 holds if one substitutes the constant $\Gamma(\bar{\beta})$ by $\Gamma(\bar{\beta})/(\sin\bar{\beta}\pi)^{\bar{\beta}}$.

For convenience we give the following estimation of the constants:

$$K_p = \frac{1}{2}(1 + \max\{\frac{1}{\alpha\lambda_1^\beta}, \frac{1}{\alpha\lambda_1^{1-\beta}}\}),$$
$$K_2 = M_\beta \times [\delta\Gamma_1\{\frac{1}{\bar{\beta}(\delta-\vartheta)} + \frac{\Gamma(\bar{\beta})}{\delta}\} + (\delta e)^{-(1-\beta)}(1-\beta)^{-\beta}\Gamma(\bar{\beta}) + \frac{\Gamma_2}{\bar{\beta}}],$$
$$K_3 = K_2 K_0$$

where
$$\Gamma_1 = \int_0^\infty s^{-\beta}e^{-\delta s}ds, \quad \Gamma_2 = \int_0^\infty s^{-\beta}e^{-\vartheta s}ds.$$

3. Quasi-periodic Solutions

Next, let $w(t)$ be a quasi-periodic function, defined by

$$w(t) = z(t, \tau_1 t, \tau_2 t) \tag{3.1}$$

where $z : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to Y$ is 1-periodic for each variable and a locally Hölder continuous function: there exist positive constants κ, σ, which satisfy

$$\begin{aligned} |z(t_1, t_2, t_3) - z(t_1', t_2', t_3')|_0 \le \kappa\{|t_1 - t_1'|^\sigma + |t_2 - t_2'|^\sigma + |t_3 - t_3'|^\sigma\}, \\ \forall t_i, t_i' : |t_i - t_i'| \le \varepsilon_0, \quad i = 1, 2, 3 \end{aligned} \tag{3.2}$$

for sufficiently small $\varepsilon_0 > 0$. As in Thorem 2.1 we also use the pair of functions $[\xi(t), \eta(t)] \in Y_1 \times Y_{1+\beta}$. Define the norm $\|[x, y]\|_{1,\beta}$ by

$$\|[x, y]\|_{1,\beta} = |Ax|_0 + |A^- y|_\beta$$

which is equivalent to the $Y_1 \times Y_{1+\beta}$ norm. Then we can show the quasi-periodicity of $[u(t), \dot{u}(t)]$ in $Y_1 \times Y_\beta$ by estimating $\|[\xi(t), \eta(t)]\|_{1,\beta}$.

Here we use the ECM sequence $[T_j]$ of $\{\tau_1, \tau_2\}$. We say that a function $y : \mathbf{R} \to Y$ has a quasi-periodic property (abr. q.p.p.) with its irrational frequencies (abr. i.f.) $\{\tau_1, \tau_2\}$ if there exists a sequence $\{\varepsilon_j\} : \lim_{j\to\infty} \varepsilon_j = 0$, which satisfies

$$|y(\cdot + T_j) - y(\cdot)|_\infty < \varepsilon_j, \quad \forall j$$

for the ECM sequence $[T_j]$ of $\{\tau_1, \tau_2\}$.

We can show that the attractor of system (2.1) has q.p.p. with i.f. $\{\tau_1, \tau_2\}$.

Theorem 3.1. *Let $w : \mathbf{R} \to Y$ be a quasi-periodic function, which satisfies* (3.1) *and* (3.2) *and assume that the same Hypotheses as Theorem 2.1. For a given constant $d > 0$:*

$$d > K_2|w|_\infty + K_3, \tag{3.3}$$

assume that

$$\delta > \vartheta' := (\frac{M_\beta k(d)\Gamma(\bar\beta)}{2\lambda(\beta)})^{1/\bar\beta} \tag{3.4}$$

where $k(\cdot)$ is the locally Lipschitz coefficient in (2.3).

Then there exists a unique solution $[\xi_\infty(\cdot), \eta_\infty(\cdot)] \in BC(\mathbf{R}, Y_1 \times Y_{1+\beta})$, which satisfies

$$\xi_\infty(t) = S_1(t)\xi_\infty(0) + \int_0^t S_1(t-s)b^{-1}(A)[F(\frac{\xi_\infty+\eta_\infty}{2}(s))+w(s)]ds, \tag{3.5}$$

$$\eta_\infty(t) = S_2(t)\eta_\infty(0) - \int_0^t S_2(t-s)b^{-1}(A)[F(\frac{\xi_\infty+\eta_\infty}{2}(s))+w(s)]ds. \tag{3.6}$$

Furthermore, the solution $[\xi_\infty(t), \eta_\infty(t)]$ has the same quasi-periodic property as the q.p. perturbation w in the following sense:

$$\sup_{t\in\mathbf{R}} \|[\xi_\infty(t+T_j), \eta_\infty(t+T_j)] - [\xi_\infty(t), \eta_\infty(t)]\|_{1,\beta} \le \varepsilon_j, \quad \forall j,$$

for the ECM sequence $[T_j]$ of $\{\tau_1, \tau_2\}$ and also, for the solution $[\xi(t), \eta(t)]$ with its initial deta $[\xi_0, \eta_0] \in Y \times Y^{1+\beta}$ we have

$$\|[\xi(t), \eta(t)] - [\xi_\infty(t), \eta_\infty(t)]\|_{1,\beta} \to 0 \ \text{ as } t \to \infty.$$

Proof. It follows from Theorem 2.1 that there exists a solution $[\xi, \eta]$ for an initial condition $[\xi_0, \eta_0] \in Y_1 \times Y_{1+\beta}$: $\|[\xi(t), \eta(t)]\|_{1,\beta} \le d,\ t \ge 0$. Then we show that $[\xi_j(t), \eta_j(t)] := [\xi(t+T_j), \eta(t+T_j)]$ converges to a q.p.p. type solution $[\xi_\infty(t), \eta_\infty(t)]$ as $j \to \infty$. Considering the estimate

$$
\begin{aligned}
&|A(\xi(t+T_j) - \xi(t+T_k))|_0 + |A^-(\eta(t+T_j) - \eta(t+T_k))|_\beta \\
&\le |A(\xi(t+T_j) - \xi(t+T_j+T_k))|_0 + |A(\xi(t+T_j+T_k) - \xi(t+T_k))|_0 \\
&+|A^-(\eta(t+T_j) - \eta(t+T_j+T_k))|_\beta + |A^-(\eta(t+T_j+T_k) - \eta(t+T_k))|_\beta
\end{aligned}
$$

and using (2.8) and (2.9), we have

$$
\begin{aligned}
&|A(\xi(t+T_j) - \xi(t+T_j+T_k))|_0 + |A^-(\eta(t+T_j) - \eta(t+T_j+T_k))|_\beta \\
&\le |S_1(T_j)A(\xi(t) - \xi(t+T_k))|_0 \\
&+ \int_0^{T_j} |AS_1(T_j - s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t)) - F(\frac{\xi+\eta}{2}(s+t+T_k))]|_0 ds \\
&+ \int_0^{T_j} |AS_1(T_j - s)b^{-1}(A)[w(s+t) - w(s+t+T_k)]|_0 ds \\
&+|S_2(T_j)A^-(\eta(t) - \eta(t+T_k))|_\beta \\
&+ \int_0^{T_j} |A^-A^\beta S_2(T_j - s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t)) \\
&-F(\frac{\xi+\eta}{2}(s+t+T_k))]|_0 ds \\
&+ \int_0^{T_j} |A^-A^\beta S_2(T_j - s)b^{-1}(A)[w(s+t) - w(s+t+T_k)]|_0 ds
\end{aligned}
$$

and

$$
\begin{aligned}
&|A(\xi(t+T_j+T_k) - \xi(t+T_k))|_0 + |A^-(\eta(t+T_j+T_k) - \eta(t+T_k))|_\beta \\
&\le |S_1(T_k)A(\xi(t) - \xi(t+T_j))|_0 \\
&+ \int_0^{T_k} |AS_1(T_k - s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t)) - F(\frac{\xi+\eta}{2}(s+t+T_j))]|_0 ds \\
&+ \int_0^{T_k} |AS_1(T_k - s)b^{-1}(A)[w(s+t) - w(s+t+T_j)]|_0 ds \\
&+|S_2(T_k)A^-(\eta(t) - \eta(t+T_j))|_\beta
\end{aligned}
$$

$$+\int_0^{T_k} |A^- A^\beta S_2(T_k - s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t))$$

$$-F(\frac{\xi+\eta}{2}(s+t+T_j))]|_0 ds$$

$$+\int_0^{T_k} |A^- A^\beta S_2(T_k - s)b^{-1}(A)[w(s+t) - w(s+t+T_j)]|_0 ds.$$

Thus, applying the argument in [3], we have

$$|A(\xi(t+T_j) - \xi(t+T_j+T_k))|_0 + |A^-(\eta(t+T_j) - \eta(t+T_j+T_k))|_\beta$$

$$\leq e^{-\alpha\lambda_1 T_j}(|A(\xi(t) - \xi(t+T_k))|_0 + |A^-(\eta(t) - \eta(t+T_k))|_\beta)$$

$$+\int_0^{T_j} M_\beta \frac{k(d)}{2\lambda(\beta)} e^{-\delta(T_j - s)}(T_j - s)^{-\beta}$$

$$\times\{|A(\xi(s+t) - \xi(s+t+T_k))|_0 + |A^-(\eta(s+t) - \eta(s+t+T_k))|_\beta\}$$

$$+\int_0^{T_j} M_\beta e^{-\delta(T_j - s)}(T_j - s)^{-\beta}|w(s+t) - w(s+t+T_k)|_0 ds$$

and we have

$$|A(\xi(t+T_j+T_k) - \xi(t+T_k))|_0 + |A^-(\eta(t+T_j+T_k) - \eta(t+T_k))|_\beta$$

$$\leq e^{-\alpha\lambda_1 T_k}(|A(\xi(t) - \xi(t+T_j))|_0 + |A^-(\eta(t) - \eta(t+T_j))|_\beta)$$

$$+\int_0^{T_k} M_\beta \frac{k(d)}{2\lambda(\beta)} e^{-\delta(T_k - s)}(T_k - s)^{-\beta}$$

$$\times\{|A(\xi(s+t) - \xi(s+t+T_j))|_0 + |A^-(\eta(s+t) - \eta(s+t+T_j))|_\beta\}$$

$$+\int_0^{T_k} M_\beta e^{-\delta(T_k - s)}(T_k - s)^{-\beta}|w(s+t) - w(s+t+T_j)|_0 ds.$$

Note that there exist integers N_j, N_k, R_j, R_k such that these values

$$|\tau_1 T_j - N_j|,\ |\tau_1 T_k - N_k|,\ |\tau_2 T_j - R_j|,\ |\tau_2 T_k - R_k|$$

are sufficiently small as in the proof of Theorem 1.2. Thus by using the Gronwall's inequality, which was introduced in Appendix of [3], and (3.2) we have

$$|A(\xi(t+T_j) - \xi(t+T_k))|_0 + |A^-(\eta(t+T_j) - \eta(t+T_k))|_\beta$$

$$\leq [\frac{e^{-(\delta-\vartheta')T_j}}{\bar{\beta}} + \Gamma(\bar{\beta})e^{-\alpha\lambda_1 T_j}](|A(\xi(t) - \xi(t+T_k))|_0$$

$$+|A^-(\eta(t) - \eta(t+T_k))|_\beta)$$

$$+\gamma(\beta)M_\beta\kappa[|\tau_1T_k-N_k|^\sigma+|\tau_2T_k-R_k|^\sigma]$$
$$+[\frac{e^{-(\delta-\vartheta')T_k}}{\bar{\beta}}+\Gamma(\bar{\beta})e^{-\alpha\lambda_1T_k}](|A(\xi(t)-\xi(t+T_j))|_0$$
$$+|A^-(\eta(t)-\eta(t+T_j))|_\beta)$$
$$+\gamma(\beta)M_\beta\kappa[|\tau_1T_j-N_j|^\sigma+|\tau_2T_j-R_j|^\sigma]$$

where
$$\gamma(\beta)=\delta\Gamma_1\{\frac{1}{\bar{\beta}(\delta-\vartheta)}+\frac{\Gamma(\bar{\beta})}{\delta}\}+(\delta e)^{-\bar{\beta}}\bar{\beta}^{-\beta}\Gamma(\bar{\beta})+\frac{\Gamma_2}{\bar{\beta}}.$$

It follows from (2.18) and Theorem 2.1 that the sequence $[\xi_j(t),\eta_j(t)]$ is a Cauchy sequence in $BC([0,+\infty):Y_1\times Y_{1+\beta})$, the space of uniformly bounded, continuous, and $Y_1\times Y_{1+\beta}$ valued functions. Thus there exists $[\xi_\infty(t),\eta_\infty(t)]$ such that, as $j\to\infty$,
$$[\xi_j,\eta_j]\to[\xi_\infty,\eta_\infty] \text{ in } BC([0,+\infty):Y_1\times Y_{1+\beta}).$$
By taking the limit $j\to\infty$ of the mild fomulas:
$$\xi(t+T_j)=S_1(t)\xi(T_j)+\int_0^t S_1(t-s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+T_j))+w(s+T_j)]ds,$$
$$\eta(t+T_j)=S_2(t)\eta(T_j)-\int_0^t S_2(t-s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+T_j))+w(s+T_j)]ds$$
we can show that $[\xi_\infty,\eta_\infty]$ satisfies the mild formulas (3.5), (3.6) with the initial state $[\xi_\infty(0),\eta_\infty(0)]$.

Furthermore, since we can extend the interval from $[0,+\infty)$ to $[s,+\infty)$ for every $s\in\mathbf{R}$ with showing the uniformly boundedness, not depending on s, of solutions, we can admit the solution $[\xi_\infty(t),\eta_\infty(t)]$ in $BC(\mathbf{R}:Y_1\times Y_{1+\beta})$. Applying the same argument and estimate as above, we can easily obtain the uniqueness of the solution.

Finally, we show the q.p.p. of $[\xi_\infty(t),\eta_\infty(t)]$ in $Y_1\times Y_{1+\beta}$ by using the previous estimate. It follows from (2.8) and (2.9) that we have
$$|A(\xi_\infty(t)-\xi_\infty(t+T_l))|_0+|A^-(\eta_\infty(t)-\eta_\infty(t+T_l))|_\beta$$
$$=\lim_{j\to\infty}[|A(\xi(t+T_j)-\xi(t+T_l+T_j))|_0+|A^-(\eta(t+T_j)-\eta(t+T_l+T_j))|_\beta]$$
$$\le\lim_{j\to\infty}[|S_1(T_j)A(\xi(t)-\xi(t+T_l))|_0$$
$$+\int_0^{T_j}|AS_1(T_j-s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t))-F(\frac{\xi+\eta}{2}(s+t+T_l))]|_0ds$$

$$+\int_0^{T_j} |AS_1(T_j-s)b^{-1}(A)|w(s+t)-w(s+t+T_l)|_0 ds$$

$$+|S_2(T_j)A^-(\eta(t)-\eta(t+T_l))|_\beta$$

$$+\int_0^{T_j} |A^-A^\beta S_2(T_j-s)b^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t))$$

$$-F(\frac{\xi+\eta}{2}(s+t+T_l))]|_0 ds$$

$$+\int_0^{T_j} |A^-A^\beta S_2(T_j-s)b^{-1}(A)|w(s+t)-w(s+t+T_l)|_0 ds].$$

By applying the pervious argument we can take the integers N_l, R_l, which satisify

$$|A(\xi_\infty(t)-\xi_\infty(t+T_l))|_0+|A^-(\eta_\infty(t)-\eta_\infty(t+T_l))|_\beta$$

$$\le \lim_{j\to\infty}[\frac{e^{-(\delta-\vartheta')T_j}}{\bar\beta}+\Gamma(\bar\beta)e^{-\alpha\lambda_1 T_j}](|A(\xi(t)$$

$$-\xi(t+T_l))|_0+|A^-(\eta(t)-\eta(t+T_l))|_\beta)$$

$$+\gamma(\beta)M_\beta\kappa[|\tau_1T_l-N_l|^\sigma+|\tau_2T_l-R_l|^\sigma]$$

$$\le \gamma(\beta)M_\beta\kappa[|\tau_1T_l-N_l|^\sigma+|\tau_2T_l-R_l|^\sigma].$$

Thus we obtain the q.p.p. of $[\xi_\infty(t),\eta_\infty(t)]$.

Next we estimate the recurrent dimensions of the orbit given by

$$\Sigma=\{[u_\infty(n),\dot u_\infty(n)]\in Y_1\times Y_\beta : n\in \mathbf{N}_0\}.$$

By applying Lemma 3.1-3.3 in [3] we can estimate

$$\begin{aligned}|u_\infty(t)-u_\infty(t')|_1 &\le \frac{1}{2}(|\xi_\infty(t)-\xi_\infty(t')|_1+|\eta_\infty(t)-\eta_\infty(t')|_1)\\ &\le \frac{1}{2}(|A(\xi_\infty(t)-\xi_\infty(t'))|_0+|A^-(\eta_\infty(t)-\eta_\infty(t'))|_\beta)\end{aligned}$$

and

$$\begin{aligned}|\dot u_\infty(t)-\dot u_\infty(t')|_\beta &\le \frac{1}{2}(|A^+(\xi_\infty(t)-\xi_\infty(t'))|_\beta+|A^-(\eta_\infty(t)-\eta_\infty(t'))|_\beta)\\ &\le \frac{1}{2\alpha\lambda_1^{\bar\beta}}|A(\xi_\infty(t)-\xi_\infty(t'))|_0+\frac{1}{2}|A^-(\eta_\infty(t)-\eta_\infty(t'))|_\beta.\end{aligned}$$

Thus we can show the q.p.p. of $[u_\infty(t), \dot{u}_\infty(t)]$.

By applying the estimate to show the q.p.p. of $[\xi_\infty(t), \eta_\infty(t)]$ we have

$$\begin{aligned}&|A(\xi_\infty(n) - \xi_\infty(n+T_l))|_0 + |A^-(\eta_\infty(n) - \eta_\infty(n+T_l))|_\beta \\ &\le \gamma(\beta) M_\beta \kappa[|\tau_1 T_l - N_l|^\sigma + |\tau_2 T_l - R_l|^\sigma],\end{aligned}$$

which also gives the similar estimate for $[u_\infty(n), \dot{u}_\infty(n)]$. By applying the argument in the proof of Theorem 1.2 we obtain the following Theorem.

Theorem 3.2. *Under the same Hypotheses as those of Theorem 3.1, let* $\{\tau_1, \tau_2\}$ *be* **(KL)** *class irrational numbers and satisfies the* δ_0*-ECM condition. Then we have*

$$\underline{D}_r(\Sigma) \le \underline{d}_r(\Sigma) \le \frac{\log E_2}{(1-\delta_0)\sigma \log E_1}. \tag{3.7}$$

4. Flexible Beams

Following the P.D.E. example in [3], we consider the equation of motion of slender and flexible structures with internal viscous damping and nonlinear forcing, determined by displacement $u(x,t)$ and bending force $u_{xx}(x,t)$,under a quasi-periodic perturbation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2\alpha \frac{\partial^5 u(x,t)}{\partial x^4 \partial t} + \frac{\partial^4 u(x,t)}{\partial x^4} = f(x, u(x,t), \frac{\partial^2 u(x,t)}{\partial x^2}) + w(x,t), \tag{4.1}$$

where $0 < x < L$. The beam is clamped at one end, $x = 0$, and at the free end, $x = L$, the bending moment and the shearing force vanish. The the boundary and the initial conditions are given by

$$\begin{aligned}&u(0,t) = u_x(0,t) = 0,\\ &u_{xx}(L,t) + 2\alpha u_{txx}(L,t) = 0,\\ &u_{xxx}(L,t) + 2\alpha u_{txxx}(L,t) = 0,\\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).\end{aligned}$$

and the quasi-periodic perturbations $w(t)$ satisfies (3.1), (3.2).

We define an operator A in $L^2(0,L)$ by

$$\begin{aligned}D(A) &= \{u \in H^4(0,L) : u(0) = u_x(0) = 0, \quad u_{xx}(L) = u_{xxx}(L) = 0\},\\ Au &= \frac{\partial^4 u}{\partial x^4}.\end{aligned}$$

Let γ_i be the solutions of

$$\cosh\gamma\cos\gamma + 1 = 0$$

such that $0 < \gamma_1 < \gamma_2 < \cdots$. Then the e.v. of A are given by

$$\lambda_i = (\frac{\gamma_i}{L})^4, \quad i = 1, 2, ...$$

We assume that the nonlinear function
$f(x, u, v) : \mathbf{R}^3 \to \mathbf{R}$ satisfies the growth condition

$$|f(x, u, v)| \leq k_0(|u| + |v|) \ \text{ for some } k_0 > 0$$

and the following Lipschitz and locally Lipschitz continuity:
$\exists k_0(c), k > 0$ such that

$$|f(x, u, v) - f(x, u', v')| \leq k_0(c)|u - u'| + k|v - v'|$$

$$\text{for } |u|, |u'| \leq c, \ v, v' \in \mathbf{R}.$$

Define $F : D(A^{1/2}) \to L^2(0, L)$ by

$$F(u)(x) = f(x, u(x), u_{xx}(x)),$$

then the condition (2.3) and (2.4) holds for the constant $\beta = 1/2$.

Following the argument in [3], we can see that the inequality conditions in Theorem 2.1 and Theorem 3.1 are satisfied for small $\alpha > 0$ as follows:
If $1/\alpha^2 - \lambda_h, \lambda_{h+1} - 1/\alpha^2 \simeq 1/\alpha^2$, then the stability inequality conditions are given by

$$C(\frac{K_0}{\alpha^2})^{\frac{2}{3}} < \lambda_1 < \frac{1}{2\alpha^2}$$

for some $C > 0$.

Remark 4.1. As another example for (2.1) we can consider a strongly damped wave equation:

$$u_{tt} - 2\alpha\Delta u_t - \Delta u = f(x, u, \nabla u) + w(t).$$

References

1. C. M. Dafermos, Almost periodic processes and almost periodic solutions of evolution equations, in: *Dynamical Systems, Proc. Univ. Florida Inter. Symp.* (A. R. Bednarek and L. Cesari, Eds.), 43-57, Academic Press, New York, 1977.
2. Y. A. Khinchin, *Continued Fractions*, the University of Chicago Press 1964. 28 # 5037.

3. K. Naito, Periodic stability of nonlinear flexible systems with damping, *SIAM J. Control and Optimization*, **33** (1995), 1778-1800.

4. ______, Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation, *J. Differential Equations*, **141** (1997), 179-200.

5. ______, Correlation dimensions of quasi-periodic orbits with frequencies given by quasi Roth numbers, *J. Korean Math. Soc.*, **37** (2000), 857-870.

6. ______, Recurrent Dimensions of Quasi-Periodic Solutions for nonlinear evolution equations, *Trans. Amer. Math. Soc.*, **354(3)** (2002), 1137-1151.

7. ______, Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations II: Gaps of dimensions and Diophantine conditions, *Discrete and Continuous Dynamical Systems*, **11(2-3)** (2004), 449-488.

8. ______, Classifications of irrational numbers and recurrent dimensions of quasi-periodic orbits, *J. Nonlinear and Convex Analysis*, **5** (2004), 169-185.

9. Y. Sakawa, Feedback control of second-order evolution equations with damping, *SIAM J. Control Optim.*, **22(3)** (1984), 343-361.

10. W. M. Schmidt, *Diophantine Approximation*, Springer Lecture Notes in Math. 785, 1980.

Koichiro Naito
Department of Mathematics and Engineering,
Kumamoto University,
Kurokami 2-39-1,
Kumamoto 860-8555,
Japan
E-mail: knaito@gpo.kumamoto-u.ac.jp