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BOLZA TYPE PROBLEMS IN DISCRETE TIME

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. In this paper we study a discrete time version of optimization problems of Bolza type. The functionals are assumed to be merely lower semicontinuous. We obtain optimality conditions which are always necessary and which are also sufficient in the convex case whenever the given problem satisfies a qualification condition.

1. Introduction

The general problem of Bolza in the Calculus of Variations (see, e.g., [6, 13, 18, 22]) can be formulated in Nonsmooth Analysis as the minimization of the functional

$$I(x) = l(x(t_0), x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt \tag{1.1}$$

over the space of all absolutely continuous arcs $x : [t_0, t_1] \to \mathbb{R}^n$. This general formulation has the advantage to incorporate the equality and inequality constraints relative to the initial/end point pair $(x(t_0), x(t_1))$ and the triple $(t, x(t), \dot{x}(t))$ since the functions l and $L(t, \cdot, \cdot)$ are extended real valued, that is, they are allowed to take the value $+\infty$. The model even permits to include nonsmooth set constraint and set-valued constraint.

In the corresponding discrete time problem, one considers in place of an arc $x : [t_0, t_1] \to \mathbb{R}^n$ a vector $x = (x_0, x_1, \cdots, x_T) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n = (\mathbb{R}^n)^{T+1}$ and in place of $\dot{x} = \frac{dx}{dt}$ the difference $\Delta x_t = x_t - x_{t-1}$ for $t = 1, \cdots, T$. The associated

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problem $(\mathcal{P}(l,L))$ takes then the form: Minimize over all $x=(x_0,x_1,\ldots,x_T)\in(\mathbb{R}^n)^{T+1}$ the function

$$\varphi(x):=l(x_0,x_T)+\sum_{t=1}^{T}L_t(x_{t-1},\Delta x_t),$$

where l and L_t for all $t=1,\cdots,T$ are functions from $\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}\cup\{+\infty\}$ which are *proper*, that is, none of which is identically $+\infty$. Throughout, unless otherwise stated, we assume that these functions are *lower semicontinuous* (lsc, for short) or locally Lipschitzian. Then φ, too, is lsc with values in $\mathbb{R}\cup\{+\infty\}$. As for the Bolza problem above in calculus of variations, it is important to observe the fact that in $(\mathcal{P}(l,L))$ the constraints are implicit in the inequality $\varphi(x)<\infty$, because only vectors x satisfying $\varphi(x)<+\infty$ are of interest in the minimization. Throughout, we assume that φ is proper, that is, there exists some $z\in(\mathbb{R}^n)^{T+1}$ such that $l(z_0,z_T)<+\infty$ and $L_t(z_{t-1},\Delta z_t)<+\infty$ for all $t=1,\cdots,T$. Letting

$$C=\{(u,v)\in\mathbb{R}^n\times\mathbb{R}^n\mid l(u,v)<\infty\}\tag{1.2}$$

(that is, C is the effective domain $\operatorname{dom} l$ of l) and

$$F_t(u)=\{v\in\mathbb{R}^n\mid L_t(u,v)<\infty\},\tag{1.3}$$

it is emphasized in [20] that, without loss of generality, one can restrict attention in $(\mathcal{P}(l,L))$ to minimizing $\varphi(x)$ over the set of all $x\in(\mathbb{R}^n)^{T+1}$ which satisfy

$$(x_0,x_T)\in C\quad\text{and}\quad\Delta x_t\in F_t(x_{t-1})\;\forall\,t=1,\cdots,T.\tag{1.4}$$

Implicit in the dynamical constraint $\Delta x_t\in F_t(x_{t-1})$ is the state constraint $x_{t-1}\in Z_t$ for $t=1,\cdots,T$, where $Z_t=\{z\in\mathbb{R}^n|F_t(z)\neq\emptyset\}$.

Conversely, given finite valued functions l and L_t, set constraint $C\subset\mathbb{R}^n\times\mathbb{R}^n$ and set-valued mapping constraints $F_t:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$, it is also of interest to study the minimization problem $(\mathcal{P}_{C,F}(l,L))$ consisting in minimizing the function $\varphi(x)$ above over all the vectors $x\in(\mathbb{R}^n)^{T+1}$ satisfying the constraints in (1.4). At a first step, this problem may be translated in the form of problem $(\mathcal{P}(l,L))$ by putting on the one hand $\tilde{l}(u,v)=l(u,v)$ if $(u,v)\in C$ and $\tilde{l}(u,v)=+\infty$ otherwise and on the other hand $\tilde{L}_t(u,v)=L_t(u,v)$ if $v\in F_t(u)$ and $\tilde{L}_t(u,v)=+\infty$ otherwise.

The discrete Bolza type problem $(\mathcal{P}(l,L))$ has been introduced and largely studied in the *convex* setting by Rockafellar and Wets [20] and they also considered the stochastic version *in discrete time* where the decision at any time t is required to depend only on past random events. This stochastic discrete Bolza type problem is related to stochastic problems with recourse. Other results of interest concerning stochastic optimization problems with recourse can be found, e.g., in [19, 10, 4] and the references therein. Results concerning the discrete Bolza problem in the form $(\mathcal{P}_{C,F}(l, L))$, with l and L locally Lipschitzian, have been also provided in Mordukhovich [13][Theorem 6.17] (see also [12]). The study of such problems by Mordukhovich was the first step of his efficient approach to derive necessary optimality conditions for optimal control problems after making appropriate discretizations of the control problems. It is also worth mentioning that, among domains of its own interest, the problem $(\mathcal{P}_{C,F}(l, L))$ contains as particular case the modelization of various economic dynamics (see, e.g., [8, 11]). In the present paper we focus our attention to the discrete problem without any convexity assumption. We first establish general necessary optimality conditions for the discrete above problem $(\mathcal{P}(l, L))$. Optimality conditions for the problem $(\mathcal{P}_{C,F}(l, L))$ is then derived when the functions l and L_t are locally Lipschitzian around the candidate point through some relative qualification condition. A particular attention is paid to the case when the images of the set-valued mappings F_t are prox-regular. The corresponding optimality conditions of [20] in the convex setting are also derived.

2. Definitions and Preliminaries

In the next section, although our necessary optimality conditions could be given with the use of *many types of subdifferentials*, we will limit ourselves to state and establish them with the basic limiting subdifferential.

Recall first that for a proper lsc function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $u \in \operatorname{dom} f$ the Fréchet subdifferential $\widehat{\partial} f(u)$ is defined by the fact that a vector $v \in \widehat{\partial} f(u)$ when for any positive number ε there exists some positive number η such that one has

$$\langle v, u' - u\rangle \leq f(u') - f(u) + \varepsilon\|u' - u\| \quad \text{for all } u' \in B(u, \eta),$$

where $B(u, \eta)$ denotes the open ball with radius η centered at the point u. One puts in general $\widehat{\partial} f(u) = \emptyset$ when $f(u)$ is not finite.

When f is the indicator function δ_S of a closed subset $S \subset \mathbb{R}^n$, that is, $\delta_S(u) = 0$ if $u \in S$ and $\delta_S(u) = +\infty$ otherwise, its Fréchet subdifferential at a point $u \in S$ is a cone. It is generally called the Fréchet normal cone to S at u and one denotes either $\widehat{N}_S(u)$ or $\widehat{N}(S, u)$.

Since the Fréchet subdifferential enjoys only fuzzy calculus rules (see, e.g., [13] for more details), one considers a limiting process of such subdifferentials yielding to the so-called limiting (basic) subdifferential. A vector v is in the limiting subdifferential $\partial f(u)$ at a point $u \in \operatorname{dom} f$ when there exists a sequence $(u_k, f(u_k))$ converging to $(u, f(u))$ and vectors $v_k \in \widehat{\partial} f(u_k)$ with $v_k \to v$. As above, one sets $\partial f(u) = \emptyset$ if $u \notin \operatorname{dom} f$. The set $\partial f(u)$ is nonconvex in general but it enjoys full pointbased calculus rules. For example, if $g : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function one has (see [13, 21]) the inclusion

$$\partial(f+g)(u) \subset \partial f(u) + \partial g(u), \tag{2.1}$$

where the addition in the second member is taken in the usual Minkowski sense, that is, $\partial f(u) + \partial g(u) := \{v + v' \mid v \in \partial f(u),\, v' \in \partial g(u)\}$.

The inclusion (2.1) can be also obtained under a much weaker condition than the local Lipschitz property of one of the functions f and g. To see that, let us recall the concept of singular limiting subdifferential. Modifying slightly the definition above, we say that a vector v belongs to the singular limiting subdifferential $\partial^\infty f(u)$ at a point $u \in \operatorname{dom} f$ when there exists a sequence $(u_k, f(u_k))$ converging to $(u, f(u))$, positive numbers $\lambda_k \downarrow 0$ and vectors $v_k \in \widehat{\partial} f(u_k)$ such that $\lambda_k v_k \to v$. So, if for two lsc functions f, g the qualification condition $\partial^\infty f(u) \cap -\partial^\infty g(u) = \{0\}$ holds, then one has (see [13, 21]) $\partial(f+g)(u) \subset \partial f(u) + \partial g(u)$. This qualification condition can be translated (see [13, 21]) in the case of any finite number of lsc functions: for a finite number of lsc functions f_i, $i = 0, 1, \cdots, m$, and for $u \in \cap_{i=0}^m \operatorname{dom} f_i$ one has

$$\partial(\sum_{i=0}^m f_i)(u) \subset \sum_{i=0}^m \partial f_i(u) \tag{2.2}$$

whenever for any $y_i \in \partial^\infty f_i(u)$ with $\sum_{i=0}^m y_i = 0$ one necessarily has $y_0 = y_1 = \cdots = y_m = 0$. The inclusion (2.1) is a particular case of (2.2) since

$$\partial^\infty g(u) = \{0\} \quad \text{whenever } g \text{ Lipschitz near u.} \tag{2.3}$$

The same qualification condition above also gives (see [13, 21])

$$\partial^\infty(\sum_{i=0}^m f_i)(u) \subset \sum_{i=0}^m \partial^\infty f_i(u). \tag{2.4}$$

Concerning the composition operation, we will recall the result with the composition with a linear mapping. If $A : \mathbb{R}^m \to \mathbb{R}^n$ is a linear *surjective* mapping, then see [13, 21]

$$\partial(f \circ A)(u) \subset A^* \partial f(Au) \text{ and } \partial^\infty(f \circ A)(u) \subset A^* \partial^\infty f(Au), \tag{2.5}$$

where A^* denotes the adjoint of A and $A^* \partial f(Au) := \{A^* v \mid v \in \partial f(Au)\}$.

As for the Fréchet normal cone (see above), the limiting normal cone to a closed subset S at $u \in S$ is defined through its indicator function by $N_S(u) := \partial \delta_S(u)$. Sometimes one write $N(S, u)$ in place of $N_S(u)$. The connexion with the singular subdifferential is provided by the equalities

$$\partial^\infty \delta_S(u) = \partial \delta_S(u) = N(S, u).$$

Of course, when the point u is a minimum point for the function f one has both $0 \in \widehat{\partial} f(u)$ and $0 \in \partial f(u)$, the first inclusion being obvious under the minimum point assumption and the second one being a consequence of the fact that one always has $\widehat{\partial} f \subset \partial f$. Further, when f is convex, the Fréchet subdifferential and the limiting subdifferential coincide with the usual Fenchel subdifferential of Convex Analysis.

In the next section, we will just say subdifferential of f and normal cone to S in place of limiting subdifferential of f and limiting normal cone to S.

3. Necessary Optimality Conditions

The following theorem states the first result of the paper. Here the functions l and L_t are neither convex nor locally Lipschitzian.

Theorem 3.1. *Let $\bar{x} \in (\mathbb{R}^n)^{T+1}$ be a solution of problem $(\mathcal{P}(l, L))$. Assume that l and L_t are proper and lsc for all $t = 1, \cdots, T$ and that the following qualification condition $Q(\bar{x})$ holds:*

$$\begin{cases} \text{the only vector } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{ for which} \\ (y_0, -y_T) \in \partial^\infty l(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta y_t, y_t) \in \partial^\infty L_t(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T \\ \text{is the zero vector in } (\mathbb{R}^n)^{T+1}. \end{cases}$$

Then there exists some vector $p = (p_0, \cdots, p_T) \in (\mathbb{R}^n)^{T+1}$ such that:
a) $(p_0, -p_T) \in \partial l(\bar{x}_0, \bar{x}_T)$
b) $(\Delta p_t, p_t) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$.

Proof.

Step 1. Consider the function $\varphi : (\mathbb{R}^n)^{T+1} \longrightarrow \mathbb{R} \cup \{+\infty\}$

$$x \mapsto \varphi(x) := l(x_0, x_T) + \sum_{t=1}^{T} L_t(x_{t-1}, \Delta x_t)$$

and put $\varphi_0(x) := l(x_0, x_T) = (l \circ A_0)(x)$ and

$$\varphi_t(x) := L_t(x_{t-1}, \Delta x_t) = (L_t \circ A_t)(x) \quad \text{for } t = 1, \cdots, T,$$

where $A_0, A_t : (\mathbb{R}^n)^{T+1} \longrightarrow (\mathbb{R}^n)^2$ are linear mappings defined by

$$A_0 x := (x_0, x_T) \text{ and } A_t x := (x_{t-1}, \Delta x_t) \text{ for all } t = 1, \cdots, T.$$

As $\bar{x}$ is a solution of the minimization problem $(\mathcal{P}(l, L))$ we have

$$0 \in \partial\varphi(\bar{x}) = \partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x}).$$

Step 2. Let us show that a corresponding qualification condition holds for the functions $\varphi_0, \varphi_1, \ldots, \varphi_T$.
In fact we are going to prove that for each $y = (y_0, \ldots, y_T) \in (\mathbb{R}^n)^{T+1}$ for which

$$\sum_{t=0}^{T} y_t = 0 \text{ with } y_t \in \partial^\infty \varphi_t(\bar{x}) \text{ for all } t = 0, \cdots, T,$$

then we necessarily have $y = 0$.

Indeed fix any such y. As the linear mappings A_t are surjective for $t = 0, \cdots, T$, then according to (2.5)

$$y_0 \in \partial^\infty \varphi_0(\bar{x}) = \partial^\infty (l \circ A_0)(\bar{x}) \subset A_0^* \partial^\infty l(A_0 \bar{x})$$

and

$$y_t \in \partial^\infty \varphi_t(\bar{x}) = \partial^\infty (L_t \circ A_t)(\bar{x}) \subset A_t^* \partial^\infty L_t(A_t \bar{x}) \quad \forall\, t = 1, \cdots, T,$$

which gives the existence of some

$$z_0 = (z_0^1, z_0^2) \in \partial^\infty l(\bar{x}_0, \bar{x}_T) \text{ such that } y_0 = A_0^* z_0 \tag{3.1}$$

and some

$$z_t = (z_t^1, z_t^2) \in \partial^\infty L_t(\bar{x}_{t-1}, \Delta \bar{x}_t) \text{ such that } y_t = A_t^* z_t \quad \text{for } t = 1, \cdots, T. \tag{3.2}$$

Now we must calculate A_0^* and A_t^* for $t = 1, \cdots, T$.
We have $A_t^* : (\mathbb{R}^n)^2 \longrightarrow (\mathbb{R}^n)^{T+1},\ t = 1, \cdots, T$ and

$$\begin{aligned} \langle A_0^*(z_1, z_2), h \rangle_{(\mathbb{R}^n)^{T+1}} &= \langle (z_1, z_2), A_0 h \rangle_{(\mathbb{R}^n)^2} = \langle (z_1, z_2), (h_0, h_T) \rangle_{(\mathbb{R}^n)^2} \\ &= \langle (z_1, 0, \ldots, 0, z_2), (h_0, h_1, \cdots, h_T) \rangle_{(\mathbb{R}^n)^{T+1}} . \end{aligned}$$

Then

$$A_0^*(z_1, z_2) = (z_1, 0, \ldots, 0, z_2) \quad \text{for all } (z_1, z_2) \in (\mathbb{R}^n)^2.$$

In the same way, for A_1^* we have

$$\begin{aligned} \langle A_1^*(z_1, z_2), h \rangle_{(\mathbb{R}^n)^{T+1}} &= \langle (z_1, z_2), A_1 h \rangle_{(\mathbb{R}^n)^2} = \langle (z_1, z_2), (h_0, h_1 - h_0) \rangle_{(\mathbb{R}^n)^2} \\ &= \langle (z_1 - z_2, z_2, 0, \cdots, 0), (h_0, h_1, \cdots, h_T) \rangle_{(\mathbb{R}^n)^{T+1}} . \end{aligned}$$

So,

$$A_1^*(z_1, z_2) = (z_1 - z_2, z_2, 0, \ldots, 0) \text{ for all } (z_1, z_2) \in (\mathbb{R}^n)^2$$

and similarly we have

$$A_t^*(z_1, z_2) = (0, \cdots, 0, z_1 - z_2, z_2, 0, \ldots, 0) \text{ for all } t = 1, \cdots, T.$$

Therefore

$$\begin{aligned} y_0 &= (z_0^1, 0, \cdots, 0, z_0^2) \\ y_1 &= (z_1^1 - z_1^2, z_1^2, 0, \cdots, 0) \\ &\vdots \\ y_t &= (0, \cdots, 0, z_t^1 - z_t^2, z_t^2, 0, \cdots, 0) \\ y_T &= (0, \cdots, 0, z_T^1 - z_T^2, z_T^2). \end{aligned}$$

As $\sum_{t=0}^{T} y_t = 0$, then we have

(a) $z_0^1 + z_1^1 - z_1^2 = 0$

(b) $z_{t-1}^2 + z_t^1 - z_t^2 = 0 \quad$ for $t = 2, \cdots, T-1$

(c) $z_0^2 + z_T^2 = 0.$

Put $q_0 = z_0^1$ and $q_t = z_t^2$ for all $t = 1, \cdots, T$. So for any $t = 2, \cdots, T-1$ we have $\Delta q_t = q_t - q_{t-1} = z_t^2 - z_{t-1}^2$ and hence from equation (b) we obtain $\Delta q_t = z_t^1$. Further from equation (a) we have $\Delta q_1 = q_1 - q_0 = z_1^2 - z_0^1 = z_1^1$ and from equation (c) we also have $q_T = z_T^2 = -z_0^2$. If we substitute in the relations (3.1) and (3.2) , we obtain

$$(q_0, -q_T) \in \partial^\infty l(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta q_t, q_t) \in \partial^\infty L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) \;\forall t = 1, \cdots, T.$$

According to the qualification condition $Q(\bar{x})$ that we have assumed, we see that $q_0 = q_1 = \cdots = q_T = 0$, and then: $0 = q_0 = z_0^1$, $0 = q_t = z_t^2$, $0 = \Delta q_t = z_t^1$ for all $t = 1, \cdots, T$, and $0 = q_T = -z_0^2$. This yields $z_0^1 = z_0^2 = z_t^1 = z_t^2 = 0$ for all $t = 1, \cdots, T$ and hence

$$y_0 = A^* z_0 = 0 \text{ and } y_t = A^* z_t = 0 \text{ for all } t = 1, \cdots, T,$$

which means

$$y = (y_0, y_1, \cdots, y_T) = 0.$$

Step 3. As the functions φ_t are lsc over a finite dimensional space for all $t = 0, \cdots, T$ and as the qualification condition in Step 2 holds, we have by the formula (2.2)

$$\partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x}) \subset \partial\varphi_0(\bar{x}) + \sum_{t=1}^{T} \partial\varphi_t(\bar{x})$$

and hence

$$0 \in \partial(l \circ A_0)(\bar{x}) + \sum_{t=1}^{T} \partial(L_t \circ A_t)(\bar{x}).$$

This ensures the existence of $\xi_0 \in \partial(l \circ A_0)(\bar{x})$ and $\xi_t \in \partial(L_t \circ A_t)(\bar{x})$ for all $t = 1, \cdots, T$ such that $\sum_{t=0}^{T} \xi_t = 0$. As the mappings A_t are surjective and the functions l, L_t are lsc for all $t = 1, \cdots, T$, according to the calculus rule of subdifferential of composition function in (2.5), we have

$$\partial(l \circ A_0)(\bar{x}) \subset A_0^* \partial l(A_0 \bar{x}) \text{ and } \partial(L_t \circ A_t)(\bar{x}) \subset A_t^* \partial L_t(A_t \bar{x}) \; \forall t = 1, \cdots, T.$$

Then $\xi_0 \in A_0^* \partial l(\bar{x}_0, \bar{x}_T)$ and $\xi_t \in A_t^* \partial l(\bar{x}_{t-1}, \Delta \bar{x}_t)$ for all $t = 1, \cdots, T$, which ensures the existence of some $u_0 \in \partial l(\bar{x}_0, \bar{x}_T)$ such that $\xi_0 = A_0^* u_0$ and some $u_t \in \partial L_t(\bar{x}_{t-1}, \Delta \bar{x}_t)$ such that $\xi_t = A_t^* u_t$ for all $t = 1, \cdots, T$. This can be translated in the form

$$u_0 = (u_0^1, u_0^2) \in \partial l(\bar{x}_0, \bar{x}_T) \text{ and } \xi_0 = (u_0^1, 0, \cdots, 0, u_0^2) \tag{3.3}$$

and for all $t = 1, \cdots, T$

$$u_t = (u_t^1, u_t^2) \in \partial L_t(\bar{x}_{t-1}, \Delta \bar{x}_t) \text{ and } \xi_t = (0, \cdots, 0, u_t^1 - u_t^2, u_t^2, 0, \cdots, 0). \tag{3.4}$$

Putting $p_0 = u_0^1$ and $p_t = u_t^2$ for all $t = 1, \cdots, T$, we see that

$$\begin{aligned} 0 = \textstyle\sum_{t=0}^{T} \xi_t = (\; & u_0^1 + u_1^1 - u_1^2, u_1^2 + u_2^1 - u_2^2, \cdots, \\ & u_{t-1}^2 + u_t^1 - u_t^2, \cdots, u_{T-1}^2 + u_T^1 - u_T^2, u_0^2 + u_T^2), \end{aligned}$$

which gives $u_0^1 + u_1^1 - u_1^2 = 0$ for the first component, $u_{t-1}^2 + u_t^1 - u_t^2 = 0$ for any $t = 2, \cdots, T$, and $u_0^2 + u_T^2 = 0$. Then

$$\Delta p_t = p_t - p_{t-1} = u_t^2 - u_{t-1}^2 = u_t^1, \; \forall t = 2, \cdots, T$$

and for $t = 1$ we also have $\Delta p_1 = p_1 - p_0 = u_1^2 - u_0^1 = u_1^1$. Observe also that $p_T = u_T^2 = -u_0^2$, so $u_0^2 = -p_T$. Finally, if we replace in (3.1) and (3.2) we obtain that the vector $p = (p_0, p_1, \cdots, p_T) \in (\mathbb{R}^n)^{T+1}$ satisfies the requirements $(p_0, -p_T) \in \partial l(\bar{x}_0, \bar{x}_T)$ and

$$(\Delta p_t, p_t) \in \partial L_T(\bar{x}_{t-1}, \Delta \bar{x}_t) \quad \forall \, t = 1, \cdots, T.$$

This completes the proof of the theorem. ■

The following corollary deals with the discrete problem $(\mathrm{P}_{C,F}(l, L))$, that is, the case of Lipschitzian functions l and L_t, explicit set constraint C and set-valued mapping constraint F_t. Before stating the corollary, we need to recall that the graph of the set-valued mapping F_t is the subset

$$\operatorname{gph} F_t := \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, v \in F_t(u)\}.$$

In the corollary we assume that the sets C and $\operatorname{gph} F_t$ are *closed* in $\mathbb{R}^n \times \mathbb{R}^n$.

Corollary 3.2. *Let $\bar{x} \in (\mathbb{R}^n)^{T+1}$ be a solution of problem $(\mathcal{P}_{C,F}(l, L))$. Assume that the functions l and L_t are locally Lipschitzian for all $t = 1, \cdots, T$ and that the following qualification condition $\tilde{Q}(\bar{x})$ holds:*

$$\begin{cases} \text{the only vector } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{ for which} \\ (y_0, -y_T) \in N_C(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta y_t, y_t) \in N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)\ \forall\, t = 1, \cdots, T \\ \text{is the zero vector in } (\mathbb{R}^n)^{T+1}. \end{cases}$$

Then there exists some vector $p = (p_0, \cdots, p_T) \in (\mathbb{R}^n)^{T+1}$ such that:

(a) $(p_0, -p_T) \in \partial l(\bar{x}_0, \bar{x}_T) + N_C(\bar{x}_0, \bar{x}_T)$

(b) $(\Delta p_t, p_t) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$ *for all* $t = 1, \cdots, T$.

Proof. Put $S_t = \operatorname{gph} F_t$ for all $t = 1, \cdots, T$. Consider the functions $\tilde{l}(x_0, x_T) = l(x_0\, x_T) + \delta_C(x_0, x_T)$ and

$$\tilde{L}_t(x_{t-1}\, \Delta x_t) = L_t(x_{t-1}\, \Delta x_t) + \delta_{S_t}(x_{t-1}, \Delta x_t),$$

and observe that they are lsc and proper. Let us show that the qualification condition $Q(\bar{x})$ of Theorem 3.1 holds for the functions $\tilde{l}$ and $\tilde{L}$ for all $t = 1, \cdots, T$. So let $y \in (\mathbb{R}^n)^{T+1}$ such that

$$(y_0, -y_T) \in \partial^\infty \tilde{l}(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta y_t, y_t) \in \partial^\infty \tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T.$$

As l, L_t are locally Lipschitzian functions for all $t = 1, \cdots, T$, we see first that by (2.4) and (2.3)

$$(y_0, -y_T) \in \partial^\infty \tilde{l}(\bar{x}_0, \bar{x}_T) \subset \partial^\infty \delta_C(\bar{x}_0, \bar{x}_T) \subset N_C(\bar{x}_0, \bar{x}_T)$$

and also

$$(\Delta y_t, y_t) \in \partial^\infty \tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t)\ \subset\ N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t), \forall\, t = 1, \cdots, T.$$

By the qualification condition $\tilde{Q}(\bar{x})$ we have $y_0 = y_1 = \ldots = y_T = 0$, that is, the qualification condition $Q(\bar{x})$ is satisfied.

Since $\tilde{l}$ and $\tilde{L}_t$ are proper and lsc for all $t = 1, \cdots, T$ and since the qualification condition $Q(\bar{x})$ relative to the problem associated with $\tilde{l}$ and $\tilde{L}$ holds, we may apply Theorem 3.1 to obtain some vector $p = (p_0 \cdots p_T) \in (\mathbb{R}^n)^{T+1}$ such that

$$(p_0, -p_T) \in \partial \tilde{l}(\bar{x}_0, \bar{x}_T) \text{ and } (\Delta p_t, p_t) \in \partial \tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T.$$

As l, L_t are locally Lipschitzian functions for all $t = 1, \cdots, T$, we have according to (2.1)

$$\partial\tilde{l}(\bar{x}_0, \bar{x}_T) \subset \partial l(\bar{x}_0, \bar{x}_T) + \partial\delta_C(\bar{x}_0, \bar{x}_T)$$

and

$$\partial\tilde{L}_t(\bar{x}_{t-1}, \Delta\bar{x}_t) \subset \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + \partial\delta_{S_t}(\bar{x}_{t-1}, \Delta\bar{x}_t),\ \forall\, t = 1, \cdots, T$$

So we conclude that

(a) $(p_0, -p_T) \in \partial l(\bar{x}_0, \bar{x}_T) + N_C(\bar{x}_0, \bar{x}_T)$ and

(b) $(\Delta p_t, p_t) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + N_{\mathrm{gph}\, F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$, $\forall\, t = 1, \cdots, T$. ■

Remark 3.3. We can take any Asplund space E in place of $\mathbb{R}^n$ in Theorem 3.1 and Corollary 3.2 to obtain the same results, but we must assume in addition some sequential normal compactness property in the qualification condition (see, e.g., [13]). We did not pursue this line since our next objective will be the study, through the present paper, of the stochastic version where naturally $x_T(\omega) \in \mathbb{R}^n$.

Let C_0 be a nonempty closed subset of $\mathbb{R}^n$. The next corollary concerns the minimization problem $(\mathcal{P}_{C_0,F}(g, L))$ where the objective is to minimize the function

$$x \mapsto g(x_T) + \sum_{t=1}^{T} L_t(x_t, \Delta x_t)$$

under the initial constraint $x_0 \in C_0$ and the inclusion constraints $\Delta x_t \in F_t(x_{t-1})$ for all $t = 1, \cdots, T$.

Corollary 3.4. *Let $\bar{x} \in (\mathbb{R}^n)^{T+1}$ be a solution of problem $(\mathcal{P}_{C_0,F}(g, L))$. Assume that the functions g and L_t are locally Lipschitzian for all $t = 1, \cdots, T$, and that the following qualification condition $\widehat{Q}(\bar{x})$ holds:*

$$\begin{cases} \text{the only vector } y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1} \text{ for which} \\ y_0 \in N_{C_0}(\bar{x}_0),\ y_T = 0,\ \text{and } (\Delta y_t, y_t) \in N_{\mathrm{gph}\, F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)\ \forall\, t = 1, \cdots, T \\ \text{is the zero vector in } (\mathbb{R}^n)^{T+1}. \end{cases}$$

Then there exists some vector $p = (p_0, \cdots, p_T) \in (\mathbb{R}^n)^{T+1}$ such that:

(a) *$p_0 \in N_{C_0}(\bar{x}_0)$, $p_T \in -\partial g(\bar{x}_T)$,*

(b) *$(\Delta p_t, p_t) \in \partial L_t(\bar{x}_{t-1}, \Delta\bar{x}_t) + N_{\mathrm{gph}\, F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$.*

Proof. Put $l(x_0, x_T) := g(x_T)$ and $C := C_0 \times \mathbb{R}^n$. Then the normal cone to C is given by $N_C(\bar{x}_0, \bar{x}_T) = N_{C_0}(\bar{x}_0) \times \{0\}$ and the function l is obviously locally

Lipschitzian with the equality $\partial l(\bar{x}_0, \bar{x}_T) = \{0\} \times \partial g(\bar{x}_T)$. Further, it is easily seen that the qualification condition $\tilde{Q}(\bar{x})$ holds. Thus, the result is a consequence of Corollary 3.2. ■

The corollary below is relative to the case when the images of the set-valued mappings F_t are prox-regular. Recall that a closed subset S of $\mathbb{R}^n$ is *ρ-prox-regular* (for some $\rho \in]0, +\infty]$) when for any point z of the ρ-open enlargement of S

$$U_\rho(S) := \{u \in \mathbb{R}^n \mid d(u, S) < \rho\}$$

(where $d(\cdot, S)$ is the distance to S with respect to the Euclidean norm), the set S has a unique nearest point (denoted by $P_S(z)$) to z. This class of sets has been introduced in $\mathbb{R}^n$ by Federer [9] under the name of positively reached sets. The local property has been proved in [16] to be related to the fact that the indicator function of S is prox-regular in the sense of Poliquin and Rockafellar [15]. So the authors of [16] used the name of ρ-prox-regular sets in the above case. A closed set $S \subset \mathbb{R}^n$ is characterized in [16] to be ρ-prox-regular if and only if

$$\langle v_1 - v_2, u_1 - u_2 \rangle \geq -\|u_1 - u_2\|^2$$

for all $v_i \in N_S(u_i)$ with $\|v_i\| \leq \rho$. Another characterization is that for any nonzero vector $v \in N_S(u)$ one has $u \in \mathrm{Proj}_S(u + \rho\frac{v}{\|v\|})$, where $\mathrm{Proj}_S(z)$ denotes the set of all nearest points in S to z. Translating the latter inclusion in the form

$$\|u - (u + \rho\frac{v}{\|v\|})\|^2 \leq \|u - u'\|^2 \quad \text{for all } u' \in S,$$

we see that it is equivalent to the inequality

$$\langle v, u' - u \rangle \leq \frac{1}{2\rho}\|v\|\,\|u' - u\|^2 \quad \text{for all } u' \in S. \tag{3.5}$$

Observe that the latter inequality still holds for $v = 0$.

In fact, those results has been proved in [16] in the setting of (infinite dimensional) Hilbert space (for which one needs to require, in addition, in the definition above the continuity of P_S over the open enlargement $U_\rho(S)$). For several other results, we refer to [9, 3, 6, 16, 7]. See also [1] for the framework of uniformly convex Banach space.

Recall also that for any set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the *coderivative* of G at a point $(u, v) \in \mathrm{gph}\, G$ is the set-valued mapping $D^*G(u, v) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by $\zeta \in D^*G(u, v)(\xi)$ if and and only if $(\zeta, -\xi) \in N(\mathrm{gph}\, G, (u, v))$.

We will also need the Lipschitz property concept for set-valued mapping. Recall that the set-valued mapping G is locally Lipschitzian around a point $\bar{u}$ with a nonnegative number γ for Lipschitz modulus provided that there exists some positive number η such that for all $u, u' \in B(\bar{u}, \eta)$ one has

$$G(u') \subset G(u) + \gamma\|u - u'\|\mathbb{B},$$

where $\mathbb{B}$ denotes the closed unit ball of $\mathbb{R}^n$ centered at the origin.

We can now state the corollary for the problem $(\mathcal{P}_{C_0,F}(g))$ where each function L_t is equal to the *null* function.

Corollary 3.5. *Let $\bar{x} \in (\mathbb{R}^n)^{T+1}$ be a solution of problem $(\mathcal{P}_{C_0,F}(g))$. Assume that the function g is locally Lipschitzian and that each set-valued mapping F_t is locally Lipschitzian around $\bar{x}_{t-1}$ for $t = 1, \cdots, T$.*

Then there exists some vector $p = (p_0, \cdots, p_T) \in (\mathbb{R}^n)^{T+1}$ such that: a) $p_0 \in N_{C_0}(\bar{x}_0)$, $p_T \in -\partial g(\bar{x}_T)$,
b) $(\Delta p_t, p_t) \in N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$, that is,

$$\Delta p_t \in D^* F_t(\bar{x}_{t-1}, \Delta\bar{x}_t)(-p_t).$$

If in addition the sets $F_t(u)$ are ρ-prox-regular for u near to $\bar{x}_{t-1}$ for each $t = 1, \cdots, T$, then one also has

$$\langle p_t, \Delta\bar{x}_t\rangle = \mathcal{H}_{t,\rho}(\bar{x}_{t-1}, \Delta\bar{x}_t, p_t)$$

for all $t = 1, \cdots, T$, where $\mathcal{H}_{t,\rho}$ may be considered as the generalized Hamiltonian up to the ρ-square, in the sense that

$$\mathcal{H}_{t,\rho}(u, v, \xi) := \sup\{\langle \xi, w\rangle - \frac{1}{2\rho}\|\xi\| \, \|w - v\|^2 \mid w \in F_t(u)\}. \tag{3.6}$$

Proof. Without loss of generality we may suppose that all the set-valued mappings F_t, for $t = 1, \cdots, T$, have the same Lipschitz modulus $\gamma \geq 0$, that is, for some real number $\eta > 0$

$$F_t(u') \subset F_t(u) + \gamma\|u' - u\|\mathbb{B} \tag{3.7}$$

for all $t = 1, \cdots, T$ and $u, u' \in \bar{x}_{t-1} + \eta\mathbb{B}$. Let us show that the qualification $\widehat{Q}(\bar{x})$ of Corollary 3.4 is satisfied. Fix any vector $y = (y_0, \cdots, y_T) \in (\mathbb{R}^n)^{T+1}$ such that $y_0 \in N_{C_0}(\bar{x}_0)$, $y_T = 0$, and $(\Delta y_t, y_t) \in N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$ for all $t = 1, \cdots, T$. It is not difficult to verify (see also [13]) that the Lipschizian property (3.4) of F_t ensures that for each $t = 1, \cdots, T$, one has $\|\zeta\| \leq \gamma\|\xi\|$ for any $(\zeta, \xi) \in N_{\operatorname{gph} F_t}(\bar{x}_{t-1}, \Delta\bar{x}_t)$. So we have $\|\Delta y_t\| \leq \gamma\|y_t\|$ and hence using the equality $y_T = 0$ we obtain $y_T = y_{T-1} = \cdots = y_0 = 0$, which says that $\widehat{Q}(\bar{x})$ holds. Therefore the assertions (a) and (b) follow from Corollary 3.4.

Assume now that in addition the prox-regularity assumption of the theorem holds and fix any $t \in \{1, \cdots, T\}$. Put $\bar{u} := \bar{x}_{t-1}$, $\bar{v} := \Delta\bar{x}_t$, and take any $(\zeta, \xi) \in N_{\operatorname{gph} F_t}(\bar{u}, \bar{v})$. Then, by definition, there exists a sequence of Fréchet normal vectors (ζ_k, ξ_k) converging to (ζ, ξ) with $(\zeta_k, \xi_k) \in \widehat{N}_{\operatorname{gph} F_t}(u_k, v_k)$ and with $(u_k, v_k) \to (\bar{u}, \bar{v})$ and $u_k \in \bar{u} + \eta\mathbb{B}$. For any $\varepsilon > 0$ there is some $\eta_k < \eta$ such that for all $(u, v) \in \operatorname{gph} F_t$ with $(u, v) \in (u_k, v_k) + \eta_k\mathbb{B}$ one has

$$\langle \zeta_k, u - u_k \rangle + \langle \xi_k, v - v_k \rangle \leq \varepsilon(\|u - u_k\| + \|v - v_k\|)$$

and hence taking $u = u_k$ we see that $\langle \xi_k, v - v_k \rangle \leq \varepsilon\|v - v_k\|$. This means that $\xi_k \in \widehat{N}(F_t(u_k), v_k)$. Our prox-regularity property entails by (3.3) and by the inclusion $\widehat{N}(\cdot,\cdot) \subset N(\cdot,\cdot)$ that

$$\langle \xi_k, v - v_k \rangle \leq \frac{1}{2\rho}\|\xi_k\| \, \|v - v_k\|^2 \;\; \forall v \in F_t(u_k). \tag{3.8}$$

Now by (3.4) we have $F_t(\bar{u}) \subset F_t(u_k) + \gamma\|\bar{u} - u_k\|\mathbb{B}$, which yields that any $w \in F_t(\bar{u})$ may be written, for each integer k, in the form $w = w_k + \gamma\|\bar{u} - u_k\|b_k$ with $b_k \in \mathbb{B}$. The latter implies on the one hand $w_k \to w$ and on the other hand according to (3.5)

$$\begin{aligned}\langle \xi_k, w - v_k \rangle &= \langle \xi_k, w_k - v_k \rangle + \langle \xi_k, w - w_k \rangle \\ &\leq \frac{1}{2\rho}\|\xi_k\| \, \|w_k - v_k\|^2 + \langle \xi_k, w - w_k \rangle.\end{aligned}$$

Passing to the limit with $k \to +\infty$ we obtain $\langle \xi, w - \bar{v} \rangle \leq \frac{1}{2\rho}\|\xi\| \, \|w - \bar{v}\|^2$, that is,

$$\langle \xi, w \rangle - \frac{1}{2\rho}\|\xi\| \, \|w - \bar{v}\|^2 \leq \langle \xi, \bar{v} \rangle.$$

Letting $(\zeta, \xi) = (\Delta p_t, p_t)$ according to (b), we see that

$$\langle p_t, \Delta\bar{x}_t \rangle = \mathcal{H}_{t,\rho}(\bar{x}_{t-1}, \Delta\bar{x}_t, p_t),$$

which completes the proof. ■

Remark 3.6. It is clear in the proof above that assertions (a) and (b) still holds when the Lipschitz property of F_t is relaxed in the Aubin property for F_t around the point $(\bar{u}, \bar{v}) := (\bar{x}_{t-1}, \Delta\bar{x}_t)$, that is, for some $\eta > 0$ and $\gamma \geq 0$ one has the inclusion

$$(\bar{v} + \eta\mathbb{B}) \cap F_t(u') \subset F_t(u) + \|u' - u\|\mathbb{B}$$

for all $u, u' \in \bar{u} + \eta\mathbb{B}$. However, the Lipschitz property is required in the proof of the equality (3.6). Nevertheless, (3.6) could be established with some other regularity properties in [2] instead of the prox-regularity.

In the case when the functions l and L_t are convex (non necessarily lsc), our approach in Theorem 3.1 allows us to provide another proof of the result of Rockafellar and Wets [20] where appears a qualification condition making use of the relative interiors of the functions l and L_t.

Theorem 3.7. *Assume that the functions l and L_t are proper and convex* (*non necessarily lsc*) *for all $t = 1, \cdots, T$ and assume also the following qualification condition holds:*

$$\begin{cases} \text{there exists some point } z \in (\mathbb{R}^n)^{T+1} \text{ such that} \\ (z_0, z_T) \in \operatorname{ri}\operatorname{dom} l \text{ and } (z_{t-1}, \Delta z_t) \in \operatorname{ri}\operatorname{dom} L_t, \ \forall\, t = 1, \cdots, T. \end{cases}$$

Then, a feasible point $\bar{x} \in (\mathbb{R}^n)^{T+1}$ of the problem $(\mathcal{P}(l, L))$ is a solution of this problem if and only if there exists some vector $p \in (\mathbb{R}^n)^{T+1}$ satisfying relations (a) and (b) of Theorem 3.1.

Proof. Assume that $\bar{x}$ is a solution of the problem $(\mathcal{P}(l, L))$ and consider the linear mappings A_0, A_t and the functions φ_0, φ_t in the first step of the proof of Theorem 3.1. We see that these functions are convex and $0 \in \partial\varphi(\bar{x}) = \partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x})$ since $\bar{x}$ is a minimum point of φ.

Let us prove that $\cap_{t=0}^{T} \operatorname{ri}\operatorname{dom}\varphi_t \neq \emptyset$. Indeed we have

$$\operatorname{dom}\varphi_0 = \operatorname{dom}(l \circ A_0) \text{ and } \operatorname{dom}\varphi_t = \operatorname{dom}(l_t \circ A_t), \ \forall\, t = 1, \cdots, T$$

and our assumption may be written in the form $A_0 z \in \operatorname{ri}\operatorname{dom} l$ and $A_t z \in \operatorname{ri}\operatorname{dom} L_t$ for $t = 1, \cdots, T$. Therefore (see [17]) $z \in \operatorname{ri}\operatorname{dom}\varphi_0$ and $z \in \operatorname{ri}\operatorname{dom}\varphi_t$ for $t = 1, \cdots, T$, which implies that $\cap_{t=0}^{T} \operatorname{ri}\operatorname{dom}\varphi_t \neq \emptyset$.

The nonemptyness of the latter intersection ensures (see [17]) the first equality below

$$\partial(\varphi_0 + \sum_{t=1}^{T} \varphi_t)(\bar{x}) = \partial\varphi_0(\bar{x}) + \sum_{t=1}^{T} \partial\varphi_t(\bar{x}) = \partial(l \circ A_0)(\bar{x}) + \sum_{t=1}^{T} \partial(l_t \circ A_t)(\bar{x}).$$

The linear mappings A_0 and A_t for $t = 1, \cdots, T$ being surjective we have $\partial(l \circ A_0)(\bar{x}) = A_0^* \partial l(A_0 \bar{x})$ and $\partial(L_t \circ A_t)(\bar{x}) = A_t^* \partial l_t(A_t \bar{x})$ for $t = 1, \cdots, T$. Thus, the proof of optimality conditions follows the arguments of the second part of Step 2 and the arguments of Step 3 in the proof of Theorem 3.1.

Finally, the sufficiency of the optimality conditions through the convexity of l and L_t is straightforward and we omit it. ■

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