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SECOND ORDER SYMMETRIC DUALITY FOR NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING INVOLVING CONES

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. We construct Mond-Weir and Wolfe types of nondifferentiable multiobjective second order symmetric dual problems with cone constraints over arbitrary closed convex cones. Weak, strong, and converse duality theorems for weakly efficient solutions are established under the assumptions of second order invex and pseudo-invex functions. Several known results are obtained as special cases of our symmetric duality.

1. Introduction

In the literature of mathematical nonlinear programming there are a large number of papers ([4-7, 14, 17-19, 21]) discussing symmetric duality theory for the first and second order cases. The symmetric duality theory for a problem involving the square root of a positive semidefinite quadratic function is introduced by Ahmad [1] and Ahmad and Husain [2]. The square root of a positive semidefinite quadratic form is one of the few cases of a nondifferentiable function for which one can write down the sub or quasi differentials explicitly. Mond and Schechter [16] replaced the square root of a positive semidefinite quadratic function by a somewhat more general function, namely the support function of a compact convex set, for which the subdifferential may be simply expressed. Some authors ([8, 11, 16, 22]) established symmetric duality relations for a problem involving the support function. Our research focus on symmetric duality in nondifferentiable multiobjective programming problems with cone constraints.

In the first order case, Suneja et al. [20] formulated a pair of multiobjective symmetric dual programs of Wolfe type over arbitrary cones in which the objective function was optimized with respect to an arbitrary closed convex cone by

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assuming the involved function to be cone-convex. Recently, Khurana [9] introduced cone-pseudo-invex and strongly cone-pseudo-invex functions and established duality theorems for a pair of Mond-Weir type multiobjective symmetric dual over arbitrary cones. More recently, Kim and Kim [10] studied two pairs of nondifferentiable multiobjective symmetric dual problems with cone constraints over arbitrary closed convex cones, which are Wolfe type and Mond-Weir type.

In the second order case, Mishra [13] formulated a pair of multiobjective second order symmetric dual nonlinear programming problems under second order pseudo-invexity assumptions on the involved functions over arbitrary cones and established duality results. Subsequently, Mishra and Lai [15] introduced the concept of cone-second order pseudo-invex and strongly cone-second order pseudo-invex functions and formulated a pair of Mond-Weir type multiobjective second order symmetric dual programs over arbitrary cones and established duality relations. Very recently, Kim et al. [12] established nondifferentiable multiobjective second order symmetric duality theorems involving cone constraints by second order invexity and pseudo-invexity conditions.

In this paper, we formulate nondifferentiable multiobjective second order symmetric dual problems with cone constraints over arbitrary closed convex cones. In order to establish the strong duality, we give necessary optimality conditions. Weak, strong, and converse duality theorems for weakly efficient solutions are established under the assumptions of second order pseudo-invex functions. And we present some special cases of our duality results.

2. Preliminaries

Now we will give some definitions and preliminary results needed in next sections.

Definition 2.1. A nonempty set K in $\mathbb{R}^k$ is said to be a cone with vertex zero if $x \in K$ implies that $\lambda x \in K$ for all $\lambda \geqq 0$. If, in addition, K is convex, then K is called a convex cone.

Consider the following multiobjective programming problem:

$$\begin{array}{lll} (\mathbf{KP}) & \text{Minimize} & f(x) \\ & \text{subject to} & -g(x) \in Q, x \in C, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $C \subset \mathbb{R}^n$, Q is a closed convex cone with nonempty interior in $\mathbb{R}^m$.

Definition 2.2. A feasible point $\overline{x}$ is a K-weakly efficient solution of (KP) if there exists no other $x \in X$ such that $f(\overline{x}) - f(x) \in intK$.

Definition 2.3. ([13]) Let $f : X(\subset \mathbb{R}^n) \times Y(\subset \mathbb{R}^m) \to \mathbb{R}$ be a twice differentiable function.

(i) f is said to be second order invex in the first variable at u for fixed v, if there exists a function $\eta_1 : X \times X \to X$ such that for $r \in \mathbb{R}^n$,

$$f(x,v) - f(u,v) \geqq \eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)r] - \frac{1}{2}r^T \nabla_{xx} f(u,v)r.$$

(ii) f is said to be second order pseudo-invex in the first variable at u for fixed v, if there exists a function $\eta_1 : X \times X \to X$ such that for $r \in \mathbb{R}^n$,

$$\eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)r] \geqq 0 \Rightarrow f(x,v) - f(u,v)$$
$$+\frac{1}{2}r^T \nabla_{xx} f(u,v)r \geqq 0.$$

f is second order pseudo-incave at $u \in C_1$ with respect to $r \in C_1$, if $-f$ is second order pseudo-invex at $u \in C_1$ with respect to $r \in C_1$.

Definition 2.4. ([16]) Let B be a compact convex set in $\mathbb{R}^n$. The support function $s(x|B)$ of B is defined by

$$s(x|B) := max\{x^T y : y \in B\}.$$

The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|B) \geq s(x|B) + z^T(y - x) \text{ for all } y \in B.$$

Equivalently,

$$z^T x = s(x|B).$$

The subdifferential of $s(x|B)$ is given by

$$\partial s(x|B) := \{z \in B : z^T x = s(x|B)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

It is readily verified that for a compact convex set B, y is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, x is in the subdifferential of s at y.

3. Mond-weir Type Symmetric Duality

We consider the following pair of second order Mond-Weir type nondifferentiable multiobjective programming problem:

$$
\begin{aligned}
(\mathbf{NMP}) \quad & \text{Minimize} \quad K(x,y,\lambda,w,p) \\
& \qquad = f(x,y) + s(x|D) - (y^T w)e - \frac{1}{2}[p^T \nabla_{yy}(\lambda^T f)(x,y)p]e \\
(1) \quad & \text{subject to} \quad -[\nabla_y(\lambda^T f)(x,y) - w + \nabla_{yy}(\lambda^T f)(x,y)p \in C_2^*, \\
(2) \quad & \qquad y^T[\nabla_y(\lambda^T f)(x,y) - w + \nabla_{yy}(\lambda^T f)(x,y)p] \geqq 0, \\
& \qquad x \in C_1, \quad w \in E_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{NMD}) \quad & \text{Maximize} \quad G(u,v,\lambda,z,r) \\
& \qquad = f(u,v) - s(v|E) + (u^T z)e - \frac{1}{2}[r^T \nabla_{xx}(\lambda^T f)(u,v)r]e \\
(3) \quad & \text{subject to} \quad \nabla_x(\lambda^T f)(u,v) + z + \nabla_{xx}(\lambda^T f)(u,v)r \in C_1^*, \\
(4) \quad & \qquad u^T[\nabla_x(\lambda^T f)(u,v) + z + \nabla_{xx}(\lambda^T f)(u,v)r] \leqq 0, \\
& \qquad v \in C_2, z \in D_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,
\end{aligned}
$$

where

(1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a three times differentiable function,
(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,
(3) C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively,
(4) K is a closed convex cone in $\mathbb{R}^k$ such that $intK \neq \emptyset$ and $\mathbb{R}^k_+ \subset K$,
(5) r, z are vectors in $\mathbb{R}^n$, p, w are vectors in $\mathbb{R}^m$,
(6) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,
(7) $D = (D_1, D_2, \cdots, D_k)^T$ and $E = (E_1, E_2, \cdots, E_k)^T, whereD_i$ and $E_i (i = 1, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and R^m, respectively.

Let $\nabla_x(\lambda^T f)(x,y)$ and $\nabla_y(\lambda^T f)(x,y)$ are gradients of $(\lambda^T f)(x,y)$ with respect to x and y. Similarly, $\nabla_{xx}(\lambda^T f)(x,y)$ and $\nabla_{yy}(\lambda^T f)(x,y)$ are the Hessian matrices of $(\lambda^T f)(x,y)$ with respect to x and y, respectively.

Now we establish the symmetric duality theorems for **(NMP)** and **(NMD)**.

Theorem 3.1. (Weak Duality) *Let* (x,y,λ,w,p) *and* (u,v,λ,z,r) *be feasible solutions of* **(NMP)** *and* **(NMD)**, *respectively. Assume that,*

(i) $(\lambda^T f)(\cdot, y)+(\cdot)^T z$ *is second order pseudo-invex in the first variable for fixed* y *with respect to* η_1*,*

(ii) $-(\lambda^T f)(x, \cdot)+(\cdot)^T w$ *is second order pseudo-invex in the second variable for fixed* x *with respect to* η_2*,*

(iii) $\eta_1(x,u)+u \in C_1$ *and* $\eta_2(v,y)+y \in C_2$*, then*

$$G(u,v,\lambda,z,r)-K(x,y,\lambda,w,p) \notin intK.$$

Proof. From (3) and $\eta_1(x,u)+u \in C_1$,

$$[\eta_1(x,u)+u]^T \left[\nabla_x(\lambda^T f)(u,v)+z+\nabla_{xx}(\lambda^T f)(u,v)r\right] \geqq 0.$$

From (4), it yields

$$\eta_1(x,u)^T \left[\nabla_x(\lambda^T f)(u,v)+z+\nabla_{xx}(\lambda^T f)(u,v)r\right] \geqq 0.$$

By the second order pseudo-invexity of $(\lambda^T f)(\cdot,y)+(\cdot)^T z$, we have

$$(\lambda^T f)(x,v)+x^T z-(\lambda^T f)(u,v)-u^T z+\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r \geqq 0. \tag{5}$$

From (1) and $\eta_2(v,y)+y \in C_2$,

$$-[\eta_2(v,y)+y]^T[\nabla_y(\lambda^T f)(x,y)-w+\nabla_{yy}(\lambda^T f)(x,y)p] \geqq 0.$$

From (2), it yields

$$\eta_2(v,y)^T[\nabla_y(\lambda^T f)(x,y)-w+\nabla_{yy}(\lambda^T f)(x,y)p] \leqq 0.$$

By the second order pseudo-invexity of $-(\lambda^T f)(x,\cdot)+(\cdot)^T w$, we obtain

$$(\lambda^T f)(x,v)-v^T w-(\lambda^T f)(x,y)+y^T w+\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p \leqq 0. \tag{6}$$

From (5) and (6), we get

$$\begin{aligned}(\lambda^T f)(u,v)-x^T z+u^T z-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r\\ \leqq (\lambda^T f)(x,y)+v^T w-y^T w-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p.\end{aligned} \tag{7}$$

Using the fact that $x^T z \leqq s(x|D_i)$ and $v^T w \leqq s(v|E_i)$ for $i=1,\cdots,k$, we get

$$x^T z \leqq \lambda^T s(x|D) \text{ and } v^T w \leqq \lambda^T s(v|E).$$

Finally, using these, we obtain

$$(8)\qquad \begin{aligned}&(\lambda^T f)(x,y)+\lambda^T s(x|D)-y^T w-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p\\ &\geqq (\lambda^T f)(u,v)-\lambda^T s(v|E)+u^T z-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r.\end{aligned}$$

But suppose that $G(u,v,\lambda,z,r)-K(x,y,\lambda,w,p)\in intK$. Since $\lambda\in K^*$, it yields

$$\begin{aligned}&[(\lambda^T f)(u,v)-\lambda^T s(v|E)+u^T z-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r]\\ &\quad -[(\lambda^T f)(x,y)+\lambda^T s(x|D)-y^T w-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p]>0,\end{aligned}$$

which is a contradiction to the inequality (8). ■

In order to prove the strong duality theorem, we can obtain the following necessary optimality conditions for a point to be a weak minimum of $(\mathbf{KP})$.

Lemma 3.1. *If $\overline{x}$ is a K-weakly efficient solution of $(\mathbf{KP})$, then there exist $\alpha\in K^*$ and $\beta\in Q^*$ not both zero such that*

$$(\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x}))(x-\overline{x})\geqq 0,\quad \textit{for all}\quad x\in C,$$

$$\beta^T g(\overline{x})=0.$$

Equivalently, there exist $\alpha\in K^*$, $\beta\in Q^*$, $\beta_1\in C^*$ and $(\alpha,\beta,\beta_1)\neq 0$ such that

$$\begin{aligned}&\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x})-\beta_1^T I=0,\\ &\beta^T g(\overline{x})=0,\\ &\beta_1^T\overline{x}=0.\end{aligned}$$

Proof. The first part of Lemma 3.1 is introduced by Bazaraa and Goode [3]. Now we prove the latter part of Lemma 3.1. Substituting $x=0$ and $x=2\overline{x}$, we get

$$(\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x}))\overline{x}=0.$$

Since $\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x})\in C^*$, let $\beta_1=\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x})$. Then

$$\begin{aligned}&\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x})-\beta_1^T I=0,\\ &\beta^T g(\overline{x})=0,\\ &\beta_1^T\overline{x}=0.\end{aligned}$$

Conversely, we obtain the following inequality by $\alpha^T\nabla f(\overline{x})+\beta^T\nabla g(\overline{x}) = \beta_1 \in C^*$,

$$(\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}))x \geqq 0, \quad \text{for all} \quad x \in C$$

and

$$\beta_1^T\overline{x} = (\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}))\overline{x} = 0.$$

Therefore,

$$(\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}))(x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C,$$

$$\beta^T g(\overline{x}) = 0. \qquad \blacksquare$$

Theorem 3.2. (Strong Duality). *Let* $(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p})$ *be a K-weakly efficient solution for* **(NMP)**. *Fix* $\lambda = \overline{\lambda}$ *in* **(NMD)**. *Assume that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ *is positive definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \geqq 0$ *or*
$\nabla_{yy}\overline{\lambda}^T f$ *is negative definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \leqq 0$,

(ii) $\nabla_y\overline{\lambda}^T f - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) *the set* $\{\nabla_y f_1, \nabla_y f_2, \cdots, \overline{w}\}$ *is linearly independent,*

where $f = f(\overline{x},\overline{y})$.

Then there exists $\overline{z} \in D_i (i = 1,\cdots,k)$ *such that* $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r} = 0)$ *is a feasible solution for* **(NMD)** *and objective values of* **(NMP)** *and* **(NMD)** *are equal. Furthermore, under the assumptions of Theorem 3.1,* $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r} = 0)$ *is a K-weakly efficient solution for* **(NMD)**.

Proof. Since $(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p})$ is a K-weakly efficient solution for **(NMP)**, by Lemma 3.1, there exist $\alpha \in K^*, \beta \in C_2, \mu \in \mathbb{R}_+, \delta \in C_1^*$, and $\rho \in K$ such that

$$\alpha^T[\nabla_x f + ze] + (\beta - \mu\overline{y})^T\nabla_{yx}(\lambda^T f) + (\beta - \mu\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p})^T\nabla_x(\nabla_{yy}(\lambda^T f))\overline{p} - \delta = 0, \tag{9}$$

$$(\alpha - \mu\overline{\lambda})^T\nabla_y f - (\alpha^T e - \mu)^T\overline{w} + (\beta - \mu\overline{y} - \mu\overline{p})^T\nabla_{yy}(\overline{\lambda}^T f) + (\beta - \mu\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p})^T\nabla_y(\nabla_{yy}(\overline{\lambda}^T f)\overline{p}) = 0, \tag{10}$$

$$-\frac{1}{2}(\alpha^T e)\overline{p}^T\nabla_{yy}f\overline{p} + (\beta - \mu\overline{y})^T[\nabla_y f + \nabla_{yy}f\overline{p}] - \rho = 0, \tag{11}$$

$$(\alpha^T e)\overline{y} - (\beta - \mu\overline{y}) \in N_{E_i}(\overline{w}), \tag{12}$$

$$(\beta - \alpha^T e\overline{p} - \mu\overline{y})^T \nabla_{yy}(\overline{\lambda}^T f) = 0, \tag{13}$$

$$\beta^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{14}$$

$$\mu\overline{y}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{15}$$

$$\delta^T\overline{x} = 0, \tag{16}$$

$$\rho^T\overline{\lambda} = 0, \tag{17}$$

$$z \in D_i, \;\; z^T\overline{x} = s(\overline{x}|D_i), \;\; i = 1, \cdots, k, \tag{18}$$

$$(\alpha, \beta, \mu, \delta, \rho) \neq 0. \tag{19}$$

As $\nabla_{yy}(\overline{\lambda}^T f)$ is positive or negative definite, (13) yields

$$\beta = (\alpha^T e)\overline{p} + \mu\overline{y}. \tag{20}$$

If $\alpha = 0$, then the above equality becomes

$$\beta = \mu\overline{y}. \tag{21}$$

From (10), we obtain

$$\mu[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \overline{p}^T\nabla_{yy}(\overline{\lambda}^T f)] = 0. \tag{22}$$

By the assumption (ii), we have $\mu = 0$. Also, from (9), (11) and (21), we get $\delta = 0, \;\rho = 0$ and $\beta = 0$, respectively. This contradicts (19). So, $\alpha > 0$. From (14) and (15), we obtain

$$(\beta - \mu\overline{y})^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0.$$

Using (20), it follows that

$$\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] + \overline{p}^T\nabla_{yy}(\overline{\lambda}^T f)\overline{p} = 0. \tag{23}$$

We now prove that $\overline{p} = 0$. Otherwise, the assumption (i) implies that

$$\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] + \overline{p}^T\nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0,$$

which contradicts (23). Hence $\overline{p} = 0$. From (20), we have

$$\beta = \mu\overline{y}. \tag{24}$$

Using (24) and $\overline{p} = 0$ in (10), we obtain

$$(\alpha - \mu\overline{\lambda})^T \nabla_y f - (\alpha^T e - \mu)\overline{w} = 0.$$

By the assumption (iii), we get

$$\alpha = \mu\overline{\lambda} \text{ and } \alpha^T e = \mu. \tag{25}$$

Therefore, $\mu > 0$, it follows that

$$\nabla_x(\overline{\lambda}^T f) + z \in C_1^*.$$

Multiplying (9) by $\overline{x}$ and using equation (16), we get

$$\overline{x}^T[\nabla_x(\overline{\lambda}^T f) + z] = 0.$$

Taking $\overline{z} := z \in D_i (i = 1, \cdots, k)$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is feasible for **(NMD)**. Moreover from (12), we get $\overline{y} \in N_{E_i}(\overline{w})$ for $i = 1, \cdots, k$, so that

$$\overline{y}^T \overline{w} = s(\overline{y}|E_i) \text{ for } i = 1, \cdots, k$$

i.e.,

$$(\overline{y}^T \overline{w})e = s(\overline{y}|E).$$

Consequently, using (18),

$$\begin{aligned} K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) &= f(\overline{x}, \overline{y}) + s(\overline{x}|D) - (\overline{y}^T \overline{w})e \\ &= f(\overline{x}, \overline{y}) - s(\overline{y}|E) + (\overline{z}^T \overline{x})e \\ &= G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0). \end{aligned}$$

Thus objective values of **(NMP)** and **(NMD)** are equal. We will now show that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a K-weakly efficient solution for **(NMD)**, otherwise there exists a feasible solution $(u, v, \overline{\lambda}, z, r = 0)$ for **(NMD)** such that

$$G(u, v, \overline{\lambda}, z, r = 0) - G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0) \in intK.$$

Since objective values of **(NMP)** and **(NMD)** are equal, we have

$$G(u, v, \overline{\lambda}, z, r = 0) - K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) \in intK,$$

which contradicts weak duality theorem. Hence the result holds. ■

Theorem 3.3. (Converse Duality). *Let* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ *be a K-weakly efficient solution for* **(NMD)**. *Fix* $\lambda = \overline{\lambda}$ *in* **(NMP)**. *Assume that*

(i) $\nabla_{xx}(\overline{\lambda}^T f)$ *is positive definite and* $\overline{r}^T[\nabla_x(\overline{\lambda}^T f) - \overline{z}] \geqq 0$ *or* $\nabla_{xx}(\overline{\lambda}^T f)$ *is negative definite and* $\overline{r}^T[\nabla_x(\overline{\lambda}^T f) - \overline{z}] \leqq 0$,

(ii) $\nabla_x(\overline{\lambda}^T f) - \overline{z} + \nabla_{xx}(\overline{\lambda}^T f)\overline{r} \neq 0$,

(iii) *the set* $\{\nabla_x f_1, \nabla_x f_2, \cdots, \overline{z}\}$ *is linearly independent where* $f = f(\overline{u}, \overline{v})$.

Then there exists $\overline{w} \in E_i(i = 1, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution for* **(NMP)** *and objective values of* **(NMP)** *and* **(NMD)** *are equal. Furthermore, under the assumptions of Theorem 3.1,* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a K-weakly efficient solution for* **(NMP)**.

Proof. It follows on the lines of Theorem 3.2. ∎

4. Wolfe Type Symmetric Duality

We consider the following pair of second order Wolfe type nondifferentiable multiobjective programming problem:

(**NWP**) Minimize

$$K(x, y, \lambda, w, p) = f(x, y) + s(x|D) - (y^T\nabla_y(\lambda^T f)(x, y))e - (y^T\nabla_{yy}(\lambda^T f)(x, y)p)e - \frac{1}{2}[p^T\nabla_{yy}(\lambda^T f)(x, y)p]e$$

(26) subject to

$$-[\nabla_y(\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y)p \in C_2^*,$$

$$x \in C_1, \quad w \in E_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,$$

(**NWD**) Maximize

$$G(u, v, \lambda, z, r) = f(u, v) - s(v|E) - (u^T\nabla_x(\lambda^T f)(u, v))e - (u^T\nabla_{xx}(\lambda^T f)(u, v)r)e - \frac{1}{2}[r^T\nabla_{xx}(\lambda^T f)(u, v)r]e$$

(27) subject to

$$\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r \in C_1^*,$$

$$v \in C_2, \quad z \in D_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,$$

where

(1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a three times differentiable function,

(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(3) C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively,

(4) K is a closed convex cone in $\mathbb{R}^k$ such that $intK \neq \emptyset$ and $\mathbb{R}_+^k \subset K$,

(5) r, z are vectors in $\mathbb{R}^n$, p, w are vectors in $\mathbb{R}^m$,

(6) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,

(7) $D = (D_1, D_2, \cdots, D_k)^T$ and $E = (E_1, E_2, \cdots, E_k)^T, where D_i$ and $E_i (i = 1, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and R^m, respectively.

Let $\nabla_x(\lambda^T f)(x, y)$ and $\nabla_y(\lambda^T f)(x, y)$ are gradients of $(\lambda^T f)(x, y)$ with respect to x and y. Similarly, $\nabla_{xx}(\lambda^T f)(x, y)$ and $\nabla_{yy}(\lambda^T f)(x, y)$ are the Hessian matrices of $(\lambda^T f)(x, y)$ with respect to x and y, respectively.

Now we establish the symmetric duality theorems for **(NWP)** and **(NWD)**.

Theorem 4.1. (Weak Duality). *Let (x, y, λ, w, p) and (u, v, λ, z, r) be feasible solutions of* **(NWP)** *and* **(NWD)***, respectively. Assume that,*

(i) $(\lambda^T f)(\cdot, y) + (\cdot)^T z$ *is second order invex in the first variable for fixed y with respect to* η_1*,*

(ii) $-(\lambda^T f)(x, \cdot) + (\cdot)^T w$ *is second order invex in the second variable for fixed x with respect to* η_2*,*

(iii) $\eta_1(x, u) + u \in C_1$ *and* $\eta_2(v, y) + y \in C_2$*. Then*

$$G(u, v, \lambda, z, r) - K(x, y, \lambda, w, p) \notin intK.$$

Proof. Since $(\lambda^T f)(\cdot, y) + (\cdot)^T z$ is second order invex with respect to η_1 for fixed y,

$$\begin{aligned}&(\lambda^T f)(x, v) + x^T z - (\lambda^T f)(u, v) - u^T z \\ &\geqq \eta_1(x, u)^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r] - \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v)r.\end{aligned}$$

From (27) and $\eta_1(x, u) + u \in C_1$,

$$[\eta_1(x, u) + u]^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_x x(\lambda^T f)(u, v)r] \geqq 0.$$

Hence

$$\begin{aligned}(28) \qquad &(\lambda^T f)(x, v) + x^T z - (\lambda^T f)(u, v) - u^T z + \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v)r \\ &\geqq -u^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r].\end{aligned}$$

Since $-(\lambda^T f)(x,\cdot)+(\cdot)^T w$ is second order invex with respect to η_2 for fixed x,

$$-(\lambda^T f)(x,v)+v^T w+(\lambda^T f)(x,y)-y^T w$$
$$\geqq -\eta_2(v,y)^T[\nabla_y(\lambda^T f)(x,y)-w+\nabla_{yy}(\lambda^T f)(x,y)p]+\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p.$$

From (26) and $\eta_2(v,y)+y\in C_2$,

$$-[\eta_2+y]^T[\nabla_y(\lambda^T f)(x,y)-w+\nabla_{yy}(\lambda^T f)(x,y)p]\geqq 0.$$

So,

$$\begin{aligned}&-(\lambda^T f)(x,v)+v^T w+(\lambda^T f)(x,y)-y^T w-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p\\&\geqq y^T[\nabla_y(\lambda^T f)(x,y)-w+\nabla_{yy}(\lambda^T f)(x,y)p].\end{aligned}\tag{29}$$

Therefore, by (28) and (29),

$$\begin{aligned}&(\lambda^T f)(x,y)+x^T z-y^T[\nabla_y(\lambda^T f)(x,y)\\&\quad+\nabla_{yy}(\lambda^T f)(x,y)p]-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p\\&\geqq(\lambda^T f)(u,v)-v^T w-u^T[\nabla_x(\lambda^T f)(u,v)\\&\quad+\nabla_{xx}(\lambda^T f)(u,v)r]-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r.\end{aligned}$$

Using the fact that $x^T z\leqq s(x|D_i)$ and $v^T w\leqq s(x|E_i)$ for $i=1,\cdots,k$, we get

$$x^T z\leqq\lambda^T s(x|D)\text{and}v^T w\leqq\lambda^T s(v|E).$$

Hence,

$$\begin{aligned}&(\lambda^T f)(x,y)+\lambda^T s(x|D)-y^T[\nabla_y(\lambda^T f)(x,y)\\&\quad+\nabla_{yy}(\lambda^T f)(x,y)p]-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p\\&\geqq(\lambda^T f)(u,v)-\lambda^T s(v|E)-u^T[\nabla_x(\lambda^T f)(u,v)\\&\quad+\nabla_{xx}(\lambda^T f)(u,v)r]-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r.\end{aligned}\tag{30}$$

But suppose that $G(u, v, \lambda, z, r) - K(x, y, \lambda, w, p) \in intK$. Since $\lambda \in K^*$, it yields

$$\begin{aligned}&\bigg[(\lambda^T f)(u, v) - \lambda^T s(v|E) - u^T[\nabla_x(\lambda^T f)(u, v) + \nabla_{xx}(\lambda^T f)(u, v)r] \\ &\qquad\qquad\qquad\qquad -\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u, v)r\bigg] \\ -&\bigg[(\lambda^T f)(x, y) - \lambda^T s(x|D) - y^T[\nabla_y(\lambda^T f)(x, y) + \nabla_{yy}(\lambda^T f)(x, y)p] \\ &\qquad\qquad\qquad\qquad -\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x, y)p\bigg] > 0,\end{aligned}$$

which is a contradiction to the inequality (30). ■

Theorem 4.2. (Strong Duality). *Let* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ *be a K-weakly efficient solution for* (NWP). *Fix* $\lambda = \overline{\lambda}$ *in* **(NWD)**. *Assume that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ *is positive definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \geqq 0$ *or* $\nabla_{yy}\overline{\lambda}^T f$ *is negative definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \leqq 0$,

(ii) $\nabla_y\overline{\lambda}^T f - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) *the set* $\{\nabla_y f_1, \nabla_y f_2, \cdots, \overline{w}\}$ *is linearly independent,*

where $f = f(\overline{x}, \overline{y})$.

Then there exists $\overline{z} \in D_i (i = 1, \cdots, k)$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a feasible solution for* **(NWD)** *and objective values of* **(NWP)** *and* **(NWD)** *are equal. Furthermore, under the assumptions of Theorem 4.1,* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a K-weakly efficient solution for* **(NWD)**.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ is a K-weakly efficient solution for (NWP), by Lemma 3.1, there exist $\alpha \in K^*$, $\beta \in C_2$, $\mu \in \mathbb{R}_+$, $\delta \in C_1^*$, and $\rho \in K$ such that

$$\begin{aligned}&\alpha^T[\nabla_x f + ze] + (\beta - (\alpha^T e)\overline{y})^T\nabla_{yx}(\overline{\lambda}^T f) \\ &\quad +(\beta - (\alpha^T e)\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p})^T\nabla_x(\nabla_{yy}(\lambda^T f))\overline{p} - \delta = 0,\end{aligned} \tag{31}$$

$$\begin{aligned}&(\beta - (\alpha^T e)\overline{y} - (\alpha^T e)\overline{p})^T\nabla_{yy}(\overline{\lambda}^T f) \\ &\quad +(\beta - (\alpha^T e)\overline{y} - (\alpha^T e)\overline{p})^T\nabla_y(\nabla_{yy}(\overline{\lambda}^T f)\overline{p}) = 0,\end{aligned} \tag{32}$$

$$\begin{aligned}&(\alpha^T e)\left[-\overline{y}^T\nabla_y f - \overline{y}^T(\nabla_{yy} f)\overline{p} - \frac{1}{2}\overline{p}^T\nabla_{yy} f\overline{p}\right] \\ &\quad +\beta^T[\nabla_y f + \nabla_{yy} f\overline{p}] - \rho = 0,\end{aligned} \tag{33}$$

$$\beta \in N_{E_i}(\overline{w}), \tag{34}$$

$$(\beta - \alpha^T e\overline{y} - (\alpha^T e)\overline{p})^T \nabla_{yy}(\overline{\lambda}^T f) = 0, \tag{35}$$

$$\beta^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{36}$$

$$\mu\overline{y}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{37}$$

$$\delta^T\overline{x} = 0, \tag{38}$$

$$\rho^T\overline{\lambda} = 0, \tag{39}$$

$$z \in D_i, \;\; z^T\overline{x} = s(\overline{x}|D_i), \;\; i = 1, \cdots, k, \tag{40}$$

$$(\alpha, \beta, \mu, \delta, \rho) \neq 0. \tag{41}$$

By the assumption (i) and (35) yields

$$\beta = (\alpha^T e)(\overline{y} + \overline{p}). \tag{42}$$

If $\alpha = 0$, then (41),(31) and (33) give $\beta = 0, \;\; \delta = 0$ and $\rho = 0$. This contradicts (40). Therefore $\alpha > 0$. Using (41) in (32)

$$\frac{1}{2}(\alpha^T e)\overline{p}^T \nabla_y(\nabla_{yy}(\overline{\lambda}^T f))\overline{p} = 0,$$

which using the assumption (ii) implies

$$\overline{p} = 0.$$

Then (41) implies $\beta = (\alpha^T e)\overline{y}$. So $\overline{y} \in C_2$. Using (42) in (31)

$$\alpha^T(\nabla_x f + ze) = \delta \in C_1^*. \tag{43}$$

Taking $\overline{z} := z \in D_i(i = 1, \cdots, k)$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is feasible for **(NWD)**.
Multiplying (43) by $\overline{x}$ and using (37), we get

$$\overline{x}^T\left[\nabla_x(\overline{\lambda}^T f) - \overline{w}\right] = 0. \tag{44}$$

Consequently, using (44), (45) and (46),

$$\begin{aligned} K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) &= f(\overline{x}, \overline{y}) + s(\overline{x}|D) - (\overline{y}^T \nabla_y((\overline{\lambda}^T f))(\overline{x}, \overline{y}))e \\ &= f(\overline{x}, \overline{y}) - (\overline{z}^T\overline{x})e - (\overline{y}^T\overline{w})e \\ &= f(\overline{x}, \overline{y}) - s(\overline{y}|E) - \overline{x}^T \nabla_x((\overline{\lambda}^T f))(\overline{x}, \overline{y})e \\ &= G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0). \end{aligned}$$

Thus objective values of **(NWP)** and **(NWD)** are equal. We will now show that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a K-weakly efficient solution for **(NWD)**, otherwise there exists a feasible solution $(u, v, \overline{\lambda}, z, r = 0)$ for **(NWD)** such that

$$G(u, v, \overline{\lambda}, z, r = 0) - G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0) \in intK.$$

Since objective values of **(NWP)** and **(NWD)** are equal.

$$G(u, v, \overline{\lambda}, z, r = 0) - K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) \in intK,$$

which contradits weak duality theorem. Hence the result holds. ∎

Theorem 4.3. (Converse Duality). *Let* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ *be a K-weakly efficient solution for* (NWD). *Fix* $\lambda = \overline{\lambda}$ *in* **(NWP)**. *Assume that*

(i) $\nabla_{xx}(\overline{\lambda}^T f)$ *is positive definite and* $\overline{r}^T[\nabla_x(\overline{\lambda}^T f) - \overline{z}] \geqq 0$ *or* $\nabla_{xx}(\overline{\lambda}^T f)$ *is negative definite and* $\overline{r}^T[\nabla_x(\overline{\lambda}^T f) - \overline{z}] \leqq 0$,

(ii) $\nabla_x(\overline{\lambda}^T f) - \overline{z} + \nabla_{xx}(\overline{\lambda}^T f)\overline{r} \neq 0$,

(iii) *the set* $\{\nabla_x f_1, \nabla_z f_2, \cdots, \overline{z}\}$ *is linearly independent,*

where $f = f(\overline{u}, \overline{v})$.

Then there exists $\overline{w} \in E_i(i = 1, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution for* **(NWP)** *and objective values of* **(NWP)** *and* **(NWD)** *are equal. Furthermore, under the assumptions of Theorem 4.1,* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a K-weakly efficient solution for* **(NWP)**.

Proof. It follows on the lines of Theorem 4.2. ∎

5. Special Cases

We give some special cases of our symmetric duality.

First of all, if $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then our results reduce to the following programming problems.

(1) Our problems $(\mathbf{NMP})$ and $(\mathbf{NMD})$ become the pair of Mond-Weir symmetric dual programs considered in X.M. Yang et al.[22] for the same B and D.

(2) If $k = 1$, then **(NMP)** and **(NMD)** are redued to the second order symmetric dual programs in Hou and Yang [8].

(3) Let $D \in \mathbb{R}^n \times \mathbb{R}^n$ and $E \in \mathbb{R}^m \times \mathbb{R}^m$ are positive semidifinite symmetric matrices. If $s(x|B) = (x^T Dx)^{\frac{1}{2}}$ where $B = \{Dz|z^T Dz \leqq 1\}$ and $s(y|C) = (y^T Ey)^{\frac{1}{2}}$ where $C = \{Ew|w^T ew \leqq 1\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then **(NMP)** and **(NMD)** become nondifferentiable second order symmetric duality in multiobjective programming in Ahmad and Husain [1].

(4) Let $D \in \mathbb{R}^n \times \mathbb{R}^n$ and $E \in \mathbb{R}^m \times \mathbb{R}^m$ are positive semidifinite symmetric matrices. If $s(x|B) = (x^T Dx)^{\frac{1}{2}}$ where $B = \{Dz|z^T Dz \leqq 1\}$ and $s(y|C) = (y^T Ey)^{\frac{1}{2}}$ where $C = \{Ew|w^T ew \leqq 1\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then **(NWP)** and **(NWD)** is reduced to nondifferentiable second order symmetric duality in multiobjective programming. In addition, if $k = 1$, then we get second order symmetric dual programs on nondifferentiable studied by Ahmad and Husain [2].

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