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SEMISYMMETRIZATIONS OF ABELIAN GROUP ISOTOPES

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Abstract. This note begins a study of the structure of quasigroup semisymmetrizations. For the class of quasigroups isotopic to abelian groups, a fairly complete description is available. The multiplication group is the split extension of the cube of the abelian group by a cyclic group whose order is identified as the semisymmetric index of the quasigroup. For a finite abelian group isotope, the dual of the semisymmetrization is isomorphic to the opposite of the semisymmetrization. The character table of the semisymmetrization is readily computed. The simplicity question for semisymmetrizations is raised. It is shown that a simple, non-abelian quasigroup need not have a simple semisymmetrization.

1. Introduction

Given a quasigroup $(Q,\cdot,/,\backslash)$, there is a semisymmetric quasigroup structure Q^{Δ}, with multiplication

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3)=(y_3/x_2,y_1\backslash x_3,x_1\cdot y_2)\,, \tag{1.1}$$

on the cube Q^3 of the underlying set Q. The semisymmetric quasigroup Q^{Δ} is known as the *semisymmetrization* of Q. The semisymmetrization is an isotopy invariant: two quasigroups are isotopic if and only if their semisymmetrizations are isomorphic [5]. Thus the semisymmetrization is a purely algebraic object playing the role that previously required use of the geometrical constructions variously described as 3-nets or 3-webs [1]. Because of their significance, semisymmetrizations warrant closer study from various directions. In this note, we begin to examine the structure of semisymmetrizations. Although the main focus is on semisymmetrizations of

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abelian groups and their isotopes, we also touch briefly on some aspects of the general case. Section 3 investigates the relationship between the simplicity of a quasigroup and the simplicity of its semisymmetrization. It is shown that a simple quasigroup may have a semisymmetrization which is not simple.

For an abelian group isotope Q, the semisymmetrization Q^{Δ} is a central pique. The final two sections apply the results of the paper [6] to this central pique. Section 4 investigates the corresponding dual. In the finite case, the dual is isomorphic to the opposite of the primal. For a finite central pique, [6, Th. 7.1] described the character table. Section 5 illustrates the application of this result to the computation of character tables of abelian group semisymmetrizations, using the smallest non-trivial semisymmetrization $\mathbb{Z}/2\mathbb{Z}^{\Delta}$ as a model. This particular example has the combinatorial structure of the Hamming cube.

For the standard concepts of quasigroup theory, readers are referred to [1] and [7].

2. Semisymmetrized Abelian Groups

Let $(A,+)$ be an abelian group. As an equationally defined quasigroup, it takes the form $(A,+,-,\backsim)$ with the left division $x \backsim y = y - x$ being the opposite of subtraction. With elements of A^3 as row vectors, the multiplication (**??**) of the semisymmetrization A^{Δ} becomes

$$[x_1,x_2,x_3]\cdot[y_1,y_2,y_3] = [y_3 - x_2, x_3 - y_1, x_1 + y_2]\,.$$

Using matrix notation, this may be written as

$$[x_1,x_2,x_3]\mathrm{P} + [y_1,y_2,y_3]\mathrm{P}^{-1} \tag{2.1}$$

with

$$\mathrm{P} = R_{A^{\Delta}}([0,0,0]) = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\,. \tag{2.2}$$

Definition 2.1. Let A be an abelian group. Then the *semisymmetric index* f of A is defined as follows:

$$f = \begin{cases} 1 & \text{if } A \text{ is trivial;} \\ 3 & \text{if } A \text{ has exponent } 2; \\ 6 & \text{otherwise.} \end{cases}$$

The semisymmetric index of an abelian group A is the order of the automorphism P of A^3. An application of [6, Th. 2.3] yields:

Theorem 2.2. *Let Q be a quasigroup isotopic to an abelian group A of semisymmetric index f. Then the multiplication group $\mathrm{Mlt}\,Q^{\Delta}$ of the semisymmetrization Q^{Δ} of Q is isomorphic to the split extension of the abelian group A^{3} by a cyclic group of order f.*

3. Simplicity Questions

Let A be an abelian group. Consider the subgroup

$$N = \{[a, a, -a] \mid a \in A\} \tag{3.1}$$

of A^3. Note that N is an eigenspace of P with eigenvalue -1. Since N is an invariant subspace under P, it forms a normal subpique of A^{Δ}. The congruence relation ν on A^{Δ} determined by N, namely

$$[x_1, x_2, x_3]\,\nu\,[y_1, y_2, y_3] \quad \Leftrightarrow \quad y_1 - x_1 = y_2 - x_2 = x_3 - y_3\,,$$

is called the *characteristic congruence* of A^{Δ}. It is relevant to the simplicity question for semisymmetrizations.

Theorem 3.1. *The semisymmetrization Q^{Δ} of a non-trivial abelian group isotope Q is not simple. In particular, a simple, non-abelian quasigroup may have a semisymmetrization which is not simple.*

Proof. If Q is isotopic to a non-trivial abelian group A, then Q^{Δ} is isomorphic to the semisymmetrization A^{Δ} which has a proper, non-trivial characteristic congruence.

Consider the set $\mathbb{Z}/n\mathbb{Z}$ for $n > 2$, equipped with the multiplication $x \cdot y = x^R + y^L$ for a pair of permutations R and L that generate the full symmetric group $(\mathbb{Z}/n\mathbb{Z})!$. Then $(\mathbb{Z}/n\mathbb{Z}, \cdot)$ is a quasigroup, with $(\mathbb{Z}/n\mathbb{Z})!$ as multiplication group (compare [3]). Since $(\mathbb{Z}/n\mathbb{Z})!$ acts primitively on $\mathbb{Z}/n\mathbb{Z}$, the non-trivial abelian group isotope $(\mathbb{Z}/n\mathbb{Z}, \cdot)$ is simple [8, I Cor.2.4.2]. ∎

Proposition 3.2. *Let Q be a quasigroup. If the semisymmetrization Q^{Δ} is simple, then so is Q itself.*

Proof. Suppose that Q^{Δ} is simple. If Q is not simple, then there is a non-constant, non-injective quasigroup homomorphism f with domain Q. This in turn would yield a non-constant, non-injective quasigroup homomorphism f^{Δ} with the semisymmetrization Q^{Δ} as its domain, contradicting the simplicity of Q^{Δ}. ∎

4. Duals of Semisymmetrizations

Let Q be a central pique, with multiplication $x \cdot y = xR + yL$ for automorphisms R and L of an abelian group $(A, +)$. Let $(\widehat{A}, +)$ be the abelian group of characters of A. Recall that a *character* may be viewed as an abelian group homomorphism $\xi : A \to \mathbb{R}/\mathbb{Z}$. The group structure on $(\widehat{A}, +)$ is pointwise, so that $a(\xi + \eta) = a\xi + a\eta$ for a in A and ξ, η in $\widehat{A}$. The *dual pique* $\widehat{Q}$ is the set $\widehat{A}$ equipped with the product $\xi \cdot \eta = R\xi + L\eta$, the left action of R and L on $\widehat{A}$ being given by the mixed associative laws $a(R\xi) = (aR)\xi$ and $a(L\xi) = (aL)\xi$ for a in A and ξ in $\widehat{A}$ [6]. Duals of semisymmetrizations of finite abelian groups and their isotopes are specified as follows.

Theorem 4.1. *Let Q be a finite abelian group isotope. Then the semisymmetrization Q^{Δ} is isomorphic to the opposite of its dual.*

Proof. Let A be the abelian group isotopic to Q. Writing the elements of A^3 as row vectors, the elements of the dual $\widehat{A^3}$ may be written as column vectors from $(\widehat{A})^3$, so that the value of a character $[\xi_1, \xi_2, \xi_3]^T$ from $(\widehat{A})^3$ on an element $[x_1, x_2, x_3]$ from A^3 is $[x_1, x_2, x_3][\xi_1, \xi_2, \xi_3]^T = \sum_{j=1}^{3} x_j \xi_j$ in $\mathbb{R}/\mathbb{Z}$. Since the abelian group A is finite, it possesses an (unnatural) isomorphism $\theta : A \to \widehat{A}$ with its dual [2, V.6.4(a)]. The linear isomorphism

$$\Theta : A^3 \to (\widehat{A})^3; [x_1, x_2, x_3] \mapsto [x_1\theta, x_2\theta, x_3\theta]^T$$

gives the required isomorphism from A^{Δ} to the opposite of its dual. ∎

Note that Theorem 4.1 does not extend to infinite quasigroups. For example, the dual of the countable semisymmetrization $\mathbb{Z}^{\Delta}$ is the uncountable 3-torus T^3.

5. The Smallest Semisymmetrization

In this section, the method of [6, §7] is used to compute the character table of the smallest non-trivial semisymmetrization $(\mathbb{Z}/2\mathbb{Z})^{\Delta}$. The computation may serve as a typical model for the computation of the character table of the semisymmetrization of any finite abelian group isotope. This particular example is interesting from the combinatorial point of view, since it exhibits the structure of the Hamming cube.

Elements of $(\mathbb{Z}/2\mathbb{Z})^3$ are written as row vectors or words of length 3 in the binary alphabet $\{0, 1\}$. The orbits of P on $(\mathbb{Z}/2\mathbb{Z})^3$ are the sets of words of given Hamming weight. The orbits are the pique conjugacy classes D_0, D_1, D_2, D_3, where D_w demotes the set of words of Hamming weight w. The pique conjugacy classes of the dual are Δ_0, Δ_1, Δ_2, Δ_3, where Δ_w denotes the set of column vectors of Hamming weight w. The multiplicities n_i and valencies f_j are given by $n_0 = f_0 = 1$, $n_1 = f_1 = 3$, $n_2 = f_2 = 1$, $n_3 = f_3 = 1$. The character table of the semisymmetrization, under the standard quasigroup normalization [4] (extending the usual normalization for group character tables) is given by Table 1 according to [6, Th. 7.1]. Note that the subpique N of (3.1) is the Hamming code $D_0 \cup D_3$ (in this case a repetition code) correcting single errors in the binary channel of length 3. The characteristic congruence ν may be considered as the kernel of the basic character ψ_3.

Table 1. Character table of $(\mathbb{Z}/2\mathbb{Z})^{\Delta}$

$(\mathbb{Z}/2\mathbb{Z})^{\Delta}$	D_0	D_1	D_2	D_3
ψ_1	1	1	1	1
ψ_2	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$
ψ_3	$\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$
ψ_4	1	-1	1	-1

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