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BASIC APPELL SEQUENCES

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Abstract. We study a subclass of Sheffer sequences denominated Appell sequences. Using methods of the so-called umbral calculus in conjunction with general results of nuclear Fréchet spaces and the theory of Köthe sequence spaces, we give necessary and sufficient conditions for an Appell sequence to be a basis.

0. Introduction

Sequences of polynomials play an important role in mathematics and physics. One of the simplest classes of polynomial sequences is the class of Sheffer sequences that contains relevant sequences such as the Laguerre, the Hermite, the Bernoulli and the Abel polynomials. The systematic study of the class of Sheffer sequences is the object of the modern classical umbral calculus started in the 1970s by Gian-Carlo Rota and his disciples [23]. Following the theory constructed by this school, a sequence of polynomials $s_n(x)$, $\deg s_n(x) = n$, related to a pair of formal series $f(t)$ and $g(t)$ by some orthogonality conditions, is called the Sheffer sequence for the pair $(g(t), f(t))$ [23]. The Sheffer sequence for $(g(t), t)$ is called the Appell sequence for $g(t)$. Although the umbral calculus is formal mathematics the object of this paper is an approximation problem involving, naturally, the convergence of infinite series. Given an invertible formal series $g(t)$ we study necessary and sufficient conditions for the Appell sequence $s_n(x)$ to be a basis in a Köthe space with its natural topology.

Several authors have dealt with similar problems, for instance, Abul-Ez [1], Boas and Buck [2], Buckholtz [4], Dragilev [6], Everitt [7], Frank and Shaw [8], Grabiner [9], Haslinger [10-12], Ibragimov [13], Kakeya [14], Kaz'min [15, 16], Nagnibida [19], Peherstorfer [20], Totik [24], Van Assche [25].

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1. Basic Results and Terminology

Let $g(t)$ be a a formal power series in the variable t over the field $\mathbb{C}$, that is

$$g(t) = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n.$$

If $b_0 \neq 0$, $g(t)$ has a formal inverse $g^{-1}(t)$ and it is called an invertible series. If $\mathcal{P}$ denotes the algebra of polynomials in the single variable x over the field $\mathbb{C}$ and $\mathcal{F}$ the algebra of formal power series in the variable t, then any element of $\mathcal{F}$ defines a linear functional on $\mathcal{P}$ by setting

$$\langle g(t)/x^n \rangle = b_n, \qquad \text{for all } n \geq 0.$$

A sequence of polynomials $\{s_n(x)\}$ is called the Appell sequence for $g(t)$ if it satisfies

$$\left\langle g(t)t^k / s_n(x) \right\rangle = n!\delta_{n,k} \text{ for all } n, k \geq 0. \tag{1}$$

Very well-known classes of Appell polynomials are the Hermite, Bernoulli and Euler polynomials, that is the sequence of polynomials corresponding to the invertible series $g(t) = e^{t^2/2}$, $g(t) = (e^t - 1)/t$ and $g(t) = \frac{1}{2}(e^t + 1)$ respectively [23].

Concerning the theory of Köthe spaces let us state the definitions and some basic results. Denote by $\lambda^1(A)$ the Köthe (echelon) space given by the matrix $A = (a_n^k)$, $n, k = 0, 1, 2, \ldots$, $a_n^k > 0$, $a_n^k \leq a_n^{k+1}$, for all k, n, that is

$$\lambda^1(A) = \left\{ (x_n),\, x_n \in \mathbb{C},\, \|(x_n)\|_k = \sum_{n=0}^{\infty} |x_n|\, a_n^k < \infty,\, \forall k = 0, 1, 2, \ldots \right\}.$$

The canonical basis in the Fréchet space $\lambda^1(A)$ is denoted by $\delta_n = (\delta_{n,k})$ and φ is the space $\varphi = span\,\{\delta_n : n \in \mathbb{N}_0\}$.

The dual space of $\lambda^1(A)$ is

$$(\lambda^1(A))' = \left\{ (x_n),\, x_n \in \mathbb{C},\, \sup_{n \geq 0} \left(\frac{|x_n|}{a_n^k} \right) < \infty, \text{ for a suitable } k \right\}$$

and the coordinate operators are continuous [17, 18].

The space $\lambda^1(A)$ is nuclear if and only if

$$\forall\, k,\, \exists\, r = r(k) \text{ such that } \left(\frac{a_n^k}{a_n^r} \right) \in \ell^1 \qquad [18].$$

Recall that a Köthe space $\lambda^1(A)$ is called an infinite power series space $\Lambda_\infty(\alpha)$ if the matrix $(a_n^k) = (e^{k\alpha_n})$, where α_n is an increasing sequence of positive numbers going to infinity. Among them the space of entire functions $\mathcal{F}_R$, $R = \infty$ and

$$s = \left\{(x_n) : \lim |x_n| n^k = 0, \forall k \in \mathbb{N}\right\} = \left\{(x_n) : \sum |x_n| e^{k\log(n)} < \infty, \forall k\right\},$$

are well-known. If $A = \left(e^{-\frac{1}{k}\alpha_n}\right)$, the Köthe space $\lambda^1(A)$ is called a finite power series sequence space, being the space of holomorphic functions on a finite disc (all such spaces are isomorphic) an important example [18].

2. Appell Basis in Köthe Spaces

Now we state the main results of the paper due to the joint work of the three authors.

Let $g(t)$ be an invertible series and $\{s_n(x)\}$ its corresponding Appell sequence. Then we have the following result

Theorem 2.1. *Let $\lambda^1(A)$ be a nuclear Köthe space. Assume that the invertible series $g(t)$ verifies: for each $n \in \mathbb{N}$, there exists a $k = k(n) \in \mathbb{N}$, such that*

$$\sup_{m\geq n}\left\{\binom{m}{n}\frac{|b_{m-n}|}{a_m^k}\right\} < \infty. \tag{2}$$

Then the Appell polynomials $\{s_n(x)\}$ constitute a basis in $\lambda^1(A)$ if and only if the following condition is satisfied: for each $k \in \mathbb{N}$, there exists a number $r = r(k) \in \mathbb{N}$ such that

$$\sup_n\left\{\sup_{m\geq n}\binom{m}{n}\frac{|b_{m-n}|}{a_m^r}\sum_{j=0}^{n} a_j^k\binom{n}{j}\left|\langle g(t)^{-1}/x^{n-j}\rangle\right|\right\} < \infty \tag{3}$$

Proof. As a consequence of (1), $\left\{s_n(x), \frac{g(t)}{n!}t^n\right\}$ is a biorthogonal system if the functionals given by the series $\frac{g(t)}{n!}t^n$ are continuous; as the monomials $x^n = \delta_n$ are a basis in any Köthe space $\lambda^1(A)$ then the biorthogonal system is complete. Recall that the functional given by a formal series $h(t) = \sum_{n=0}^{\infty}\frac{c_n}{n!}t^n$ is continuous if and only if the sequence $(c_n) \in (\lambda^1(A))'$ and therefore by condition (2) the functionals $\frac{g(t)t^n}{n!}$, $n \in \mathbb{N}$, are continuous.

From [10, proposition 2] a complete biorthogonal system $\xi_n = (\xi_n^p)$, $\eta_n = (\eta_n^p)$, constitutes a basis in the nuclear Köthe space $\lambda^1(A)$ if and only if, for each $k \in \mathbb{N}$, there exists a number $r = r(k) \in \mathbb{N}$ such that

$$\sup_{n,m\geq 0} \left\{ \frac{|\eta_n^m|}{a_m^r} \sum_{j=0}^{\infty} \left|\xi_n^j\right| a_j^k \right\} < \infty.$$

As

$$\frac{g(t)t^n}{n!} = \sum_{j=0}^{\infty} \binom{n+j}{j} b_j \frac{t^{j+n}}{(n+j)!},$$

the result follows from [23, Theorem 2.5.4]. ∎

A formal power series g in the variable t can be considered as a linear operator on $\mathcal{P}$ [23], using the notation t^k for the kth derivative operator on $\mathcal{P}$, that is

$$g(t)x^n = \sum_{s=0}^{n} \binom{n}{s} b_s x^{n-s}.$$

The juxtaposition $g(t)p(x)$ denote the action of the operator $g(t)$ on the polynomial $p(x)$. It is, therefore, important to notice that any formal power series $g(t)$ behaves, algebraically, as an operator invariant by differentiation, that is, $Tt = tT$. The operator t can be extended to any Köthe space $\lambda^1(A)$ if and only if

$$\forall k,\ \exists N(k) \text{ such that } n a_{n-1}^k \leq a_n^{N(k)} \text{ for all } n.$$

The previous observations lead, in a natural way, to the following

Theorem 2.2. *Let $g(t)$ be an invertible formal power series and T the corresponding invariant-differentiation operator. Assume that the condition: $\forall k$, $\exists\, N(k)$, $\exists\, C(k)$ such that*

$$\sum_{s=0}^{n} \binom{n}{s} |b_s|\, a_{n-s}^k \leq C(k) a_n^{N(k)}, \text{ for all } n, \tag{4}$$

where $A = (a_n^k)$ is a Köthe matrix, is satisfied. Then the Appell sequence $\{s_n(x)\}$ for $g(t)$ is a basis in $\lambda^1(A)$ if and only if the operator T is an isomorphism from $\lambda^1(A)$ onto $\lambda^1(A)$.

Proof. Given the formal power series

$$g(t) = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n$$

condition (4) implies that the operator T

$$T = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n$$

is continuous.

Assume that T is an isomorphism from $\lambda^1(A)$ onto $\lambda^1(A)$. Then the inverse operator T^{-1} of T is an invariant differentiation operator

$$T^{-1} = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

where the coefficients (c_n) are given by the series $g^{-1}(t)$.

As T^{-1} is, obviously, an isomorphism and $T^{-1}x^n = s_n(x)$ by [23, Theorem 2.5.5], it follows that the polynomials $\{s_n(x)\}$ are a basis in $\lambda^1(A)$.

The converse is obvious. ■

Let us consider now infinite power series spaces and assume that the differentiation operator t is continuous.

As it is well-known there is a strong relationship between the isomorphisms invariant by differentiation and the eigenvalues and eigenvectors of the differentiation operator t [5, 19, 21, 22, 26]. In the spaces $\mathcal{F}_R$, $0 < R \leq \infty$, all complex numbers μ are eigenvalues of t and the corresponding exponential functions $e^{\mu x}$ (that constitute a dense set) are eigenfunctions.

Theorem 2.3. *Let $\Lambda_\infty(\alpha)$ be an infinite power series space such that*

$$\limsup_n \left(\frac{e^{k\alpha_n}}{n!} \right)^{\frac{1}{n}} = 0, \quad \textit{for every } k \tag{5}$$

and let $g(t) = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n$ be an invertible formal power series such that

$$\exists C > 0, \exists M > 0 \text{ such that } \sum_{s=0}^{n} |b_s| \binom{n}{s} \leq C e^{M\alpha_n}, \text{ for all } n \tag{6}$$

Then the Appell polynomials $\{s_n(x)\}$ given by $g(t)$ constitute a basis in $\Lambda_\infty(\alpha)$ if and only if the formal power series $g(t)$ is of the form $g(t) = e^{\,at+b}$, a and b constants.

Proof. Assume that $\{s_n(x)\}$ is an Appell basis on $\Lambda_\infty(\alpha)$. Then the linear operator T given by $g(t)$ which is continuous by condition (6) is an isomorphism (commuting with differentiation) from $\Lambda_\infty(\alpha)$ onto $\Lambda_\infty(\alpha)$. As

$$\limsup_n \left(\frac{e^{k\alpha_n}}{n!}\right)^{\frac{1}{n}} = 0, \quad \text{for every } k$$

or, equivalently, all complex numbers are eigenvalues of t [21], the function

$$g(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$$

is entire without zeros, that is $g(z) = e^{f(z)}$, where $f(z)$ is an entire function. If ρ is the order of $g(z)$ the condition

$$\lim \frac{\log(n^n)}{\log(n!)} = 1$$

implies that $\rho \leq 1$ [21] and, therefore, the operator $T = e^{P(t)}$, where $P(t)$ is a polynomial such that $\deg(P(t)) \leq 1$.

Conversely, if the operator

$$T = e^{at+b} = e^b \sum_{n=0}^{\infty} \frac{a^n}{n!} t^n$$

is continuous then it is an isomorphism and the polynomials

$$s_n(x) = g^{-1} x^n = e^{-b} \sum_{m=0}^{\infty} \binom{n}{m} (-a)^m x^{n-m} = e^{-b}(x-a)^n$$

are a basis on $\Lambda_\infty(\alpha)$. ■

Theorem 2.4. *Suppose that $\Lambda_\infty(\alpha)$ is an infinite power series space such that*

$$\lim_n \left(\frac{e^{\alpha_n}}{n!}\right)^{\frac{1}{n}} = \infty$$

and let $g(t) = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^n$ be an invertible formal power series such that the complex function $g(z) = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n$ is holomorphic in a neighborhood of zero. Then the Appell polynomials corresponding to $g(t)$ constitute a basis on $\Lambda_\infty(\alpha)$.

Proof. Consider the invertible power series $g(t)$ and its formal inverse $g(t)^{-1}$ and let T and T^{-1} be the corresponding operators. T is continuous on $\Lambda_\infty(\alpha)$ if and only if the following condition holds [21, proposition 2]

$$\exists\, C > 0, \exists\, M > 0, \text{ such that } |b_k| \leq C e^{M\alpha_n}, \text{ for all } n.$$

Using the Cauchy's inequalities and the assumption on (α_n) the previous condition is fulfilled. Therefore T is continuous.

Proceeding in analogous way T^{-1} is continuous too (note that the complex function given by $g(t)^{-1}$ is, as well, holomorphic in a neighborhood of zero). As T is then an isomorphism the result follows. ■

Corollary 2.5. *The Hermite, Bernoulli and Euler polynomials ($H_m(x)$, $B_m(x)$ and $E_m(x)$ respectively) constitute a basis in any infinite power series space $\Lambda_\infty(\alpha_n)$ with the condition*

$$\lim_n \left(\frac{e^{\alpha_n}}{n!}\right)^{\frac{1}{n}} = \infty.$$

Example 2.6. Take $\alpha_n = n^2$. Any sequence of the space $\Lambda_\infty(n^2)$ can be identified with an entire function, so we write $\Lambda_\infty(n^2) \subset \mathcal{F}_R$, $R = \infty$, but topologically speaking $\Lambda_\infty(n^2)$ is not a subspace of $\mathcal{F}_R$, $R = \infty$, as the natural topology of $\Lambda_\infty(n^2)$ is finer that the open-compact topology. Anyway any entire function $f(z) = \sum a_n z^n$ where $(a_n) \in \ell^1(e^{kn^2})$ for every k, is the limit of a sequence

$$\sum_{m=0}^{n} c_m H_m(x)$$

or

$$\sum_{m=0}^{n} c_m B_m(x)$$

or

$$\sum_{m=0}^{n} c_m E_m(x),$$

in the usual topology of $\Lambda_\infty(n^2)$ and consequently in the open-compact topology (see [8] for similar results for Abel-Gončarov polynomial expansions).

3. Remarks

We include in this section a compilation of nontrivial remarks and observations.

With reference to theorem 2.1 we have the following remark:

Remark 3.1. It is possible to prove, as in the case of Gončarov polynomials [12], that if a sequence of Appell polynomials is a basis in $\mathcal{F}_{R_1}$ is also a basis in the space $\mathcal{F}_{R_2}$, $R_2 > R_1$. In [12], given a sequence (z_k) of complex numbers such that the corresponding Gončarov polynomials constitute a basis in a space $\mathcal{F}_R$, $R < \infty$, a constant $W((z_k))$ is defined. In the same way, for any invertible formal power series $g(t)$ satisfying (2) and any space $\mathcal{F}_{R_1}$ we define

$$W(g) = \inf \{R \text{ such that condition (3) is valid } \}$$

and so if $W(g) > 0$ there exists a function holomorphic in a disc of radius R^*, $0 < R^* < W(g)$ which is not representable in the series of Appell polynomials corresponding to $g(t)$. In connection with these problems see [6] for a necessary and sufficient condition for a sequence of polynomials (p_n) of degree $n = 0, 1, 2, \dots$ to be a basis in all spaces $\mathcal{F}_r$, $r > R$, where R is a fixed positive number.

In relation to theorem 2.3 we state the next remark

Remark 3.2. The operator e^t is well-defined from φ to φ but for the extension to $\Lambda_\infty(\alpha)$ is necessary the condition

$$\exists\, M > 0 \ \text{ such that } \sup_n \left\{ \frac{2^n}{e^{M\alpha_n}} \right\} < \infty.$$

Note that in the space s condition (5) is satisfied but the operator e^t cannot be extended to s. There are many examples of infinite power series spaces satisfying condition (5) and e^t giving a continuous linear operator, for instance the space of entire functions. We get another simple example taking $\alpha_n = n\log(\log(n))$, n large enough. In $\mathcal{F}_R$, $R < \infty$, the only Appell basis corresponding to a continuous linear operator $g(t)$ is the canonical one x^n [19, cor. 20].

Let us now make some comments about the existence of Appell bases in certain weighted Banach spaces.

Remark 3.3. Consider a sequence (w_n) of positive numbers such that $w_{n+m} \leq w_n w_m$, $\forall n, m$. As it is well-known the weighted Banach space $\ell^1(w_n)$ is a Banach algebra [3], dual to the weighted Banach space $c_0(\frac{n!}{w_n})$. The characterization of differentiation-invariant isomorphisms from $c_0\left(\frac{n!}{w_n}\right)$ to $c_0\left(\frac{n!}{w_n}\right)$ given in [22] states that any $g(t) = e^{f(t)}$, $f(t) \in \ell^1(w_n)$, (or equivalently any $g(t) \in \ell^1(w_n)$ such that the function $g(z) \neq 0$ for all $|z| \leq \rho$, $\rho = \lim(w_n)^{1/n}$) gives an isomorphism and, consequently, the corresponding Appell polynomials are a basis. Then we can easily find examples of basis. For instance, taking

$$g(t) = \frac{1}{2+t}$$

the corresponding polynomials

$$s_n(x) = 2x^n + nx^{n-1}$$

are a basis in the space $c_0(n!)$. Using a matrix representation the basis is given by

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We get a simple example of a basis on the space $c_0\left(\frac{n!}{w_n}\right)$ taking $g(t) = e^{-t}$, so the polynomials

$$s_n(x) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} = (1+x)^n$$

are a basis and using a matrix representation we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-1} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A larger class of sequences (w_n) can be considered. For instance taking $\left(\frac{w_{n+1}}{w_n}\right)$ increasing and bounded such that $\lim\left(\frac{w_{n+1}}{w_n}\right) = A$, then the Appell sequence $\{s_n(x)\}$ corresponding to any element $g(t) \in \exp\left(\ell^1(A^n)\right)$ is a basis in $c_0\left(\frac{n!}{w_n}\right)$ [22]. For more sophisticated conditions we refer the interested reader to [22].

Remark 3.4. Finally let us point out that using the Gel'fond -Leont'ev derivative D, that is

$$Dx^n = d_n x^{n-1}$$

where (d_n) denotes a nondecreasing sequence of positive numbers and the nonclassical umbral calculi, that is, taking a sequence of nonzero constants $c_n$such that $n!$ is replaced by c_n [23], similar results for Appell sequences to be basis on Köthe spaces can be derived.

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