TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 4, pp. 989-997, September 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

SOME SHARP UPPER BOUNDS ON THE SPECTRAL RADIUS OF GRAPHS

Lihua Feng, Qiao Li and Xiao-Dong Zhang

Abstract. In this paper, we first give a relation between the adjacency spectral radius and the Q-spectral radius of a graph. Then using this result, we further give some new sharp upper bounds on the adjacency spectral radius of a graph in terms of degrees and the average 2-degrees of vertices. Some known results are also obtained.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. For $v \in V$, the degree of v is denoted by d_v, the average 2-degree of v, denoted by m_v, equals to $(\sum_{uv \in E(G)} d_u)/d_v$. Then $d_v m_v$ is the 2-degree of v. For two vertex u and v, we use $u \sim v$ to mean that these two vertices are adjacent. Let $A(G)$ be the *adjacency matrix* of G and $D(G)$ be the diagonal degree matrix of G, respectively. We call the matrix $L(G) = D(G) - A(G)$ the *Laplacian matrix* of G, while call the matrix $Q(G) = D(G) + A(G)$ the *Q-matrix* of G. We denote the largest eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ by $\rho(G)$, $\lambda(G)$ and $\mu(G)$, respectively, and call them the adjacency spectral radius, the Laplacian spectral radius, the Q-spectral radius of G, respectively.

If X is a real symmetric matrix, its eigenvalues must be real, and the largest one is denoted in this paper by $\rho(X)$. It follows immediately that if G is a simple graph, then $A(G)$, $L(G)$ and $Q(G)$ are symmetric matrices, moreover $A(G)$ and $Q(G)$ are irreducible nonnegative matrices.

Let $K = K(G)$ be a vertex-edge incidence matrix of G. Thus $Q(G) = D(G) + A(G) = KK^t$ and $K^tK = 2I_m + A(L_G)$, where L_G is the line graph of G. Since KK^t and K^tK have the same nonzero eigenvalues, we can get that

$$\mu(G) = 2 + \rho(L_G).$$

Received June 6, 2005, accepted October 24, 2005.
Communicated by Xuding Zhu.
2000 *Mathematics Subject Classification*: 05C50.
Key words and phrases: Adjacency matrix, Q-matrix, Line graph, Spectral radius.
Supported by National Natural Science Foundation of China (No. 10371075 and No. 10531070).

For other terminologies, we follow [1] and [6].

Recently, the following results on the spectral radius were obtained:

(i) [10] Let G be a simple connected graph with n vertices, the maximum degree Δ and the second largest degree Δ', here $\Delta \neq \Delta'$. If there are p vertices with degree Δ , then

$$\rho(G) \leq \frac{\Delta' - 1 + \sqrt{(\Delta' + 1)^2 + 4p(\Delta - \Delta')}}{2}. \tag{1}$$

The equality holds if and only if G is Δ regular or $G \cong K_p \oplus H$, the join of the two graphs, H is a $(\Delta' - p)$ regular graph with $n - p$ vertices.

(ii) [2] For any connected graph, we have

$$\rho(G) \leq \sqrt{2m - \delta(n-1) + \Delta(\delta - 1)}. \tag{2}$$

The equality holds if and only if G is regular or a star plus copies of K_2 , or a complete graph plus a regular graph with smaller degree of vertices.

(iii) [2] For any connected graph, we have

$$\rho(G) \leq max_{u \in V} \sqrt{d_u m_u}. \tag{3}$$

The equality holds if and only if G is a regular graph or a semiregular bipartite graph.

Recently, Shu et al. got the following relation between $\lambda(G)$ and $\mu(G)$.

Lemma 1.1. [9] *For a connected graph G, we have*

$$\lambda(G) \leq \mu(G),$$

with equality if and only if G is bipartite.

This paper is organized as follows. In Section 2, we present a relation between $\rho(G)$ and $\mu(G)$. In Section 3, we give some sharp upper bounds on the adjacency spectral radius by using the results included in Section 2.

2. Some Lemmas

Lemma 2.1. [12] *Let G be a simple connected graph and L_G be the line graph of G, then the spectral radius of L_G satisfies*

$$\rho(L_G) \leq max_{u \sim v} \left(\sqrt{d_u(d_u + m_u) + d_v(d_v + m_v)} - 2 \right),$$

the equality holds if and only if G is regular or semiregular.

Lemma 2.2. *Let G be a connected graph, then $\rho(G) \leq \frac{1}{2}\mu(G)$, and the equality holds if and only if G is regular.*

Proof. Let $X = (x_1, x_2, ..., x_n)^t \in R^n$ be a unit eigenvector that belongs to $\rho(G)$, $X' = (x'_1, x'_2, ..., x'_n) \in R^n$ be a unit eigenvector that belongs to $\mu(G)$.

First, we will show that $\rho(G) \leq \frac{1}{2}\mu(G)$. Since

$$\mu(G) = X'^t(D + A)X' = \sum_{v_i \sim v_j, i<j} (x'_i + x'_j)^2,$$

while

$$\begin{aligned}\rho(G) &= X^tAX = \sum_{v_i \sim v_j, i<j} 2x_ix_j \\ &\leq \frac{1}{2}\sum_{v_i \sim v_j, i<j} (x_i + x_j)^2 \\ &\leq \frac{1}{2}\sum_{v_i \sim v_j, i<j} (x'_i + x'_j)^2 \\ &= \frac{1}{2}\mu(G).\end{aligned}$$

So we get the desired inequality.

If the equality holds, all the inequalities above must be equalities. So we have

$$\rho(G) = \sum_{v_i \sim v_j, i<j} 2x_ix_j = \frac{1}{2}\sum_{v_i \sim v_j, i<j} (x_i + x_j)^2.$$

Since $2x_ix_j \leq x_i^2 + x_j^2$, we must have $x_i = x_j$ when v_i and v_j are adjacent. Since G is connected, we get that X is the multiple of all one vector. From $AX = \rho X$, we get that G must be regular. Conversely, it is easy to check that when G is regular, the equality holds. ■

Remark 1. Since $\mu(G) = \rho(D+A) = 2+\rho(L_G)$, where L_G is the line graph of G. So this Lemma gives a relation between the spectral radius of a graph and its line graph, namely,

$$\rho(G) \leq \frac{1}{2}\rho(L_G) + 1,$$

the equality holds if and only if G is a regular graph.

Corollary 2.3. *If G is a bipartite graph with degree sequence $d_1, d_2, \cdots, d_n$. Then*

$$\lambda(G) \geq 2\sqrt{\frac{1}{n}\sum d_i^2},$$

where the sum is taken over all the $1 \leq i \leq n$. Moreover, the equality holds if and only if G is regular bipartite.

Proof. From [5] or [11], we know $\rho(G) \geq \sqrt{\frac{1}{n}\sum d_i^2}$, and the equality holds if and only if G is a regular graph or a semiregular bipartite graph. Together with Lemma 1.1 and Lemma 2.2, we get the result. ∎

Lemma 2.4. [7] *Let G be a graph of order n and $Q = Q(G)$, let P be any polynomial, then*

$$min_{v\in V(G)} s_v(P(Q)) \leq P(\mu) \leq max_{v\in V(G)} s_v(P(Q)),$$

where $s_v(X)$ denotes the vth row sum of a matrix X. The equality holds if and only if the row sums of Q are equal.

Lemma 2.5. *Let G be a simple graph and Δ be the maximum degree, then*

$$\mu(G) \leq \frac{\Delta + \sqrt{\Delta^2 + 8T}}{2},$$

where $T = max\{d_u m_u | u \in V\}$ is the maximum 2-degree. Moreover, the equality holds if and only if G is regular.

Proof. Since $Q = D + A$, by a simple calculation, we have $s_v(Q) = 2d_v$ and $s_v(AD) = s_v(A^2) = \sum_{u\sim v} d_u = d_v m_v$. Then

$$\begin{aligned} s_v(Q^2) &= s_v(D^2 + DA + AD + A^2) \\ &= s_v(D(D+A)) + s_v(AD) + s_v(A^2) \\ &= d_v s_v(Q) + 2d_v m_v \\ &\leq \Delta s_v(Q) + 2d_v m_v \end{aligned}$$

So we have

$$s_v(Q^2 - \Delta Q) \leq max_{v\in V(G)} 2d_v m_v = 2T.$$

By Lemma 2.4, we have

$$\mu^2 - \Delta\mu - 2T \leq 0,$$

and then

$$\mu \leq \frac{\Delta + \sqrt{\Delta^2 + 8T}}{2}.$$

In order to get the equality above, all inequalities in the above should be equalities. That is $d_v = \Delta$ and $d_v m_v = T$ hold for any vertex v. So by Lemma 2.4, G is regular. Conversely, when G is regular, it is easy to check that the equality holds. ∎

Remark 2. From the proof of this Lemma, we can get easily that $\mu(G) \leq max_{u \in V(G)} \sqrt{2d_u^2 + 2d_u m_u}$, and this is one of the main results in [8].

Corollary 2.6. *Let G be a simple connected graph, then*

$$\lambda(G) \leq \frac{1}{2}\Big(\Delta + \sqrt{\Delta^2 + 8\big(2m - (n-1)\delta + (\delta - 1)\Delta\big)}\ \Big).$$

Equality holds if and only if G is regular bipartite.

Proof. Since $d_u m_u \leq 2m - (n-1)\delta + (\delta - 1)\Delta$, together with Lemma 1.1 and Lemma 2.5, we can get the result. ∎

Lemma 2.7. *Let G be a simple connected graph, then*

$$\mu(G) \leq max_{u \in V(G)} \Big(d_u + \sqrt{d_u m_u}\ \Big),$$

and the equality holds if and only if G is regular or semiregular bipartite.

Proof. Let $x = (x_v, v \in V)$ be a unit vector such that $\mu x = Qx$, then for any $u \in V$, $\mu x_u = d_u x_u + \sum_{v \sim u} x_v = \sum_{v \sim u}(x_v + x_u)$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mu^2 x_u^2 &\leq d_u \sum_{v \sim u}(x_u + x_v)^2 \\ &= d_u^2 x_u^2 + 2d_u x_u^2(\mu - d_u) + d_u \sum_{v \sim u} x_v^2. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{u \in V} \mu^2 x_u^2 &= \sum_{u \in V}(2d_u\mu - d_u^2)x_u^2 + \sum_{u \in V} d_u \sum_{v \sim u} x_v^2 \\ &= \sum_{u \in V}(2d_u\mu - d_u^2)x_u^2 + \sum_{u \in V} d_u m_u x_u^2. \end{aligned}$$

Thus we have

$$\sum_{u \in V}(\mu^2 - 2d_u\mu + d_u^2 - d_u m_u)x_u^2 \leq 0.$$

So there exists one vertex u such that

$$\mu^2 - 2d_u\mu + d_u^2 - d_u m_u \leq 0,$$

which implies our result.

If the equality holds, then we have for any $u \in V$, $uv, uw \in E$, $x_u + x_v = x_u + x_w$, that is, for any $u \in V$, the neighbors of u have equal eigencomponents. While from $\mu x_u = d_u x_u + \sum_{v \sim u} x_v$, we have

$$\begin{aligned}(\mu - d_u)x_u &= d_u x_v,\\ (\mu - d_v)x_v &= d_v x_u,\\ (\mu - d_w)x_w &= d_w x_u.\end{aligned}$$

So $d_v = d_w$ and for any $uv \in E$,

$$\mu = d_u + d_v. \tag{a}$$

Since G is connected, there are at most two different degrees in G. If the subgraph $G[N(u)]$ induced by $N(u)$ has at least one edge, then from (a) we have $d_u = d_v$ and G must be regular. If $G[N(u)]$ contains no edge, then G must be semiregular bipartite. So we get the desired result. Conversely, it is easy to check that when G is either regular or semiregular bipartite, the result holds. ■

Corollary 2.8. *Let G be a simple connected graph, then $\lambda \leq max_{u \in V(G)}\big(d_u + \sqrt{d_u m_u}\ \big)$. Equality holds if and only if G is regular bipartite or semiregular bipartite.*

3. Main Results and Examples

In this section, we establish some upper bounds for the adjacency spectral radius of a graph G by the matrix $Q(G)$ and its line graph L_G. Finally, we give two examples to illustrate that our results are sometimes better than the known ones in Section 1.

Theorem 3.1. *Let G be a simple connected graph, then*

$$\rho(G) \leq max_{u \in V(G)} \sqrt{\frac{d_u^2 + d_u m_u}{2}}. \tag{4}$$

The equality holds if and only if G is a regular graph.

Proof. From the proof of Lemma 2.5, we have $s_v(Q^2) = s_v(D^2 + DA + AD + A^2) = 2d_v^2 + 2d_v m_v$, from Lemma 2.4, we have $\mu(G) \leq max_{u \in V(G)} \sqrt{2d_u^2 + 2d_u m_u}$, and by Lemma 2.2, we get the result. ■

Theorem 3.2. *Let G be a simple connected graph, then*

$$\rho(G) \leq max_{u\sim v}\frac{\sqrt{d_u(d_u+m_u)+d_v(d_v+m_v)}}{2}, \tag{5}$$

the equality holds if and only if G is a regular graph.

Proof. By Lemmas 2.1 and 2.2 and Remark 1, we can get the result. ∎

Theorem 3.3. *Let G be a simple connected graph, then*

$$\rho(G) \leq \frac{\Delta+\sqrt{\Delta^2+8T}}{4}, \tag{6}$$

where $T = max\{d_u m_u | u \in V\}$ is the maximum 2-degree in V. Moreover, the equality holds if and only if G is regular.

Proof. By Lemmas 2.2 and 2.5, we can get the desired result. ∎

Theorem 3.4. *Let G be a simple connected graph, then*

$$\rho(G) \leq max_{u\in V(G)}\frac{1}{2}\Big(d_u+\sqrt{d_u m_u}\ \Big). \tag{7}$$

The equality holds if and only if G is a regular graph.

Proof. From Lemma 2.2 and Lemma 2.7, we can get the desired result. ∎

Remark 3. It is easy to check that (4), (7) and (3) are incomparable, since d_u and m_u are incomparable in general. While a simple computation shows that (7) is better than (4).

Theorem 3.5. *Let G be a simple connected graph, then*

$$\rho(G) \leq max\Big\{\frac{1}{4}\Big((d_u+d_v)+\sqrt{(d_u-d_v)^2+4m_u m_v}\ \Big) : uv \in E(G)\Big\}. \tag{8}$$

The equality holds if and only if G is a regular graph.

Proof. Consider $D^{-1}QD$, by modifying the proof of Theorem 2.14 in [4] and a similar argument in Lemma 2.7 for the case when the equality holds, we can get

$$\mu(G) \leq max\Big\{\frac{1}{2}\Big((d_u+d_v)+\sqrt{(d_u-d_v)^2+4m_u m_v}\ \Big) : uv \in E(G)\Big\},$$

equality holds if and only if G is regular or semiregular bipartite. From Lemma 2.2, we can get the result. ∎

Remark 4. It is easy to check that

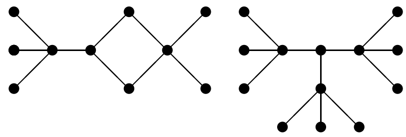
$$\frac{1}{2}\Big((d_u+d_v)+\sqrt{(d_u-d_v)^2+4m_um_v}\ \Big) \le \sqrt{d_u^2+d_v^2+2m_um_v} \le \frac{d_u^2+d_v^2+m_um_v+d_ud_v}{d_u+d_v}.$$

So by Lemma 2.2, we can get that

$$\begin{aligned}\rho(G) &\le \frac{1}{4}\Big((d_u+d_v)+\sqrt{(d_u-d_v)^2+4m_um_v}\ \Big)\\ &\le \frac{1}{2}\Big(\sqrt{d_u^2+d_v^2+2m_um_v}\ \Big)\\ &\le \frac{1}{2}\Big(\frac{d_u^2+d_v^2+m_um_v+d_ud_v}{d_u+d_v}\ \Big).\end{aligned}$$

Even for these two new bounds, they are incomparable with (4), (5), (7).

At last, we give two examples to illustrate the results in this section are sometimes better than those cited in Section 1. Let G_1 and G_2 be two graphs as shown in the following figures.



G_1 G_2

We summarize all the upper bounds for the spectral radius of graphs considered in this paper as follows:

	ρ	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
G_1	2.450	3.450	3.317	2.828	3.317	3.236	3.123	3.225	2.781
G_2	2.330	3.646	3.464	3.464	3.317	3.646	3.279	3.232	3.000

The table above shows in a general sense, these bounds are incomparable except for (4) and (7). (8) is the best of all.

Acknowledgment

The authors are grateful to the referees for their valuable comments, corrections and suggestions which lead to an improvement of this paper.

REFERENCES

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, New York, Macmilan Press, 1976.
2. D. S. Cao, Bounds on the eigenvalues and chromatic number, *Linear Algebra Appl.*, **270** (1998), 1-13.
3. K. ch. Das, An improved upper bound for the Laplacian graph eigenvalues, *Linear Algebra Appl.*, **368** (2003), 269-278.
4. K. ch. Das, A characterization on graphs which achieved the upper bound for the largest Laplacian eigenvalue of graphs, *Linear Algebra Appl.*, **376** (2004), 173-186.
5. M. Hofmeister, Spectral radius and degree sequence, *Math. Nachr.*, **139** (1988), 37-44.
6. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.
7. H. Q. Liu, M. Lu and F. Tian, On the Laplacian spectral radius of a graph, *Linear Algebra Appl.*, **376** (2004), 135-141.
8. J. S. Li and Y. L. Pan, de Caen's inequality and bound on the largest Laplacian eigenvalue of a graph, *Linear Algebra Appl.*, **328** (2001), 153-160.
9. J. L. Shu, Y. Hong and K. Wen-Ren, A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph, *Linear Algebra Appl.*, **347** (2002), 123-129.
10. J. L. Shu and Y. R. Wu, Sharp upper bounds on the spectral radius of graphs, *Linear Algebra Appl.*, **377** (2004), 241-248.
11. A. M. Yu, M. Lu and F. Tian, On the spectral radius of graphs, *Linear Algebra Appl.*, **387** (2004), 41-49.
12. X. D. Zhang, Two sharp upper bounds for the Lapalacian eigenvalues, *Linear Algebra Appl.*,**362** (2003), 207-213.

Lihua Feng
School of Mathematics,
Shandong Institute of Business and Technology,
Yantai, Shndong 264005,
P. R. China
E-mail: fenglh@163.com

Qiao Li and Xiao-Dong Zhang
Department of Mathematics,
Shanghai Jiao Tong University,
800 Dongchuan Road,
Shanghai 200240,
P. R. China
E-mail: qiaoli@sjtu.edu.cn; xiaodong@sjtu.edu.cn