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ON ISOMETRIC EXTENSIONS AND DISTANCE ONE PRESERVING MAPPINGS

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Abstract. In this survey article, we introduce the isometric extension problem of isometric mapping between the unit spheres and the distance one preserving problem . Some important results in the development of the related problems are outlined in this paper and some recent advancement and open problems are repointed.

1. Introduction

The classical Mazur-Ulam theorem tells us that a surjective isometry V between two real normed spaces E and F is affine (i.e. $V(\lambda x+(1-\lambda)y)=\lambda V(x)+(1-\lambda)V(y)$ for all x, y in E and λ in $\mathbb{R}$). So, it is important to determine the isometry of an operator on the whole space E by its isometry on some region of the space or its preserving some distance properties. Thus the two problems, the extension problem of isometries and the relationship between the "distance one preserving property" (or DOPP for short) and isometry, are posed.

In this paper, we introduce the two problems and some recent advances and open problems are presented.

2. The Extension Problem of Isometries Between Unit spheres

If we consider our earth to be an (unit) ball, then every "equidistant movement" is a isometry on this unit sphere. For the isometric extension problem, concerning the mapping from the unit sphere $S_1(E)$ of the Banach space E onto the unit sphere

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$S_1(F)$ of another Banach space F, we consider the spaces all over the real field. In 1987, D.Tingley raised the following problem in [20]:

Problem. Suppose that $V_0 : S_1(E) \to S_1(F)$ is a surjective isometric mapping, does there exist a linear isometric mapping $V : E \to F$ such that $V|_{S_1(E)} = V_0$?

D. Tingley, who is the first one to study the problem, gave some results in [20] under the condition that E and F are finite dimensional spaces. More precisely, the answer can be formulated as follows: Suppose that $V_0 : S_1(E) \to S_1(F)$ is a surjective isometric mapping, then $V_0(-x) = -V_0(x)$, i.e. V_0 is an odd mapping.

For the infinite dimensional strictly convex space and the space ℓ_1, Y. Ma also obtained above result in [17]. On the other hand, R. Wang obtained the positive answer to the original problem for the spaces of type $C_0(\Omega, E,)$ including $C_0^{(n)}(X)$, the space of type $[\oplus C_0(\Omega, E)]_1$, the space of type $[\oplus C_0(\Omega, E)]_p$ with some additional condition on E, the ℓ_1-sum of strictly convex normed spaces $\cdots$. (cf. [22–27]).

Y. Xiao has studied the isometric extension problem for a class of atomic AL_p-space (i.e. abstract L^p-space) and got some meaningful conclusion. But there is some essential fallacy in his paper [31].

D. Zhan in the papers [33, 34] has discussed the isometric extension problem of the unit spheres of type $L^p(\Omega, H)$, where H is a Hilbert space , and got the positive answer.

Y. Xiao, and R. Wang, both discussed the isometric extension problem of the unit sphere of $AM-$space (i.e. abstract M-space) and obtained the positive answer in [32].

G. Ding is the first one to study the problem between the spaces of different type, which can be found in [12]. More precisely, if $V_0 : S_1(E) \to S_1(C(\Omega))$ is a surjective mapping satisfying the condition that

$$||V_0(x) - |\lambda| \cdot V_0(x_0)|| \leq ||x - |\lambda| x_0||, forall \lambda \in \mathbb{R}, x \in S_1(E), x_0 \in sm[S_1(E)]$$

and the set $sm[S_1(E)]$ of the smooth points in $S_1(E)$ is dense in $S_1(E)$, then V_0 can be extended to an isometric operator defined on the whole space. Recently, X. Fang proved that the above conclusion holds if we replace $C(\Omega)$ with any normed space F in [15].

G. Ding in [13] has further discussed the problem for the case that the isometric operator between unit spheres of the two Hilbert spaces is not “surjective” and obtained the following result:

Suppose that $V_0 : S_1(E) \to S_1(F)$ satisfies 1-Lipschitz condition and $-V_0[S_1(E)] \subset V_0[S_1(E)]$. Then V_0 can be extended to an isometric real linear operator defined on the whole space.

The first counterexample was given by L. Zhang in [35], where he gave a mapping from $S_1(\ell^\infty_{(2)})$ into $S_1(\ell^\infty_{(3)})$ which shows that the isometric extension problem fails. Recently, L. Zhang also constructed a counterexample for the infinite-dimensional spaces ℓ' in [36]. By the same idea, Y. Dai gave another counterexample in which he constructed a mapping from $S_1(\ell^1_{(2)})$ into $S_1(\ell^1_{(3)})$.

J. Wang found, in [21], some conditions under which an into isometry V_0 between two unit spheres of atomic AL^p-spaces ($1 < p < \infty, p \neq 2$) can be linearly isometrically extended. It is a pity that those conditions are too abstract and formal. These conditions are as follows:

$$P_\gamma[V_0(S_1(E))] \subset span\{V_0(e_\gamma)\} \quad \text{and } V_0(-e_\gamma) \in \text{span}\{V_0(e_\gamma)\}, \text{ for all } \gamma \in \Gamma.$$

where, $V_0 : S_1(E) \to S_1(F)$ is an isometry, P_γ is the principal band projection from F onto $B_{V_0(e_\gamma)}$, and $B_{V_0(e_\gamma)}$ denotes the principal projection band generated by $V_0(e_\gamma)$ $(E = F = \ell^p(\Gamma))$.

I conjectured that the above conditions are not required, this conjecture was proved by Z. Hou. He obtained in [16] an affirmative answer for the "into" isometry between the unit spheres of "any" two AL^p-spaces ($1 < p < \infty, p \neq 2$).

Recently, in [14] ,G. Ding obtained some conditions under which an into isometric mapping from the unit sphere of the $\mathcal{L}^\infty(\Gamma)$-type space, in particular, the atomic AM-space, to the unit sphere of some Banach space E can be linearly extended (Where, the $\mathcal{L}^\infty(\Gamma)$-type space denotes such normed space in which the norm is defined by "sup" and Γ is an index set. For example, the spaces $\ell^\infty(\Gamma), c(\Gamma)$ and $c_0(\Gamma)$ (in particular, ℓ^∞, c and c_0) etc. all belong to the $\mathcal{L}^\infty(\Gamma)$-type spaces). These conditions are as follows:

(i) For every x_1 and x_2 in $S_1(\mathcal{L}^\infty(\Gamma))$, λ_1 and λ_2 in $\mathbb{R}$,

$$\|\lambda_1 V_0(x_1)+\lambda_2 V_0(x_2)\| = 1 \quad \Longrightarrow \quad \lambda_1 V_0(x_1)+\lambda_2 V_0(x_2) \in V_0[S_1(\mathcal{L}^\infty(\Gamma))].$$

(ii) If $V_0(x) = \sum_{k=1}^{n} \lambda_k V_0(\chi_{\Gamma_k})$, then $x = \sum_{k=1}^{n} \lambda'_k \chi_{\Gamma_k} + x_0$. Here, $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ are mutual disjoint subsets of Γ and $supp.x_0 \subset (\bigcup_{k=1}^{n} \Gamma_k)^c$.

On the other hand, in [9], we obtained a method by which we can get the representation theorem of surjective isometric mapping between the unit spheres of atomic AL^p-spaces. In §5 of Chapter 11 of Banach's monograph [3], there is some representation theorem for isometric mappings $V \in \mathcal{B}(c \to c)$ and $\mathcal{B}(\ell^p \to \ell^p)$ $(p \geq 1)$ etc. But these operators are linear and are defined on the whole space. Moreover, there does not exist representation theorem for the space ℓ^∞ since there is no basis in ℓ^∞. In this aspect, the newest result is the following theorem (in the real spaces):

Suppose that $V_0 : S_1[\ell^p(\Gamma)] \to S_1[\ell^p(\Delta)]$ $(1 \leq p \leq \infty,\ p \neq 2)$ is an surjective isometric mapping. Then there exist a permutation $\pi : \Delta \to \Gamma$ and the number set

$\{\theta_\delta\}_{\delta\in\Delta}$ with $|\theta_\delta| = 1$ for all $\delta \in \Delta$, such that

$$V_0(x) = \sum_{\delta\in\Delta} \theta_\delta \cdot \xi_{\pi(\delta)} d_\delta, \quad \forall\, x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma \in S_1[\ell^p(\Gamma)].$$

(cf. [10, 11]).

Open Problem.

1. Can we give a counterexample for the isometric extension problem from $S_1(E)$ "onto" itself?
2. Can we formulate an isometric extension problem which maps $S_1(E)$ "into" $S_1(F)$? That is, can we find some conditions under which an into isometry V_0 from the unit sphere $S_1(E)$ to the unit sphere $S_1(F)$ can be linearly isometrically extended to the whole space E (where, E can be F or not)?

3. DOPP (or SDOPP) + (?) $\Longrightarrow$ Isometry

Thirty five years ago, the famous former Soviet mathematician A. D. Alexandrov, in [1] , posed a question as follows:

Question. Whether or not the mapping with DOPP is an isometry? where, DOPP denotes the distance one preserving mapping, i.e.

$$d(x,y) = 1 \implies d_1(T(x), T(y)) = 1, \quad \text{for all } x, y \in E,$$

and SDOPP denotes the strong DOPP, i.e.

$$d(x,y) = 1 \iff d_1(T(x), T(y)) = 1, \quad \text{for all } x, y \in E.$$

Up to now, there are only a few good results to be obtained though more than 35 years have passed. As early as, in 1953, F. S. Beckman and D. A. Quarles proved in [4] that if T is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ with DOPP ($n \geq 2$), then T is an isometry . In 1985, W. Benz, in [5], pointed out that suppose E and F are two normed spaces, $dim\, E \geq 2$, F is strictly convex, and if the mapping $T : E \to F$ preserves two distances ρ_0 and $n_0\rho_0$ for some $\rho_0 > 0$ and $n_0 \in \mathbb{N}$, then T is an isometry.

In 1993, T.M. Rassias and P. Semrl obtained, in [19], the following results:

(1) Suppose that E and F are two normed spaces, $\max(dim\, E, dim\, F) \geq 2,\ T : E \to F$ is a surjective 1-Lipschitz mapping. If T has SDOPP then T is an isometry.

In particular, by (1), they obtained the following corollary:

(2) In (1), when F is a strictly convex space, without the condition of 1-Lipschitz, the conclusion is still true. (Indeed, the move is merely formal because in the case of (2), the condition of 1-Lipschitz can be led out.)

In [18], Y. Ma improved the above (2) a little, she obtained that suppose the above F is strictly convex, T is a 1-Lipschitz mapping, and if T has DOPP then T is an isometry.

In [30], S. Xiang has shown that if E, E_1 are Hilbert spaces, then

1. when $dim\, E \geq 2$ and $f : E \to E_1$ is a mapping preserving 1 and $\sqrt{3}$, then f is an affine isometric operator.
2. when $dim\, E \geq 3$ and $f : E \to E_1$ is a mapping preserving 1 and $\sqrt{2}$, then f is an affine isometric operator.

In [28], R. Wang constructed many counterexamples concerning DOPP, SDOPP and isometry on $\mathbb{R}$. Moreover, his another paper in [29] is to give a representation theorem for DOPP and SDOPP mappings from $\mathbb{R}$ to $\mathbb{R}$.

In the newest papers [6] and [7], the authors introduced the 2-normed and n-normed spaces and the corresponding definitions of 2-isometry, n-isometry, AOPP (i.e. area one preserving property), nDOPP (i.e. n-distance one preserving property), strictly convex, 2-Lipschitz mapping, n-Lipschitz mapping and 2-colinear and n-colinear in the above spaces. Then they obtained some results similar to those found in paper [19].

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