TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 1, pp. 147-185, February 2014
DOI: 10.11650/tjm.18.2014.3208
This paper is available online at http://journal.taiwanmathsoc.org.tw

WEIGHTED INEQUALITIES ON MORREY SPACES FOR LINEAR AND MULTILINEAR FRACTIONAL INTEGRALS WITH HOMOGENEOUS KERNELS

Takeshi Iida

Abstract. In this paper, we consider weighted inequalities for linear and multilinear fractional integrals with homogeneous kernels on Morrey spaces. Recently, weighted inequalities without homogeneous kernels were proved by the authors. In this paper, we generalize ones with homogeneous kernels.

1. INTRODUCTION

The purpose of this paper is to unify some inequalities on multi-Morrey spaces for linear and multilinear fractional operators with homogeneous kernels. For simplification, we assume that all the functions are non-negative. We first recall some standard notations. All cubes in $\mathbb{R}^n$ are supposed by the definition to have their sides parallel to the coordinate axes. For a cube $Q \subset \mathbb{R}^n$, we use $l(Q)$ to denote the side-length of Q and cQ to denote the cube with the same center as Q but with side-length $cl(Q)$. Let $|E|$ denote the Lebesgue measure of E. The integral average of a measurable function u over E is written

$$m_E(u) = \fint_E u(x)dx = \frac{1}{|E|}\int_E u(x)dx.$$

For $1 \leq p < \infty$, p' is the conjugate index namely, $\frac{1}{p} + \frac{1}{p'} = 1$. Let f be a locally integrable function on $\mathbb{R}^n$. The fractional integral operator $I_\alpha f(x)$, $0 < \alpha < n$, is given by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Moreover, for $0 \leq \alpha < n$, $M_\alpha f(x)$ denotes the fractional maximal function:

Received April 18, 2013, accepted July 4, 2013.
Communicated by Yong-Sheng Han.
2010 *Mathematics Subject Classification*: 26A33, 42B25.
Key words and phrases: Multilinear fractional integral operators, Morrey spaces, Weights, Homogeneous kernels.

$$M_\alpha f(x) := \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|\le r} |f(x-y)|\, dy.$$

For $0 < p \le p_0 < \infty$, the Morrey space $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ is defined by the norm (or quasi-norm)

$$\|f\|_{\mathcal{M}^{p_0}_p} := \sup_{\substack{Q\subset\mathbb{R}^n \\ Q:cubes}} |Q|^{\frac{1}{p_0}} \left(⨍_Q |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

There is a remarkable result on the Morrey boundedness of I_α. It is due to Adams [1] (see also [2, 4]):

Theorem A. *Let $0 < \alpha < n$, $1 < p \le p_0 < \infty$ and $1 < q \le q_0 < \infty$. The inequality holds*

$$\|I_\alpha f\|_{\mathcal{M}^{q_0}_q} \le C \|f\|_{\mathcal{M}^{p_0}_p}$$

if $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ and $\frac{q}{q_0} = \frac{p}{p_0}$.

We introduced the multi-Morrey space (in our earlier paper [11]), which we recall now.

Definition 1. Let $0 < p \le p_0 < \infty$, $1 < p_1, \dots, p_m < \infty$, $\vec{P} := (p_1, \dots, p_m)$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. One says that $\vec{f} = (f_1, \dots, f_m) \in \mathcal{M}^{p_0}_{\vec{P}}(\mathbb{R}^n)$ if the following quantity is finite:

$$\left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} := \sup_{\substack{Q\subset\mathbb{R}^n \\ Q:cubes}} |Q|^{\frac{1}{p_0}} \prod_{j=1}^{m} \left(⨍_Q |f_j(y_j)|^{p_j} \, dy_j \right)^{\frac{1}{p_j}} < \infty. \tag{1}$$

Remark 1. The expression defined in (1) is not a norm on $\mathcal{M}^{p_0}_{\vec{P}}(\mathbb{R}^n)$ as long as $m \ge 2$. In fact, since, $m \ge 2$, then

$$\left\|\vec{f}+\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} = 2^m \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} > 2 \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} = \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} + \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}},$$

the triangle inequality fails. Moreover, it may be equal to zero when only one of the components is zero (see [11]).

Remark 2. When $p = p_0$, we have

$$\left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}} = \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

Hence, the results will cover ones in Lebesgue spaces.

Moreover we shall define linear and multilinear fractional operators with kernels.

Definition 2. Let f be a locally integrable function on $\mathbb{R}^n$.

(1) Given $0 < \alpha < n$ and a measurable function Ω on $\mathbb{R}^n \setminus \{0\}$, define

$$I_{\Omega,\alpha} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y) f(y)}{|x-y|^{n-\alpha}} dy.$$

(2) Given $0 \le \alpha < n$ and a measurable function Ω on $\mathbb{R}^n \setminus \{0\}$, define

$$M_{\Omega,\alpha} f(x) := \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y| \le r} |\Omega(y)|\, |f(x-y)|\, dy.$$

Let $\vec{f} = (f_1, \ldots, f_m)$ be a collection of m locally integrable functions on $\mathbb{R}^n$.

(3) Given $0 < \alpha < mn$ and a measurable function Ω_* on $\mathbb{R}^{mn} \setminus \{\vec{0}\}$, define

$$I_{\Omega_*,\alpha,m}\left(\vec{f}\right)(x) := \int_{\mathbb{R}^{mn}} \frac{\Omega_*(x-y_1, \ldots, x-y_m) \prod_{j=1}^m f_j(y_j)}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y},$$

where $d\vec{y} = dy_1 \cdots dy_m$. Moreover put $I_{\alpha,m}(\vec{f})(x) := I_{1,\alpha,m}(\vec{f})(x)$.

(4) Given $0 \le \alpha < mn$ and a measurable function Ω_* on $\mathbb{R}^{mn} \setminus \{\vec{0}\}$, define

$$\mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right)(x) := \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}| \le r} |\Omega_*(\vec{y})| \prod_{j=1}^m |f_j(x-y_j)|\, d\vec{y},$$

where $|\vec{y}| = |(y_1, \ldots, y_m)| = \sqrt{|y_1|^2 + \cdots + |y_m|^2}$. Moreover, put $\mathcal{M}_{\alpha,m}\left(\vec{f}\right)(x) := \mathcal{M}_{1,\alpha,m}\left(\vec{f}\right)(x)$.

Remark 3. Let $0 < \alpha < n$ and Ω be a kernel as above. Then,

$$|I_{\Omega,\alpha}(f)(x)| \le I_{|\Omega|,\alpha}(|f|)(x).$$

In the actual proof, $I_{|\Omega|,\alpha}(|f|)(x)$ will be controlled and as a consequence $I_{\Omega,\alpha}$ is proven to be bounded. In view of this pointwise inequality, there is no need to take care of the problem of the absolute convergence of the integral defining $I_{\Omega,\alpha}(f)(x)$.

In these frameworks, we investigate some weighted inequalities. The boundedness of $I_\alpha f$ on weighted Lebesgue spaces was investigated by Muckenhoupt and Wheeden [17]. The inequalities of $I_\alpha f$ on Morrey spaces were discovered by Adams [1] and Olsen [18]. Using the inequality of $I_\alpha f$ on Morrey spaces, Olsen investigated the Schrödinger equation. The boundedness of $I_{\Omega,\alpha} f$ on weighted Lebesgue spaces was investigated by Ding and Lu (see [5] and also [15]). In 2010, Chen and Xue [3] extended the Ding and Lu result to as a sort of multilinear version.

On the other hand, Komori and Shirai [13] introduced the weighted Morrey spaces and showed the boundedness of $I_\alpha f$ on weighted Morrey spaces. The boundedness is not the Adams type, but the Spanne type (see [19]). Chiarenza and Frasca [4] showed that the Adams inequality is more precise than the Spanne inequality.

Hence, it is natural to consider the Adams inequality on weighted Morrey spaces. In [12], we considered weight condition

$$[w]_{p_0,p,p} := \sup_{\substack{Q \subset Q' \\ Q,Q':cubes}} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{p_0}} \left(\fint_Q w(x)^p dx\right)^{\frac{1}{p}} \left(\fint_{Q'} w(x)^{-p'} dx\right)^{\frac{1}{p'}} < \infty,$$

which is related to the boundedness of the Hardy-Littlewood maximal operator M on weighted Morrey spaces. In [12], we showed that the Adams inequality and the Olsen type inequality on weighted Morrey spaces for linear and multilinear fractional integral operators. In this paper, we extend the results to linear and multilinear fractional integral operators with homogeneous kernels. The rest of the present paper is organized as follows: In Section 2, we state main results. In Section 3, we list some lemmas to prove main results. In Section 4, we prove main results. In Section 5, we take up an example which does not belong to the product on m Morrey spaces but the multi-Morrey quantity is finite.

2. Main Results

2.1. Linear operators

We state a fundamental result which is the Adams inequality with homogeneous kernels. As far as we know, the result is even new. Firstly, let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere.

Proposition 1. *Suppose that we are given parameters,* α, s, p, p_0, q, q_0 *satisfying* $0 < \alpha < n$, $1 < s \leq \infty$, $(1 \leq)s' < p \leq p_0 < \infty$ *and* $1 < q \leq q_0 < \infty$. *Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Moreover suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$ *is homogeneous of order* 0*: For any* $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. *Then we have*

$$\|I_{\Omega,\alpha}(f)\|_{\mathcal{M}_q^{q_0}} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{\mathcal{M}_p^{p_0}}.$$

Remark 4. The condition of index $s' \geq 1$ is related to the integrablity of homogeneous kernels Ω. If the homogeneous kernel Ω satisfies $\Omega \in L^\infty(\mathbb{S}^{n-1})$, then we obtain the condition of indices which is completely corresponding to the condition of the Adams inequality.

Suppose that a quantity of weights $[w]_{q_0,q,p}$ is finite:

$$[w]_{q_0,q,p} := \sup_{\substack{Q \subset Q' \\ Q,Q':cubes}} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{q_0}} \left(⨍_Q w(x)^q dx\right)^{\frac{1}{q}} \left(⨍_{Q'} w(x)^{-p'} dx\right)^{\frac{1}{p'}}.$$

The following theorem is, so to speak, the Adams inequality on weighted Morrey spaces with homogeneous kernels (see [1, 2]). Because of the complicated condition of weights, the proof of Theorem 1 is not as simple as that of Proposition 1.

Theorem 1. *Suppose that we are given parameters α, s, p, p_0, q, q_0 satisfying $0 < \alpha < n$, $1 < s \le \infty$, $(1 \le)s' < p \le p_0 < \infty$ and $1 < q \le q_0 < \infty$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Moreover suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$ is homogeneous of order 0: For any $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$. If there exists $a > 1$ such that

$$[w^{s'}]_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}} < \infty, \tag{2}$$

then we have

$$\|I_{\Omega,\alpha}(f)w\|_{\mathcal{M}^{q_0}_q} \le C[w^{s'}]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

Moreover we obtain the following inequality.

Theorem 2. *Suppose that we are given parameters α, s, p, p_0, q, q_0 satisfying $0 < \alpha < n$, $1 < s \le \infty$, $(1 \le)s' < p \le p_0 < \infty$ and $1 < q \le q_0 < \infty$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \textit{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Moreover suppose that $\Omega \in L^s(\mathbb{S}^{n-1})$ is homogeneous of order 0. If $[w^{s'}]_{\frac{q_0}{s'},\frac{q}{s'},\frac{p}{s'}} < \infty$, then we have

$$\|M_{\Omega,\alpha}(f)w\|_{\mathcal{M}^{q_0}_q} \le C[w^{s'}]^{\frac{1}{s'}}_{\frac{q_0}{s'},\frac{q}{s'},\frac{p}{s'}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|fw\|_{\mathcal{M}^{p_0}_p}.$$

In order to state the Ding and Lu result, we recall the class A_p and the class $A_{p,q}$(cf. [6, 7, 15]). Firstly, we recall the definition of the class A_p.

Definition 3. Let $1 < p < \infty$. A positive weight function w defined on $\mathbb{R}^n$ belongs to the class $A_p(\mathbb{R}^n)$ and is called an A_p-weight if

$$[w]_{A_p} := \sup_{\substack{Q \subset \mathbb{R}^n \\ Q:cubes}} \left(⨍_Q w(x) dx\right) \left(⨍_Q w(x)^{\frac{1}{1-p}} dx\right)^{p-1} < \infty.$$

Moreover put $A_\infty(\mathbb{R}^n) := \bigcup_{p>1} A_p(\mathbb{R}^n)$.

Next, we recall the definition of the class $A_{p,q}(\mathbb{R}^n)$.

Definition 4. Let $1 < p < \infty$ and $0 < q < \infty$. A positive weight function w defined on $\mathbb{R}^n$ belongs to the class $A_{p,q}(\mathbb{R}^n)$ and is called an $A_{p,q}$-weight if

$$[w]_{A_{p,q}} := \sup_{\substack{Q\subset\mathbb{R}^n \\ Q:cubes}} \left(\fint_Q w(x)^q dx\right)^{\frac{1}{q}} \left(\fint_Q w(x)^{-p'} dx\right)^{\frac{1}{p'}} < \infty.$$

Remark 5. It is well known that if a weight w satisfies $w \in A_{p,q}(\mathbb{R}^n)$ if and only if

$$\begin{cases} w^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n), \\ w^{-p'} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n). \end{cases}$$

Similarly, a weight w satisfies $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}(\mathbb{R}^n)$ if and only if

$$\begin{cases} w^q \in A_{1+\frac{q/s'}{(p/s')'}}(\mathbb{R}^n), \\ w^{-s'\left(\frac{p}{s'}\right)'} \in A_{1+\frac{(p/s')'}{q/s'}}(\mathbb{R}^n). \end{cases}$$

These properties allow us to use the reverse Hölder inequality (see Lemma 3).

When $q = q_0$ and $p = p_0$, Theorems 1 and 2 are reduced to weighted L^p- inequalities.

Corollary 1. [5, 15]. *Let* $0 \le \alpha < n$, $1 < s \le \infty$, $1 \le s' < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose that* $\Omega \in L^s(\mathbb{S}^{n-1})$, *for all* $\lambda > 0$, $\Omega(\lambda x) = \Omega(x)$ *and* $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}(\mathbb{R}^n)$.
(1) *If* $0 < \alpha < n$, *then*

$$\|I_{\Omega,\alpha} f\|_{L^q(w^q)} \le C[w^{s'}]^{\frac{1}{s'}}_{A_{\frac{p}{s'},\frac{q}{s'}}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^p(w^p)}.$$

(2) *If* $0 \le \alpha < n$, *then*

$$\|M_{\Omega,\alpha} f\|_{L^q(w^q)} \le C[w^{s'}]^{\frac{1}{s'}}_{A_{\frac{p}{s'},\frac{q}{s'}}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^p(w^p)}.$$

Proof of Corollary 1. Under the condition of Corollary 1, we check that, if the weight w satisfies $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}(\mathbb{R}^n)$, then w satisfies the condition (2). In fact, by the reverse Hölder inequality, for every cube $Q \subset \mathbb{R}^n$ we have

$$\left(\fint_Q w(x)^{aq} dx\right)^{\frac{1}{aq}} \le C \left(\fint_Q w(x)^q dx\right)^{\frac{1}{q}}.$$

Hence for pair of cubes $Q \subset Q'$, we have

$$\begin{aligned}
&\left(\frac{|Q|}{|Q'|}\right)^{\frac{s'}{aq}} \left(⨍_Q w(x)^q dx\right)^{\frac{s'}{q}} \left(⨍_{Q'} w(x)^{-s'(\frac{p}{s'})'} dx\right)^{\frac{1}{(\frac{p}{s'})'}} \\
&\le \left(\frac{|Q|}{|Q'|}\right)^{\frac{s'}{aq}} \left(⨍_Q w(x)^{aq} dx\right)^{\frac{s'}{aq}} \left(⨍_{Q'} w(x)^{-s'(\frac{p}{s'})'} dx\right)^{\frac{1}{(\frac{p}{s'})'}} \\
&= \left(⨍_Q w(x)^{aq} dx\right)^{\frac{s'}{aq}} \left(⨍⨍_{Q'} w(x)^{-s'(\frac{p}{s'})'} dx\right)^{\frac{1}{(\frac{p}{s'})'}} \\
&\le C \left(⨍_{Q'} w(x)^{q} dx\right)^{\frac{s'}{q}} \left(⨍_{Q'} w(x)^{-s'(\frac{p}{s'})'} dx\right)^{\frac{1}{(\frac{p}{s'})'}} \\
&\le C[w^{s'}]_{A_{\frac{p}{s'},\frac{q}{s'}}} < \infty.
\end{aligned}$$

Therefore by Theorems 1 and 2, we obtain Corollary 1. ■

2.2. Multilinear operators

We pass to the multilinear case. The next theorem is the Adams inequality on weighted Morrey spaces for multilinear fractional operators with homogeneous kernels. Firstly, let $\mathbb{S}_{m,n} := \mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$. Suppose that a quantity of multiple weights $[\vec{w}]_{q_0,q,\vec{P}}$ given below is finite:

$$[\vec{w}]_{q_0,q,\vec{P}} := \sup_{\substack{Q \subset Q' \\ Q,Q':cubes}} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{q_0}} \left(⨍_Q (w_1 \cdots w_m)(x)^q dx\right)^{\frac{1}{q}} \prod_{j=1}^{m} \left(⨍_{Q'} w_j(y_j)^{-p_j'} dy_j\right)^{\frac{1}{p_j'}}.$$

Theorem 3. *Suppose that we are given parameters α, s, $p_1, \ldots, p_m$ p, p_0, q, q_0, $\vec{P} = (p_1, \ldots, p_m)$ satisfying $0 < \alpha < n$, $1 < s \le \infty$, $(1 \le)s' < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $0 < p \le p_0 < \infty$ and $0 < q \le q_0 < \infty$. Assume that*

$$\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} \quad \text{and} \quad \frac{q}{q_0} = \frac{p}{p_0}.$$

Moreover assume that $\Omega_* \in L^s(\mathbb{S}_{m,n})$ satisfies the following homogeneity: For any $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$. If there exists $a > 1$ such that

$$\left[\vec{w}^{s'}\right]_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}} < \infty, \tag{3}$$

where $\vec{w}^{s'} := \left(w_1^{s'}, \ldots, w_m^{s'}\right)$. Then we have the following inequality:

$$\left\| I_{\Omega_*,\alpha,m}\left(\vec{f}\right)(w_1 \cdots w_m)\right\|_{\mathcal{M}^{q_0}_q} \le C \left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Remark 6. In a series of main results, we can replace the kernel Ω_* with the following kernels Ω_{**}: $\Omega_{**} \in L^s\left(\mathbb{S}^{mn-1}\right)$ and for any $\lambda > 0$, $\Omega_{**}(\lambda x_1, \ldots, \lambda x_m) = \Omega_{**}(x_1, \ldots, x_m)$. However, this case does not cover results due to Chen and Xue. Hence we use the kernel Ω_*.

We obtain the following inequality with respect to the multilinear fractional maximal operator.

Theorem 4. *Let* $1 < s \leq \infty$, $1 \leq s' < p_j < \infty$, $0 \leq \alpha < mn$, $0 < p \leq p_0 < \infty$, $0 < q \leq q_0 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$. *Suppose that* $\left[\vec{w}^{s'}\right]_{\frac{q_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'}} < \infty$. *Then*

$$\begin{aligned}&\left\| \mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right)(w_1 \cdots w_m) \right\|_{\mathcal{M}^{q_0}_q} \\ &\leq C \left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{q_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.\end{aligned}$$

Moen [16] introduced the multiple weights class $A_{\vec{P},q}(\mathbb{R}^n)$ (see also [14]):

Definition 5. Let $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $0 < q < \infty$. One says that a vector of weights $\vec{w}$ is in the multiple weights class $A_{\vec{P},q}(\mathbb{R}^n)$ if

$$[\vec{w}]_{A_{\vec{P},q}} := \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: cubes}} \left(\fint_Q \prod_{j=1}^m w_j(x)^q dx \right)^{\frac{1}{q}} \prod_{j=1}^m \left(\fint_Q w_j(y_j)^{-p'_j} dy_j \right)^{\frac{1}{p'_j}} < \infty.$$

In Theorems 3 and 4, if we take $q = q_0$ and $p = p_0$, we obtain the following corollary which generalizes the result due to Chen and Xue [3].

Corollary 2. *Let* $1 < s \leq \infty$, $1 \leq s' < p_j < \infty$, $0 \leq \alpha < mn$, $0 < q < \infty$ *and* $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n}$. *Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$. *Suppose that* $\vec{w}^{s'} \in A_{\frac{\vec{P}}{s'}, \frac{q}{s'}}(\mathbb{R}^n)$.

(1) *If* $0 < \alpha < mn$, *then*

$$\left\| I_{\Omega_*,\alpha,m}\left(\vec{f}\right) \right\|_{L^q((w_1 \cdots w_m)^q)} \leq C \left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{A_{\frac{\vec{P}}{s'}, \frac{q}{s'}}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j^{p_j})}.$$

(2) *If* $0 \leq \alpha < mn$, *then*

$$\left\| \mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right) \right\|_{L^q((w_1 \cdots w_m)^q)} \leq C \left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{A_{\frac{\vec{P}}{s'}, \frac{q}{s'}}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j^{p_j})}.$$

In fact, for $\Omega_j \in L^s(\mathbb{S}^{n-1})$ and $\Omega_j(\lambda x) = \Omega_j(x)$, if we take

$$\Omega_*(x_1, \ldots, x_m) = \begin{cases} \prod_{j=1}^m \Omega_j(x_j) & (\text{for all } j = 1, \ldots, m;\ x_j \neq 0), \\ 0 & (\text{otherwise}), \end{cases}$$

then Corollary 2 corresponds to the result due to Chen and Xue [3]. For the sake of the convenience, we shall state the unweighted version of Theorems 3 and 4, which are new results as well.

Corollary 3. *Let* $1 < s \le \infty$, $1 \le s' < p_j < \infty$, $0 < \alpha < mn$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$.

(*1*) *If* $0 < \alpha < mn$, *then*

$$\left\| I_{\Omega_*,\alpha,m}\left(\vec{f}\right) \right\|_{\mathcal{M}^{q_0}_q} \le C \left\|\Omega_*\right\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

(*2*) *If* $0 \le \alpha < mn$, *then*

$$\left\| \mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right) \right\|_{\mathcal{M}^{q_0}_q} \le C \left\|\Omega_*\right\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Remark 7. Another proof of Corollary 3 is obtained by the boundedness of $I_{\alpha,m}$ and a standard argument. However, the proof of Theorem 3 is not as simple as the proof of Corollary 3.

We can extend Theorems 3 and 4 to two-weighted versions. Firstly, suppose that a quantity of two-weight type multiple weights $[v, \vec{w}]_{q_0,q,\vec{P}}$:

$$[v, \vec{w}]_{q_0,q,\vec{P}} := \sup_{\substack{Q \subset Q' \\ Q,Q':cubes}} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{q_0}} \left(\fint_Q v(x)^q dx\right)^{\frac{1}{q}} \prod_{j=1}^m \left(\fint_{Q'} w_j(y_j)^{-p_j'} dy_j\right)^{\frac{1}{p_j'}}.$$

By the same argument as Theorem 3, we obtain the following inequalities.

Theorem 5. *Let* $1 < s \le \infty$, $1 \le s' < p_j < \infty$, $0 < \alpha < mn$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$.

Case 1. *Let* $q > 1$. *Suppose that there exists* $a > 1$ *such that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} < \infty$. *Then*

$$\left\| I_{\Omega_*,\alpha,m}(\vec{f}) v \right\|_{\mathcal{M}^{q_0}_q} \le C \left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} \left\|\Omega_*\right\|_{L^s(\mathbb{S}_{m,n})} \left\|(f_1 w_1, \ldots, f_m w_m)\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Case 2. *Let* $0 < q \le 1$. *Suppose that there exists* $a > 1$ *such that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} < \infty$. *Then*

$$\left\| I_{\Omega_*,\alpha,m}(\vec{f})v \right\|_{\mathcal{M}^{q_0}_{q}} \le C \left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

On the other hand, we have the following inequality.

Theorem 6. *Let* $1 < s \le \infty$, $1 \le s' < p_j < \infty$, $0 \le \alpha < mn$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$. *Suppose that there exists* $a > 1$ *such that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{q_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} < \infty$. *Then*

$$\left\| \mathcal{M}_{\Omega_*,\alpha,m}(\vec{f})v \right\|_{\mathcal{M}^{q_0}_{q}} \le C \left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{q_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Moreover, we can generalize Theorems 5 and 6 in order to include the Olsen type inequality. Suppose that another quantity of two-weight type multiple weights $[v, \vec{w}]_{q_0, r_0, q, \vec{P}}$:

$$[v, \vec{w}]_{q_0, r_0, q, \vec{P}} := \sup_{\substack{Q \subset Q' \\ Q, Q' : cubes}} \left(\frac{|Q|}{|Q'|}\right)^{\frac{1}{q_0}} |Q'|^{\frac{1}{r_0}} \left(⨍_Q v(x)^q dx\right)^{\frac{1}{q}} \prod_{j=1}^{m} \left(⨍_{Q'} w_j(y_j)^{-p'_j} dy_j\right)^{\frac{1}{p'_j}}.$$

Then we obtain the following inequalities.

Theorem 7. *Let* $1 < s \le \infty$, $0 < \alpha < mn$, $1 \le s' < p_j < \infty$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < r_0 \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $1 < a < \min\left\{\frac{r_0}{q_0}, \frac{p'_1}{s'}, \ldots, \frac{p'_m}{s'}\right\}$, $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$ *and* $r_0 \ge \frac{n}{\alpha}$. *Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$.

Case 1. *Let* $q > 1$. *Suppose that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} < \infty$. *Then*

$$\left\| I_{\Omega_*,\alpha,m}\left(\vec{f}\right) v \right\|_{\mathcal{M}^{q_0}_{q}} \le C \left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Case 2. *Let* $0 < q \le 1$. *Suppose that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} < \infty$. *Then*

$$\left\| I_{\Omega_*,\alpha,m}\left(\vec{f}\right) v \right\|_{\mathcal{M}^{q_0}_{q}} \le C \left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

On the other hand, we have the following inequality.

Theorem 8. *Let* $1 < s \leq \infty$, $0 \leq \alpha < mn$, $1 \leq s' < p_j < \infty$, $0 < p \leq p_0 < \infty$, $0 < q \leq q_0 < r_0 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$ *and* $r_0 \geq \frac{n}{\alpha}$*. Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1, \ldots, \lambda_m > 0$, $\Omega_*(\lambda_1 x_1, \ldots, \lambda_m x_m) = \Omega_*(x_1, \ldots, x_m)$*. Suppose that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{q_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} < \infty$*. Then*

$$\left\|\mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right) v\right\|_{\mathcal{M}^{q_0}_q} \leq C \left[v^{s'}, \vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{q_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \|(f_1 w_1, \ldots, f_m w_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Theorem 7 has two corollaries. In Theorem 7, if we take $v = g$ and $\vec{w} = (1, \ldots, 1)$, then we have the Olsen type inequality. For the linear case, we refer to [8, 9, 18, 24, 25, 26, 27, 28, 29].

Corollary 4. *Assume that the conditions of Theorem 7.*

Case 1. *Let* $q > 1$*. For* $g \in \mathcal{M}^{r_0}_{aq}(\mathbb{R}^n)$,

$$\left\|g \cdot I_{\Omega_*,\alpha,m}\left(\vec{f}\right)\right\|_{\mathcal{M}^{q_0}_q} \leq C \|g\|_{\mathcal{M}^{r_0}_{aq}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Case 2. *Let* $0 < q \leq 1$*. For* $g \in \mathcal{M}^{r_0}_q(\mathbb{R}^n)$,

$$\left\|g \cdot I_{\Omega_*,\alpha,m}\left(\vec{f}\right)\right\|_{\mathcal{M}^{q_0}_q} \leq C \|g\|_{\mathcal{M}^{r_0}_q} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

The following inequalities are understood as a sort of the Fefferman-Stein inequality (see [15], Theorem 1.3.2, p. 17).

Corollary 5. *Under the conditions of Theorem 7, moreover assume that* $0 < q_i < r_i \leq \infty$ $(i = 1, \ldots, m)$*. Let* $\frac{1}{aq_0} = \sum_{i=1}^m \frac{1}{q_i}$, $\frac{1}{r_0} = \sum_{i=1}^m \frac{1}{r_i}$ *and*

$$W_i(x) := \sup_{Q \ni x} |Q|^{\frac{1}{r_i}} \left(⨍_Q v_i(y_i)^{q_i} dy_i\right)^{\frac{1}{q_i}} \quad (i = 1, \ldots, m),$$

where Q *runs over all cubes.*

Case 1. *Let* $q > 1$*. If* $[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} < \infty$*, then*

$$\begin{aligned}\left\|I_{\Omega_*,\alpha,m}\left(\vec{f}\right) v_1 \cdots v_m\right\|_{\mathcal{M}^{q_0}_q} \leq & C \left[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \\ & \times \|(f_1 W_1, \ldots, f_m W_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.\end{aligned}$$

Case 2. *Let* $0 < q \le 1$. *If* $[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} < \infty$, *then*

$$\left\| I_{\Omega_*,\alpha,m}\left(\vec{f}\right) v_1 \cdots v_m \right\|_{\mathcal{M}^{q_0}_q} \le C \left[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \times \|(f_1 W_1, \ldots, f_m W_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

For the sake of convenience, we prove Corollary 5.

Proof of Corollary 5. Write

$$\left[v_1 \cdots v_m, \overrightarrow{W}\right] := \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: cubes}} |Q|^{\frac{s'}{r_0}} \left(\fint_Q (v_1 \cdots v_m)(x)^{aq_0} dx \right)^{\frac{s'}{aq_0}} \prod_{j=1}^m \left(\fint_Q W_j(x)^{-s'\left(\frac{p_j}{as'}\right)'} dx \right)^{\frac{1}{\left(\frac{p_j}{as'}\right)'}}.$$

In Theorem 7, we take $v = v_1 \cdots v_m$ and $w_i = W_i$. Since $0 < q \le q_0 < \infty$ and $a > 1$, by Hölder's inequality, we have

$$\left[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}\right]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}}, \left[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}\right]_{\frac{aq_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'a}} \le \left[v_1 \cdots v_m, \overrightarrow{W}\right].$$

It suffices to show that $\left[v_1 \cdots v_m, \overrightarrow{W}\right] < \infty$. By Hölder's inequality, for every cube $Q \subset \mathbb{R}^n$ we have

$$|Q|^{\frac{s'}{r_0}} \left(\fint_Q (v_1 \cdots v_m)(x)^{aq_0} dx \right)^{\frac{s'}{aq_0}} \le \prod_{j=1}^m |Q|^{\frac{s'}{r_j}} \left(\fint_Q v_j(x)^{q_j} dx \right)^{\frac{s'}{q_j}}.$$

On the other hand, by definition of W_i, for all $y_i \in Q$,

$$W_i(y_i) \ge |Q|^{\frac{1}{r_i}} \left(\fint_Q v_i(z_i)^{q_i} dz_i \right)^{\frac{1}{q_i}}.$$

This implies that, for all $y_i \in Q$,

$$W_i(y_i)^{-s'\left(\frac{p_i}{s'a}\right)'} \le \left(|Q|^{\frac{1}{r_i}} \left(\fint_Q v_i(z_i)^{q_i} dz_i \right)^{\frac{1}{q_i}} \right)^{-s'\left(\frac{p_i}{s'a}\right)'}.$$

Hence we obtain

$$\begin{aligned} &|Q|^{\frac{s'}{r_0}} \left(\fint_Q (v_1 \cdots v_m)(x)^{aq_0} dx \right)^{\frac{s'}{aq_0}} \prod_{j=1}^m \left(\fint_Q W_j(y_j)^{-s'\left(\frac{p_i}{s'a}\right)'} dy_j \right)^{\frac{1}{\left(\frac{p_i}{s'a}\right)'}} \\ &\le \prod_{j=1}^m |Q|^{\frac{s'}{r_j}} \left(\fint_Q v_j(x)^{q_j} dx \right)^{\frac{s'}{q_j}} \times \prod_{i=1}^m \left[|Q|^{\frac{1}{r_i}} \left(\fint_Q v_i(z_i)^{q_i} dz_i \right)^{\frac{1}{q_i}} \right]^{-s'} \\ &= 1 < \infty. \end{aligned}$$

Therefore we conclude

$$\left[v_1 \cdots v_m, \overrightarrow{W}\right] \leq 1 < \infty.$$

By Theorem 7, we obtain the desired inequality. ■

A similar argument yields two corollaries of Theorem 8.

Corollary 6. *Assume that the conditions of Theorem 8. For* $g \in \mathcal{M}_q^{r_0}(\mathbb{R}^n)$*, we have the following inequality:*

$$\left\| g \cdot \mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right) \right\|_{\mathcal{M}_q^{q_0}} \leq C \|g\|_{\mathcal{M}_q^{r_0}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\vec{f}\right\|_{\mathcal{M}_{\vec{P}}^{p_0}}.$$

Corollary 7. *Under the conditions of Theorem 8, moreover assume that* $0 < q_i < r_i \leq \infty$ $(i = 1, \ldots, m)$*. Let* $\frac{1}{q_0} = \sum_{i=1}^m \frac{1}{q_i}$, $\frac{1}{r_0} = \sum_{i=1}^m \frac{1}{r_i}$ *and*

$$W_i(x) := \sup_{Q \ni x} |Q|^{\frac{1}{r_i}} \left(\fint_Q v_i(y_i)^{q_i} dy_i \right)^{\frac{1}{q_i}} \quad (i = 1, \ldots, m).$$

If $[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}]_{\frac{q_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'}} < \infty$*, then*

$$\begin{aligned} \left\| \mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right) v_1 \cdots v_m \right\|_{\mathcal{M}_q^{q_0}} \leq & C \left[(v_1 \cdots v_m)^{s'}, \overrightarrow{W}^{s'}\right]_{\frac{q_0}{s'}, \frac{r_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'}}^{\frac{1}{s'}} \|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \\ & \times \|(f_1 W_1, \ldots, f_m W_m)\|_{\mathcal{M}_{\vec{P}}^{p_0}}. \end{aligned}$$

3. Some Lemmas

3.1. Fractional integral operators

Lemma 1. *If* $0 < \alpha < n$ *and* $\Omega \in L^s(\mathbb{S}^{n-1})$ *then, we have*

$$|I_{\Omega,\alpha}(f)(x)| \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} I_\alpha(F)(x),$$

where

$$F(x) := \left(M\left(|f|^{s'}\right)(x) \right)^{\frac{1}{s'}}.$$

Proof. Since

$$|I_{\Omega,\alpha}(f)(x)| \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k \in \mathbb{Z}} 2^{k\alpha} \inf_{y \in B(x, 2^{k+1} r)} M\left[|f|^{s'}\right](x)^{\frac{1}{s'}},$$

we have

$$\begin{aligned} |I_{\Omega,\alpha}(f)(x)| &\leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k \in \mathbb{Z}} 2^{k\alpha} \frac{1}{|B(x, 2^{k+1})|} \int_{y \in B(x, 2^{k+1})} M\left[|f|^{s'}\right](y)^{\frac{1}{s'}} dy \\ &= C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k \in \mathbb{Z}} 2^{k\alpha} \frac{1}{|B(x, 2^{k+1})|} \int_{y \in B(x, 2^{k+1})} F(y) dy. \end{aligned}$$

Therefore

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha}\frac{1}{|B(x,2^{k+1})|}\int_{y\in B(x,2^{k+1})} F(y)dy = C\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\frac{\chi_{B(x,2^{k+1})}(y)}{2^{k(n-\alpha)}}F(y)dy$$
$$\leq C\int_{\mathbb{R}^n}\sum_{k=\lfloor\log_2|x-y|\rfloor-1}^{\infty}\frac{F(y)}{2^{k(n-\alpha)}}dy,$$

where $\lfloor\cdot\rfloor$ is the floor function. Note that the series can be calculated explicitly; we obtain

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha}\frac{1}{|B(x,2^{k+1})|}\int_{y\in B(x,2^{k+1})} F(y)dy \leq C\int_{\mathbb{R}^n}\frac{F(y)}{\left(2^{\lfloor\log_2|x-y|\rfloor-1}\right)^{n-\alpha}}dy.$$

Since

$$\lfloor\log_2|x-y|\rfloor \geq \log_2|x-y|-1,$$

we have

$$2^{\lfloor\log_2|x-y|\rfloor-1} \geq 2^{\log_2|x-y|-2} = \frac{1}{4}|x-y|.$$

Hence we have

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha}\frac{1}{|B(x,2^{k+1})|}\int_{y\in B(x,2^{k+1})} F(y)dy \leq C\int_{\mathbb{R}^n}\frac{F(y)}{|x-y|^{n-\alpha}}dy = CI_\alpha F(x).$$

Therefore we obtain Lemma 1. ■

Chiarenza and Frasca [4] proved the following inequality:

Lemma 2. *If* $1 < p \leq p_0 < \infty$, *then*

$$\|Mf\|_{\mathcal{M}^{p_0}_p(\mathbb{R}^n)} \leq C\,\|f\|_{\mathcal{M}^{p_0}_p(\mathbb{R}^n)}\,.$$

We recall the reverse Hölder inequality (see [6, 7, 15]):

Lemma 3. *Let* $1 < p < \infty$. *If* $w \in A_p(\mathbb{R}^n)$ *then there exists constants* C *and* $\varepsilon > 0$, *depending only on* p *and the* A_p *constant of* w, *such that for every cube* Q,

$$\left(⨍_Q w(x)^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}} \leq C\left(⨍_Q w(x)dx\right).$$

In [10], the author completely characterized the multiple weights class $A_{\vec{P},q}(\mathbb{R}^n)$ in terms of A_p-weights (see also [3]):

Lemma 4. *Let* $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $0 < q < \infty$. *A vector* $\vec{w} = (w_1, w_2, \ldots, w_m)$ *of weights satisfies* $\vec{w} \in A_{\vec{P},q}(\mathbb{R}^n)$ *if and only if*

$$\begin{cases} (w_1 \cdots w_m)^q \in A_{1+q\left(m-\frac{1}{p}\right)}(\mathbb{R}^n), \\ w_j^{-p'_j} \in A_{1+p'_j \cdot s_j}(\mathbb{R}^n) \qquad\qquad (j = 1, \ldots, m), \end{cases}$$

where $s_j = \frac{1}{q} + m - \frac{1}{p} - \frac{1}{p'_j}$ $(j = 1, \ldots, m)$.

Lemma 4 gives us that if $\vec{w} \in A_{\vec{P},q}(\mathbb{R}^n)$ then $(w_1 \cdots w_m)^q$ and $w_j^{-p'_j}$ $(j = 1, \ldots, m)$ have the property of the reverse Hölder inequality. We need the following Lemma 5 in order to prove Theorems 3, 5 and 7.

Lemma 5. [12]. *Let* $0 \leq \alpha < mn$, $\vec{P} = (p_1, \ldots, p_m)$, $\vec{R} = (r_1, \ldots, r_m)$, $0 < r_i < p_i < \infty$, $0 < q \leq q_0 < \infty$, $0 < p \leq p_0 < \infty$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Then we have the following inequality:*

$$\left\| \mathcal{M}_{\alpha,\vec{R}}(\vec{f}) \right\|_{\mathcal{M}^{q_0}_q} \leq C \left\| \vec{f} \right\|_{\mathcal{M}^{p_0}_{\vec{P}}},$$

where

$$\mathcal{M}_{\alpha,\vec{R}}(\vec{f})(x) := \sup_{Q \ni x} l(Q)^\alpha \prod_{j=1}^m \left(\fint_Q |f_j(y_j)|^{r_j} \, dy_j \right)^{\frac{1}{r_j}},$$

and Q *runs over all cubes.*

Next, we define a maximal function as follows:

$$\tilde{M}^q_{\alpha,s}\left(\vec{f}, v\right)(x) := \sup_{Q \ni x} l(Q)^\alpha m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \left(\fint_Q v(y)^q dy \right)^{\frac{1}{q}}, \tag{4}$$

where Q runs over all cubes and

$$m_{3Q}\left(\vec{f}^{s'}\right) := \prod_{j=1}^m \left(\fint_{3Q} |f_j(y_j)|^{s'} \, dy_j \right).$$

We derive the following pointwise inequalities by Hölder's inequality and the corresponding conditions of weights. We take advantage of the following Lemma 6 in order to show Theorem 5.

Lemma 6. *Under the condition of Theorem 5, we have the following inequality.*

Case 1. *Let* $q > 1$. *Suppose that* $\left[v^{s'}, \vec{w}^{s'}\right]_{\frac{aq_0}{s'}, \frac{aq}{s'}, \frac{\vec{P}}{s'a}} < \infty$, *then*

$$\tilde{M}^{aq}_{\alpha,s}(\vec{f},v)(x) \le C\left[v^{s'},\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{aq}{s'},\frac{\vec{P}}{s'a}} \mathcal{M}_{\alpha,\frac{\vec{P}}{a}}(f_1w_1,\dots,f_mw_m)(x).$$

Case 2. *Let* $0 < q \le 1$*. Suppose that* $\left[v^{s'},\vec{w}^{s'}\right]_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'a}} < \infty$*, then*

$$\tilde{M}^{q}_{\alpha,s}(\vec{f},v)(x) \le C\left[v^{s'},\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'a}} \mathcal{M}_{\alpha,\frac{\vec{P}}{a}}(f_1w_1,\dots,f_mw_m)(x).$$

We take advantage of the following Lemma 7 in order to show Theorem 7.

Lemma 7. *Under the condition of Theorem 7, we have the following inequality:*
Case 1. *Let* $q > 1$*. Suppose that* $\left[v^{s'},\vec{w}^{s'}\right]_{\frac{aq_0}{s'},\frac{r_0}{s'},\frac{aq}{s'},\frac{\vec{P}}{s'a}} < \infty$*, then*

$$\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},v\right)(x) \le C\left[v^{s'},\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{r_0}{s'},\frac{aq}{s'},\frac{\vec{P}}{s'a}} \mathcal{M}_{\alpha-\frac{n}{r_0},\frac{\vec{P}}{a}}(f_1w_1,\dots,f_mw_m)(x).$$

Case 2. *Let* $0 < q \le 1$*. Suppose that* $\left[v^{s'},\vec{w}^{s'}\right]_{\frac{aq_0}{s'},\frac{r_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'a}} < \infty$*, then*

$$\tilde{M}^{q}_{\alpha,s}\left(\vec{f},v\right)(x) \le C\left[v^{s'},\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{r_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'a}} \mathcal{M}_{\alpha-\frac{n}{r_0},\frac{\vec{P}}{a}}(f_1w_1,\dots,f_mw_m)(x).$$

By Hölder's inequality and the corresponding conditions of weights, we obtain Lemmas 6 and 7.

3.2. Fractional maximal functions

By Hölder's inequality, we obtain Lemma 8.

Lemma 8. *Let* $0 \le \alpha < mn$*,* $1 < s \le \infty$ *and* $0 < \alpha s' < mn$*. Moreover assume that* $\Omega_* \in L^s(\mathbb{S}_{m,n})$ *satisfies the following homogeneity: For any* $\lambda_1,\dots,\lambda_m > 0$*,* $\Omega_*(\lambda_1x_1,\dots,\lambda_mx_m) = \Omega_*(x_1,\dots,x_m)$*. Then we have*

$$\mathcal{M}_{\Omega_*,\alpha}\left(\vec{f}\right)(x) \le C\left\|\Omega_*\right\|_{L^s(\mathbb{S}_{m,n})} \mathcal{M}_{\alpha s',m}\left(f_1^{s'},\dots,f_m^{s'}\right)(x)^{\frac{1}{s'}}.$$

In [12], we have the following inequalities. We use the following inequality in order to prove Theorem 2.

Lemma 9. [12]. *Let* $0 \le \alpha < n$*,* $1 < p \le p_0 < \infty$*,* $1 < q \le q_0 < \infty$*,* $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$*. If* $[w]_{q_0,q,p} < \infty$*, then*

$$\left\|M_\alpha(f)w\right\|_{\mathcal{M}^{q_0}_q} \le C[w]_{q_0,q,p}\left\|fw\right\|_{\mathcal{M}^{p_0}_p}.$$

We use the following inequality in order to prove Theorem 4.

Lemma 10. [12]. *Let* $0 \le \alpha < mn$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$ *and* $\frac{q}{q_0} = \frac{p}{p_0}$. *If* $[\vec{w}]_{q_0,q,\vec{P}} < \infty$, *then*

$$\left\| \mathcal{M}_{\alpha,m}\left(\vec{f}\right)(w_1 \cdots w_m)\right\|_{\mathcal{M}_q^{q_0}} \le C\,[\vec{w}]_{q_0,q,\vec{P}} \left\|(f_1 w_1, \ldots, f_m w_m)\right\|_{\mathcal{M}_{\vec{P}}^{p_0}}.$$

We use the following inequality in order to prove Theorem 6.

Lemma 11. [12]. *Let* $1 < p_j < \infty$, $0 \le \alpha < mn$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$. *If for* $a > 1$, $[v, \vec{w}]_{q_0,q,\vec{P}/a} < \infty$, *then*

$$\left\| \mathcal{M}_{\alpha,m}(\vec{f})v\right\|_{\mathcal{M}_q^{q_0}} \le C\,[v,\vec{w}]_{q_0,q,\vec{P}/a} \left\|(f_1 w_1, \ldots, f_m w_m)\right\|_{\mathcal{M}_{\vec{P}}^{p_0}}.$$

Lastly, we use the following inequality in order to prove Theorem 8.

Lemma 12. [12]. *Let* $0 \le \alpha < mn$, $1 < p_j < \infty$, $0 < p \le p_0 < \infty$, $0 < q \le q_0 < r_0 \le \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\frac{1}{q_0} = \frac{1}{p_0} + \frac{1}{r_0} - \frac{\alpha}{n}$, $\frac{q}{q_0} = \frac{p}{p_0}$ *and* $r_0 \ge \frac{n}{\alpha}$. *Suppose that for* $a > 1$, $[v, \vec{w}]_{q_0,r_0,q,\vec{P}/a} < \infty$. *Then we have*

$$\left\| \mathcal{M}_{\alpha,m}\left(\vec{f}\right)v\right\|_{\mathcal{M}_q^{q_0}} \le C\,[v,\vec{w}]_{q_0,r_0,q,\vec{P}/a} \left\|(f_1 w_1, \ldots, f_m w_m)\right\|_{\mathcal{M}_{\vec{P}}^{p_0}}.$$

4. Proofs

4.1. Proof of Proposition 1

Firstly, we give the proof of Proposition 1.

Proof of Proposition 1. By Lemma 1, we obtain

$$\|I_{\Omega,\alpha}(f)\|_{\mathcal{M}_q^{q_0}(\mathbb{R}^n)} \le C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|I_\alpha F\|_{\mathcal{M}_q^{q_0}(\mathbb{R}^n)}.$$

Theorem A yields

$$\|I_\alpha F\|_{\mathcal{M}_q^{q_0}(\mathbb{R}^n)} \le C\,\|F\|_{\mathcal{M}_p^{p_0}(\mathbb{R}^n)} = C\left\| M\left[|f|^{s'}\right]\right\|^{\frac{1}{s'}}_{\mathcal{M}^{\frac{p_0}{s'}}_{\frac{p}{s'}}(\mathbb{R}^n)}.$$

By Lemma 2, we obtain

$$\left\| M\left[|f|^{s'}\right]\right\|^{\frac{1}{s'}}_{\mathcal{M}^{\frac{p_0}{s'}}_{\frac{p}{s'}}(\mathbb{R}^n)} \le C\left\| |f|^{s'}\right\|^{\frac{1}{s'}}_{\mathcal{M}^{\frac{p_0}{s'}}_{\frac{p}{s'}}(\mathbb{R}^n)} = C\,\|f\|_{\mathcal{M}_p^{p_0}(\mathbb{R}^n)}.$$

Therefore we obtain Proposition 1. ∎

However, the proofs of the other theorems are not as simple as Proposition 1.

Since the proofs of that of Theorems 5 and 7 are similar to Theorem 3, we omit their details. Moreover since the proofs of Theorems 2, 4 and 6 are the same argument as that of Theorem 8, we concentrate on Theorem 8.

Therefore, we prove Theorems 1, 3 and 8. The proofs method are straight.

4.2. Proofs of main theorems for linear operators

We will prove Theorem 1.

Proof of Theorem 1. For every cube $Q_0 = Q(x_0, r) \subset \mathbb{R}^n$, we decompose $f = f\chi_{3Q_0} + f\chi_{(3Q_0)^c} = f_0 + f_\infty$. Firstly we estimate $|I_{\Omega,\alpha}(f_\infty)(x)|$. For $x \in Q_0$, we have

$$\begin{aligned}
|I_{\Omega,\alpha} f_\infty(x)| &\le \int_{y\in(3Q_0)^c} \frac{|\Omega(x-y)|\,|f(y)|}{|x-y|^{n-\alpha}} dy \\
&\le \int_{|x-y|>r} \frac{|\Omega(x-y)|\,|f(y)|}{|x-y|^{n-\alpha}} dy \\
&= \sum_{k=0}^{\infty} \int_{2^k r<|x-y|\le 2^{k+1}r} \frac{|\Omega(x-y)|\,|f(y)|}{|x-y|^{n-\alpha}} dy \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{n-\alpha}} \int_{|x-y|\le 2^{k+1}r} |\Omega(x-y)|\,|f(y)|\,dy.
\end{aligned}$$

By Hölder's inequality, we obtain

$$\begin{aligned}
&|I_{\Omega,\alpha} f_\infty(x)| \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{n-\alpha}} \left(\int_{|x-y|\le 2^{k+1}r} |\Omega(x-y)|^s\,dy \right)^{\frac{1}{s}} \left(\int_{|x-y|\le 2^{k+1}r} |f(y)|^{s'}\,dy \right)^{\frac{1}{s'}} \\
&= \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{n-\alpha}} \left(\int_{\mathbb{S}^{n-1}} \left(\int_0^{2^{k+1}r} |\Omega(l\xi)|^s\, l^{n-1} dl \right) d\xi \right)^{\frac{1}{s}} \left(\int_{|x-y|\le 2^{k+1}r} |f(y)|^{s'}\,dy \right)^{\frac{1}{s'}}.
\end{aligned}$$

By the homogeneity of Ω,

$$\begin{aligned}
&|I_{\Omega,\alpha} f_\infty(x)| \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{n-\alpha}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left(\int_0^{2^{k+1}r} l^{n-1} dl \right)^{\frac{1}{s}} \left(\int_{|x-y|\le 2^{k+1}r} |f(y)|^{s'}\,dy \right)^{\frac{1}{s'}} \\
&\le C\, \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k=0}^{\infty} \left(2^k r \right)^{\frac{n}{s}-n+\alpha} \left(\int_{2^{k+3}\sqrt{n}Q_0} |f(y)|^{s'}\,dy \right)^{\frac{1}{s'}}.
\end{aligned}$$

Hence, for $x \in Q_0$, we obtain

$$|I_{\Omega,\alpha} f_\infty(x)| \le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k=0}^{\infty} \left(2^k r\right)^{\frac{n}{s}-n+\alpha} \left(\int_{2^{k+3}\sqrt{n}Q_0} |f(y)|^{s'} dy\right)^{\frac{1}{s'}}.$$

Therefore, we can estimate $I_{\Omega,\alpha} f_\infty(x)$;

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \left(\int_{Q_0} |I_{\Omega,\alpha} f_\infty(x)|^q w(x)^q dx\right)^{\frac{1}{q}} \\
&\le C |Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left(\int_{Q_0} \left[\sum_{k=0}^{\infty} (2^k r)^{\frac{n}{s}-n+\alpha} \left(\int_{2^{k+3}\sqrt{n}Q_0} |f(y)|^{s'} dy\right)^{\frac{1}{s'}}\right]^q w(x)^q dx\right)^{\frac{1}{q}} \\
&\le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \\
&\times \sum_{k=0}^{\infty} \left|2^{k+3}\sqrt{n}Q_0\right|^{-\frac{1}{s'}+\frac{\alpha}{n}} \left(\int_{2^{k+3}\sqrt{n}Q_0} |f(y)|^{s'} dy\right)^{\frac{1}{s'}} \left(\int_{Q_0} w(x)^q dx\right)^{\frac{1}{q}}.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \left(\int_{Q_0} |I_{\Omega,\alpha} f_\infty(x)|^q w(x)^q dx\right)^{\frac{1}{q}} \\
&\le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \sum_{k=0}^{\infty} \left|2^{k+3}\sqrt{n}Q_0\right|^{-\frac{1}{s'}+\frac{\alpha}{n}} \left(\int_{2^{k+3}\sqrt{n}Q_0} |f(y)|^p w(y)^p dy\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{2^{k+3}\sqrt{n}Q_0} w(y)^{-s'(\frac{p}{s'})'} dy\right)^{\frac{1}{s'}\frac{1}{(\frac{p}{s'})'}} \left(\int_{Q_0} w(x)^q dx\right)^{\frac{1}{q}} \\
&= C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \sum_{k=0}^{\infty} \left|2^{k+3}\sqrt{n}Q_0\right|^{-\frac{1}{s'}+\frac{\alpha}{n}+\frac{1}{p_0}-\frac{1}{p}} \left(\int_{2^{k+3}\sqrt{n}Q_0} |f(y)|^p w(y)^p dy\right)^{\frac{1}{p}} \\
&\quad \times \left|2^{k+3}\sqrt{n}Q_0\right|^{\frac{1}{p}-\frac{1}{p_0}} \left(\int_{2^{k+3}\sqrt{n}Q_0} w(y)^{-s'(\frac{p}{s'})'} dy\right)^{\frac{1}{s'}\frac{1}{(\frac{p}{s'})'}} \left(\int_{Q_0} w(x)^q dx\right)^{\frac{1}{q}}.
\end{aligned}$$

By virtue of the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^{p_0}_p}$, we have

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \left(\int_{Q_0} |I_{\Omega,\alpha} f_\infty(x)|^q w(x)^q dx\right)^{\frac{1}{q}} \\
&\le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|fw\|_{\mathcal{M}^{p_0}_p(\mathbb{R}^n)} \sum_{k=0}^{\infty} \left(\frac{|Q_0|}{|2^{k+3}\sqrt{n}Q_0|}\right)^{\frac{1}{q_0}} \left(\frac{|Q_0|}{|2^{k+3}\sqrt{n}Q_0|}\right)^{-\frac{1}{a q_0}} \\
&\left[2^{-k\cdot\frac{s'}{a q_0}} \left(⨍_{Q_0} w(x)^q dx\right)^{\frac{s'}{q}} \left(⨍_{2^{k+3}\sqrt{n}Q_0} w(y)^{-s'(\frac{p}{s'})'} dy\right)^{\frac{1}{(\frac{p}{s'})'}}\right]^{\frac{1}{s'}}.
\end{aligned}$$

By virtue of the condition (2) and $a>1$, we obtain

$$\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left(\int_{Q_0}|I_{\Omega,\alpha}f_\infty(x)|^q\,w(x)^q dx\right)^{\frac{1}{q}}\\
&\le C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\,[w^{s'}]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}\,\|fw\|_{\mathcal{M}^{p_0}_p}\sum_{k=0}^{\infty}2^{-k\cdot\frac{n}{q_0}\left(1-\frac{1}{a}\right)}\\
&\le C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\,[w^{s'}]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}\,\|fw\|_{\mathcal{M}^{p_0}_p}\,.
\end{aligned}$$

Next, we estimate $I_{\Omega,\alpha}f_0(x)$. For $x\in Q_0$, we have the following inequality:

$$|I_{\Omega,\alpha}f_0(x)|\le C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\sum_{Q\in\mathcal{D}(Q_0)}l(Q)^\alpha m_{3Q}(|f|^{s'})^{\frac{1}{s'}}\chi_Q(x),$$

where $\mathcal{D}(Q_0)$ is the collection of all dyadic subcubes of Q_0 and

$$m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}}:=\left(⨍_{3Q}|f(y)|^{s'}\,dy\right)^{\frac{1}{s'}}.$$

That is, $\mathcal{D}(Q_0)$ denotes the set of all those cubes obtained by dividing Q_0 into 2^n congruent cubes of half its length, dividing each of those into 2^n congruent cubes. By convention, Q_0 itself belongs to $\mathcal{D}(Q_0)$. In fact, for $x\in Q_0$,

$$\begin{aligned}
|I_{\Omega,\alpha}f_0(x)|&\le\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|\,|f_0(y)|}{|x-y|^{n-\alpha}}dy\\
&=\sum_{k\in\mathbb{Z}}\int_{2^{k-1}r\le|x-y|<2^kr}\frac{|\Omega(x-y)|\,|f_0(y)|}{|x-y|^{n-\alpha}}dy\\
&\le\sum_{k\in\mathbb{Z}}\frac{1}{(2^{k-1}r)^{n-\alpha}}\int_{|x-y|<2^kr}|\Omega(x-y)|\,|f_0(y)|\,dy.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
&|I_{\Omega,\alpha}f_0(x)|\\
&\le\sum_{k\in\mathbb{Z}}\frac{1}{(2^{k-1}r)^{n-\alpha}}\left(\int_{|x-y|<2^kr}|\Omega(x-y)|^s\,dy\right)^{\frac{1}{s}}\left(\int_{|x-y|<2^kr}|f_0(y)|^{s'}\,dy\right)^{\frac{1}{s'}}.
\end{aligned}$$

By the homogeneity of Ω,

$$
\begin{aligned}
&|I_{\Omega,\alpha}f_0(x)| \\
&\le \sum_{k\in\mathbb{Z}} \frac{1}{(2^{k-1}r)^{n-\alpha}} \left(\int_{\mathbb{S}^{n-1}} |\Omega(\xi)|^s \, d\xi \cdot \int_0^{2^k r} l^{n-1} dl \right)^{\frac{1}{s}} \left(\int_{3Q_0\cap\{|x-y|<2^k r\}} |f(y)|^{s'} \, dy \right)^{\frac{1}{s'}} \\
&= C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{Z}} (2^k r)^{-\frac{n}{s'}+\alpha} \left(\int_{3Q_0\cap\{|x-y|<2^k r\}} |f(y)|^{s'} \, dy \right)^{\frac{1}{s'}} \\
&\le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{D}(Q_0),\\ l(Q)=2^k r}} l(Q)^{-\frac{n}{s'}+\alpha} \left(\int_{3Q_0\cap\{|x-y|<l(Q)\}} |f(y)|^{s'} \, dy \right)^{\frac{1}{s'}} \chi_Q(x).
\end{aligned}
$$

A geometric observation shows

$$
\begin{aligned}
|I_{\Omega,\alpha}f_0(x)| &\le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{D}(Q_0),\\ l(Q)=2^k r}} l(Q)^{-\frac{n}{s'}+\alpha} \left(\int_{3Q} |f(y)|^{s'} \, dy \right)^{\frac{1}{s'}} \chi_Q(x) \\
&= C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^{\alpha} m_{3Q}(|f|^{s'})^{\frac{1}{s'}} \chi_Q(x).
\end{aligned}
$$

Let $\gamma_0 := m_{3Q_0}\left(|f|^{s'}\right)^{\frac{1}{s'}}$. We choose $A_* = (2\cdot 18^n)^{\frac{1}{s'}} > 1$.

For $k \in \mathbb{N}$, let

$$
D_k := \bigcup \left\{ Q \in \mathcal{D}(Q_0);\, m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}} > \gamma_0 A_*^k \right\}.
$$

Considering the maximal cubes with respect to inclusion, we can write $D_k = \bigcup_j Q_{k,j}$. By the maximality of $Q_{k,j}$, we obtain the following:

$$
\gamma_0 A_*^k < m_{3Q_{k,j}}\left(|f|^{s'}\right)^{\frac{1}{s'}} \le 2^{\frac{n}{s'}} \gamma_0 A_*^k. \tag{5}
$$

On the other hand, let $E_0 = Q_0 \backslash D_1$ and $E_{k,j} = Q_{k,j} \backslash D_{k+1}$. Then $\{E_0\}$ and $\{E_{k,j}\}$ are disjoint and $E_0 \cup \left(\bigcup_{k,j} E_{k,j}\right)$ recovers Q_0. Moreover if we take sufficently large A_*, then we have

$$
|Q_0| \le 2\,|E_0| \quad \text{and} \quad |Q_{k,j}| \le 2\,|E_{k,j}|. \tag{6}
$$

In [12, 24, 25, 26, 27, 29], the above relationships (6) are shown. For the sake of self-containedness, we shall check (6) . Firstly, fix a cube $Q_{k,j}$. Since $Q_{k,j} = (Q_{k,j} \cap D_{k+1}) \cup E_{k,j}$, we estimate $|Q_{k,j} \cap D_{k+1}|$. For $x \in Q_{k,j} \cap D_{k+1}$, there exists $Q \ni x$ such that $3Q \subset 3Q_{k,j}$. Therefore we have

$$
\gamma_0 A_*^{k+1} < M(|f|^{s'} \chi_{3Q_{k,j}})(x)^{\frac{1}{s'}}.
$$

Hence we have

$$Q_{k,j} \cap D_{k+1} \subset \left\{ x \in Q_{k,j}; M(|f|^{s'} \chi_{3Q_{k,j}})(x)^{\frac{1}{s'}} > \gamma_0 A_*^{k+1} \right\}.$$

By the weak-boundedness of M on $L^1(\mathbb{R}^n)$ and (5), the above inclusion implies that

$$\begin{aligned}
|Q_{k,j} \cap D_{k+1}| &\leq \left| \left\{ x \in Q_{k,j}; M(|f|^{s'} \chi_{3Q_{k,j}})(x) > (\gamma_0 A_*^{k+1})^{s'} \right\} \right| \\
&\leq \frac{3^n}{(\gamma_0 A_*^{k+1})^{s'}} \int_{3Q_{k,j}} |f(y)|^{s'} \, dy \\
&= \frac{3^n}{(\gamma_0 A_*^{k+1})^{s'}} \left(⨍_{3Q_{k,j}} |f(y)|^{s'} \, dy \right) |3Q_{k,j}| \\
&< \frac{18^n}{A_*^{s'}} |Q_{k,j}| = \frac{1}{2} |Q_{k,j}|.
\end{aligned}$$

A similar argument yields

$$|Q_0| \leq 2|E_0|.$$

Moreover let

$$\mathcal{D}_0(Q_0) = \left\{ Q \in \mathcal{D}(Q_0); m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}} \leq \gamma_0 A_* \right\}$$

and

$$\mathcal{D}_{k,j}(Q_0) = \left\{ Q \in \mathcal{D}(Q_0); Q \subset Q_{k,j}, \gamma_0 A_*^k < m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}} \leq \gamma_0 A_*^{k+1} \right\}.$$

Then $\mathcal{D}_0(Q_0) \cup \left(\bigcup_{k,j} \mathcal{D}_{k,j}(Q_0) \right)$ recovers $\mathcal{D}(Q_0)$;

$$\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \left(\bigcup_{k,j} \mathcal{D}_{k,j}(Q_0) \right). \tag{7}$$

By duality, we have

$$\|I_{\Omega,\alpha}(f_0)w\|_{L^q(Q_0)} = \sup_{\|g\|_{L^{q'}(Q_0)}=1} \|I_{\Omega,\alpha}(f_0)wg\|_{L^1(Q_0)}.$$

So we take $g \in L^{q'}(Q_0)$, $g(x) \geq 0$ a.e. $x \in Q_0$ and $\|g\|_{L^{q'}(Q_0)} = 1$. Moreover let

$$I = \sum_{Q \in \mathcal{D}_0(Q_0)} l(Q)^\alpha m_{3Q}(|f|^{s'})^{\frac{1}{s'}} \int_Q w(x)g(x)dx \tag{8}$$

and

$$II_{k,j} = \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q)^\alpha m_{3Q}(|f|^{s'})^{\frac{1}{s'}} \int_Q w(x)g(x)dx. \tag{9}$$

By (7), (8) and (9), we have

$$\begin{aligned}
&\int_{Q_0} |I_{\Omega,\alpha}(f_0)(x)|\, w(x)g(x)dx \\
&\le C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha m_{3Q}(|f|^{s'})^{\frac{1}{s'}} \int_Q w(x)g(x)dx \\
&= C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left(\sum_{Q\in\mathcal{D}_0(Q_0)} + \sum_{k,j}\sum_{Q\in\mathcal{D}_{k,j}(Q_0)}\right) l(Q)^\alpha m_{3Q}(|f|^{s'})^{\frac{1}{s'}} \int_Q w(x)g(x)dx \\
&= C\,\|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left(I + \sum_{k,j} II_{k,j}\right).
\end{aligned}$$

Next, since I is controlled in a manner similar to $II_{k,j}$, we estimate $II_{k,j}$.

$$\begin{aligned}
II_{k,j} &= \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q)^\alpha m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}} \int_Q w(x)g(x)dx \\
&\le \gamma_0 A_*^{k+1} \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q)^\alpha \int_Q w(x)g(x)dx \\
&\le A_* m_{3Q_{k,j}}\left(|f|^{s'}\right)^{\frac{1}{s'}} \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q)^\alpha \int_Q w(x)g(x)dx \\
&\le A_* m_{3Q_{k,j}}\left(|f|^{s'}\right)^{\frac{1}{s'}} l(Q_{k,j})^\alpha \int_{Q_{k,j}} w(x)g(x)dx \\
&\le A_*\,|Q_{k,j}|\, m_{3Q_{k,j}}\left(|f|^{s'}\right)^{\frac{1}{s'}} l(Q_{k,j})^\alpha \left(⨍_{Q_{k,j}} w(x)^{aq}dx\right)^{\frac{1}{aq}} \\
&\quad \times \left(⨍_{Q_{k,j}} g(x)^{(aq)'}dx\right)^{\frac{1}{(aq)'}}.
\end{aligned}$$

By virtue of (6) we have

$$\begin{aligned}
II_{k,j} &\le 2A_*\,|E_{k,j}|\, m_{3Q_{k,j}}\left(|f|^{s'}\right)^{\frac{1}{s'}} l(Q_{k,j})^\alpha \left(⨍_{Q_{k,j}} w(x)^{aq}dx\right)^{\frac{1}{aq}} \\
&\quad \times \left(⨍_{Q_{k,j}} g(x)^{(aq)'}dx\right)^{\frac{1}{(aq)'}} \qquad (10)\\
&= 2A_* \int_{E_{k,j}} l(Q_{k,j})^\alpha m_{3Q_{k,j}}\left(|f|^{s'}\right)^{\frac{1}{s'}} \left(⨍_{Q_{k,j}} w(y)^{aq}dy\right)^{\frac{1}{aq}}
\end{aligned}$$

$$\times \left(\fint_{Q_{k,j}} g(y)^{(aq)'} dy \right)^{\frac{1}{(aq)'}} dx$$
$$\le 2A_* \int_{E_{k,j}} \tilde{M}^{aq}_{\alpha,s}(f,w)(x) \cdot M\left[|g|^{(aq)'}\right](x)^{\frac{1}{(aq)'}} dx,$$

where we recall

$$\tilde{M}^{aq}_{\alpha,s}(f,w)(x) = \sup_{Q \ni x} l(Q)^\alpha m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}} \left(\fint_Q w(y)^{aq} dy \right)^{\frac{1}{aq}}.$$

Similarly, we obtain

$$I \le 2A_* \int_{E_0} \tilde{M}^{aq}_{\alpha,s}(f,w)(x) \cdot M\left[|g|^{(aq)'}\right](x)^{\frac{1}{(aq)'}} dx. \tag{11}$$

Hence from (10) and (11) we have

$$I + \sum_{k,j} II_{k,j} \le 2A_* \int_{Q_0} \tilde{M}^{aq}_{\alpha,s}(f,w)(x) \cdot M\left[|g|^{(aq)'}\right](x)^{\frac{1}{(aq)'}} dx.$$

By Hölder's inequality and the Hardy-Littlewood maximal theorem about the $L^{(aq)'}$-boundedness of the maximal operator, we have

$$\begin{aligned} I + \sum_{k,j} II_{k,j} &\le C\left(\int_{Q_0} \tilde{M}^{aq}_{\alpha,s}(f,w)(x)^q dx \right)^{\frac{1}{q}} \left(\int_{Q_0} M\left[|g|^{(aq)'}\right](x)^{\frac{q'}{(aq)'}} dx \right)^{\frac{1}{q'}} \\ &\le C\left(\int_{Q_0} \tilde{M}^{aq}_{\alpha,s}(f,w)(x)^q dx \right)^{\frac{1}{q}} \left(\int_{Q_0} |g(x)|^{q'} dx \right)^{\frac{1}{q'}} \\ &= C\left\| \tilde{M}^{aq}_{\alpha,s}(f,w) \right\|_{L^q(Q_0)}, \end{aligned}$$

where we used $\|g\|_{L^{q'}(Q_0)} = 1$ and $\mathrm{supp}(g) \subset Q_0$ for the equality of the last line. Therefore, we obtain the inequality:

$$\|I_{\Omega,\alpha}(f_0)w\|_{L^q(Q_0)} \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left\| \tilde{M}^{aq}_{\alpha,s}(f,w) \right\|_{L^q(Q_0)}.$$

By Lemma 3 and the condition (2), we can obtain the pointwise inequality;

$$\tilde{M}^{aq}_{\alpha,s}(f,w)(x) \le C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}} M_{\frac{\alpha p}{a}}\left[(fw)^{\frac{p}{a}}\right](x)^{\frac{a}{p}},$$

where

$$M_{\frac{\alpha p}{a}}\left[(fw)^{\frac{p}{a}}\right](x) = \sup_{Q \ni x} l(Q)^{\frac{\alpha p}{a}} \fint_Q |f(y)|^{\frac{p}{a}} w(y)^{\frac{p}{a}} dy.$$

In fact, since $a > 1$, we have $\left(\frac{p}{s'a}\right)' > \left(\frac{p}{s'}\right)'$. Thus, we take advantage of Hölder's inequality and Lemma 3 for their indices;

$$m_{3Q}\left(|f|^{s'}\right)^{\frac{1}{s'}}\left(\fint_{Q} w(y)^{aq}dy\right)^{\frac{1}{aq}} \leq C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}\left(\fint_{3Q}|f(y)|^{\frac{p}{a}}w(y)^{\frac{p}{a}}dy\right)^{\frac{a}{p}}.$$

This implies that

$$\tilde{M}^{aq}_{\alpha,s}(f,w)(x) \leq C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}} M_{\frac{\alpha p}{a}}\left[(fw)^{\frac{p}{a}}\right](x)^{\frac{a}{p}}.$$

Therefore we obtain

$$\begin{aligned}|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left\|\tilde{M}^{aq}_{\alpha,s}(f,w)\right\|_{L^q(Q_0)} &\leq C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left\|M_{\frac{\alpha p}{a}}\left[(fw)^{\frac{p}{a}}\right]^{\frac{a}{p}}\right\|_{L^q(Q_0)}\\ &\leq C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}\left\|M_{\frac{\alpha p}{a}}\left[(fw)^{\frac{p}{a}}\right]\right\|^{\frac{a}{p}}_{\mathcal{M}^{\frac{aq_0}{p}}_{\frac{aq}{p}}}.\end{aligned}$$

Because we have the following relationship with indices;

$$\frac{p}{aq_0} = \frac{p}{ap_0} - \frac{\frac{\alpha p}{a}}{n}, \quad \frac{a}{\frac{aq}{p}} = \frac{\frac{ap_0}{p}}{\frac{aq_0}{p}},$$

we can take advantage of Theorem A. This implies that $M_{\frac{\alpha p}{a}} : \mathcal{M}^{ap_0/p}_a(\mathbb{R}^n) \to \mathcal{M}^{aq_0/p}_{aq/p}(\mathbb{R}^n)$. Therefore we obtain

$$\begin{aligned}&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left\|\tilde{M}^{aq}_{\alpha,s}(f,w)\right\|_{L^q(Q_0)}\\ &\leq C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}\left\|(fw)^{\frac{p}{a}}\right\|^{\frac{a}{p}}_{\mathcal{M}_a^{\frac{ap_0}{p}}} = C\left[w^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{p}{s'}}\|fw\|_{\mathcal{M}^{p_0}_p}.\end{aligned}$$

Hence we obtain the desired inequality. ■

4.3. Proofs of main theorems for multilinear operators

Next we shall prove Theorem 3.

Proof of Theorem 3. For $Q_0 = Q(x_0, r)$, we decompose $f_j = f_j\chi_{3Q_0} + f_j\chi_{(3Q_0)^c} = f_j^0 + f_j^\infty$. Then we have the following;

$$\begin{aligned}I_{\Omega_*,\alpha,m}\left(\vec{f}\right)(x) &= I_{\Omega_*,\alpha,m}\left(f_1^0,\ldots,f_m^0\right)(x) + \sum_{(l_1,\ldots,l_m)\neq(0,\ldots,0)} I_{\Omega_*,\alpha,m}\left(f_1^{l_1},\ldots,f_m^{l_m}\right)(x)\\ &= A + \sum_{(l_1,\ldots,l_m)\neq(0,\ldots,0)} B_{(l_1,\ldots,l_m)}.\end{aligned}$$

By a standard argument, we estimate $B_{\vec{l}}$, where $\vec{l} = (l_1, \ldots, l_m)$. For $x \in Q_0$, we have

$$
\begin{aligned}
\left|B_{\vec{l}}\right| &\le \int_{\mathbb{R}^{mn}} \frac{\left|f_1^{l_1}(y_1)\cdots f_m^{l_m}(y_m)\right|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} \left|\Omega_*(x-y_1,\ldots,x-y_m)\right| d\vec{y} \\
&\le \int_{|(x-y_1,\ldots,x-y_m)|>r} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} \left|\Omega_*(x-y_1,\ldots,x-y_m)\right| d\vec{y} \\
&\le \sum_{k=0}^{\infty} \int_{2^k r<|(x-y_1,\ldots,x-y_m)|\le 2^{k+1}r} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} \left|\Omega_*(x-y_1,\ldots,x-y_m)\right| d\vec{y} \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{mn-\alpha}} \int_{|(x-y_1,\ldots,x-y_m)|<2^{k+1}r} \left|\Omega_*(x-y_1,\ldots,x-y_m)\right| |f_1(y_1)\cdots f_m(y_m)|\, d\vec{y}.
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
\left|B_{\vec{l}}\right| &\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{mn-\alpha}} \left(\int_{|(x-y_1,\ldots,x-y_m)|<2^{k+1}r} \left|\Omega_*(x-y_1,\ldots,x-y_m)\right|^s d\vec{y} \right)^{\frac{1}{s}} \\
&\quad \times \left(\int_{|(x-y_1,\ldots,x-y_m)|<2^{k+1}r} |f_1(y_1)\cdots f_m(y_m)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{mn-\alpha}} \left(\int_{|x-y_m|<2^{k+1}r} \cdots \int_{|x-y_2|<2^{k+1}r} \right. \\
&\quad \left. \left[\int_{|x-y_1|<2^{k+1}r} \left|\Omega_*(x-y_1,\ldots,x-y_m)\right|^s dy_1 \right] dy_2 \cdots dy_m \right)^{\frac{1}{s}} \\
&\quad \times \left(\int_{|(x-y_1,\ldots,x-y_m)|<2^{k+1}r} |f_1(y_1)\cdots f_m(y_m)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&= C \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{mn-\alpha}} \left(\int_{\mathbb{S}_{m,n}} \left(\int_0^{2^{k+1}r} \cdots \int_0^{2^{k+1}r} \right. \right. \\
&\quad \left. \left. \left|\Omega_*(l_1\xi_1,\ldots,l_m\xi_m)\right|^s \prod_{j=1}^{m} l_j^{n-1} dl_1 \cdots dl_m \right) d\vec{\xi} \right)^{\frac{1}{s}} \\
&\quad \times \left(\int_{|(x-y_1,\ldots,x-y_m)|<2^{k+1}r} |f_1(y_1)\cdots f_m(y_m)|^{s'} d\vec{y} \right)^{\frac{1}{s'}},
\end{aligned}
$$

where $d\vec{\xi} = d\xi_1 \cdots d\xi_m$. By the homogeneity of $\Omega_*(\lambda_1 x_1,\ldots,\lambda_m x_m) = \Omega_*(x_1,\ldots, x_m)$, we have

$$
\left|B_{\vec{l}}\right| \le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{mn-\alpha}} \left(\int_{\mathbb{S}_{m,n}} \left|\Omega_*(\xi_1,\ldots,\xi_m)\right|^s d\vec{\xi} \cdot \prod_{j=1}^{m} \int_0^{2^{k+1}} l_j^{n-1} dl_j \right)^{\frac{1}{s}}
$$

$$
\begin{aligned}
&\times\left(\int_{|(x-y_1,\ldots,x-y_m)|<2^{k+1}r}|f_1(y_1)\cdots f_m(y_m)|^{s'}\,d\vec{y}\right)^{\frac{1}{s'}}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{(2^kr)^{mn-\alpha}}\left(2^{k+1}r\right)^{\frac{mn}{s}}\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\prod_{j=1}^{m}\left(\int_{|x-y_j|<2^{k+1}r}|f_j(y_j)|^{s'}\,dy_j\right)^{\frac{1}{s'}}.
\end{aligned}
$$

Hence we have the following:

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left(\int_Q\left|B_{\vec{l}}\right|^q(w_1\cdots w_m)(x)^q dx\right)^{\frac{1}{q}}\\
&\le C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\sum_{k=0}^{\infty}(2^kr)^{\alpha-\frac{mn}{s'}}\prod_{j=1}^{m}\left(\int_{2^{k+3}\sqrt{n}Q_0}|f_j(y_j)|^{s'}\,dy_j\right)^{\frac{1}{s'}}\\
&\qquad\times|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left(\int_{Q_0}(w_1\cdots w_m)(x)^q dx\right)^{\frac{1}{q}}.
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left(\int_Q\left|B_{\vec{l}}\right|^q(w_1\cdots w_m)(x)^q dx\right)^{\frac{1}{q}}\\
&\le C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\sum_{k=0}^{\infty}(2^kr)^{\alpha-\frac{mn}{s'}}\prod_{j=1}^{m}\left(\int_{2^{k+3}\sqrt{n}Q_0}|f_j(y_j)|^{p_j}\,w_j(y_j)^{p_j}dy_j\right)^{\frac{1}{p_j}}\\
&\quad\times\left(\fint_{2^{k+3}\sqrt{n}Q_0}w_j(y_j)^{-s'\left(\frac{p_j}{s'}\right)'}dy_j\right)^{\frac{1}{s'}\cdot\frac{1}{\left(\frac{p_j}{s'}\right)'}}|Q_0|^{\frac{1}{q_0}}\\
&\quad\times\left(\fint_{Q_0}(w_1\cdots w_m)(x)^q dx\right)^{\frac{1}{q}}\prod_{j=1}^{m}\left|2^{k+3}\sqrt{n}Q_0\right|^{\frac{1}{s'}\cdot\frac{1}{\left(\frac{p_j}{s'}\right)'}}.
\end{aligned}
$$

By considering the multi-Morrey quantity, we obtain

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}}\left(\int_Q\left|B_{\vec{l}}\right|^q(w_1\cdots w_m)(x)^q dx\right)^{\frac{1}{q}}\\
&\le C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\|(f_1w_1,\ldots,f_mw_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}\sum_{k=0}^{\infty}\left(\frac{|Q_0|}{|2^{k+3}\sqrt{n}Q_0|}\right)^{\frac{1}{q_0}\left(1-\frac{1}{a}\right)}\\
&\quad\times\left[\left(\frac{|Q_0|}{|2^{k+3}\sqrt{n}Q_0|}\right)^{\frac{s'}{aq_0}}\left(\fint_{Q_0}(w_1\cdots w_m)(x)^q dx\right)^{\frac{s'}{q}}\right]^{\frac{1}{s'}}\\
&\quad\times\left[\prod_{j=1}^{m}\left(\fint_{2^{k+3}\sqrt{n}Q_0}w_j(y_j)^{-s'\cdot\left(\frac{p_j}{s'}\right)'}dy_j\right)^{\frac{1}{\left(\frac{p_j}{s'}\right)'}}\right]^{\frac{1}{s'}}.
\end{aligned}
$$

By the condition (3) and $a>1$, we obtain

$$
\begin{aligned}
&|Q_0|^{\frac{1}{q_0}-\frac{1}{q}} \left(\int_Q \left| B_{\vec{l}} \right|^q (w_1 \cdots w_m)(x)^q dx \right)^{\frac{1}{q}} \\
&\leq C \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \left\| (f_1 w_1, \ldots, f_m w_m) \right\|_{\mathcal{M}^{p_0}_{\vec{P}}} \left[\vec{w}^{s'} \right]^{\frac{1}{s'}}_{\frac{a q_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'}} \sum_{k=0}^{\infty} \left(\frac{|Q_0|}{|2^{k+3}\sqrt{n} Q_0|} \right)^{\frac{1}{q_0}\left(1-\frac{1}{a}\right)} \\
&\leq C \left[\vec{w}^{s'} \right]^{\frac{1}{s'}}_{\frac{a q_0}{s'}, \frac{q}{s'}, \frac{\vec{P}}{s'}} \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \left\| (f_1 w_1, \ldots, f_m w_m) \right\|_{\mathcal{M}^{p_0}_{\vec{P}}} .
\end{aligned}
$$

Therefore we obtain the desired inequality for $B_{\vec{l}}$. Next, we shall estimate A: For $x \in Q_0$, we have

$$
\begin{aligned}
|A| &\leq \sum_{k \in \mathbb{Z}} \int_{2^{k-1} r \leq |(x-y_1, \ldots, x-y_m)| < 2^k r} \frac{|\Omega_*(x-y_1, \ldots, x-y_m)| \prod_{j=1}^m \left| f_j^0(y_j) \right|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y} \\
&\leq C \sum_{k \in \mathbb{Z}} \frac{1}{(2^k r)^{mn-\alpha}} \left(\int_{|(x-y_1, \ldots, x-y_m)| < 2^k r} |\Omega_*(x-y_1, \ldots, x-y_m)|^s d\vec{y} \right)^{\frac{1}{s}} \\
&\quad \times \left(\int_{|(x-y_1, \ldots, x-y_m)| < 2^k r} \left| f_1^0(y_j) \cdots f_m^0(y_m) \right|^{s'} d\vec{y} \right)^{\frac{1}{s'}} .
\end{aligned}
$$

By the homogeneity,

$$
\begin{aligned}
|A| &\leq C \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \sum_{k \in \mathbb{Z}} (2^k r)^{\alpha - \frac{mn}{s'}} \prod_{j=1}^m \left(\int_{3Q_0 \cap \{|x-y_j| < 2^k r\}} |f_j(y_j)|^{s'} dy_j \right)^{\frac{1}{s'}} \\
&\leq C \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \sum_{k \in \mathbb{Z}} \sum_{\substack{Q \in \mathcal{D}(Q_0), \\ l(Q) = 2^k r}} l(Q)^{\alpha - \frac{mn}{s'}} \prod_{j=1}^m \left(\int_{|x-y_j| < l(Q)} |f_j(y_j)|^{s'} dy_j \right)^{\frac{1}{s'}} \chi_Q(x) \\
&\leq C \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha - \frac{mn}{s'}} |3Q|^{\frac{m}{s'}} \prod_{j=1}^m \left(\fint_{3Q} |f_j(y_j)|^{s'} dy_j \right)^{\frac{1}{s'}} \chi_Q(x) \\
&= C \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha} m_{3Q} \left(\vec{f}^{s'} \right)^{\frac{1}{s'}} \chi_Q(x),
\end{aligned}
$$

where for the last equality we have used

$$
m_{3Q}\left(\vec{f}^{s'} \right) = \prod_{j=1}^m \left(\fint_{3Q} |f_j(y_j)|^{s'} dy_j \right).
$$

This implies that

$$
|A| \leq C \left\| \Omega_* \right\|_{L^s(\mathbb{S}_{m,n})} \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha} m_{3Q} \left(\vec{f}^{s'} \right)^{\frac{1}{s'}} \chi_Q(x).
$$

Let $\gamma_0 = m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}}$. We choose that $A_* = (2m)^{\frac{m}{s'}} \cdot 18^{\frac{mn}{s'}} > 1$.

For $k \in \mathbb{N}$, let

$$D_k = \bigcup \left\{ Q \in \mathcal{D}(Q_0); m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} > \gamma_0 A_*^k \right\}.$$

Considering the maximal cubes with respect to inclusion, we can write $D_k = \bigcup_j Q_{k,j}$. Moreover, by maximality of $Q_{k,i}$, we obtain the following inequality;

$$\gamma_0 A_*^k < m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \leq 2^{\frac{mn}{s'}} \gamma_0 A_*^k. \tag{12}$$

Let $E_0 = Q_0 \backslash D_1$ and $E_{k,i} = Q_{k,i} \backslash D_{k+1}$. Then E_0 and $E_{k,i}$ are disjoint. Moreover E_0 and $\bigcup_{k,i} E_{k,i}$ recover Q_0 and we have the following inequality:

$$|Q_0| \leq 2\,|E_0| \quad \text{and} \quad |Q_{k,i}| \leq 2\,|E_{k,i}|\,. \tag{13}$$

In [12, 24, 25, 26, 27, 29], the above relationships (13) are shown. For the sake of self-containedness, we shall check (13).

Since $Q_{k,i} = (Q_{k,i} \cap D_{k+1}) \cup E_{k,i}$, firstly we estimate $|Q_{k,i} \cap D_{k+1}|$. For $x \in Q_{k,i} \cap D_{k+1}$, there exists $Q \ni x$ such that $3Q \subset 3Q_{k,i}$. Therefore we have

$$\begin{aligned}
\gamma_0 A_*^{k+1} < m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} &= \prod_{j=1}^{m} \left(\fint_{3Q} |f_j(y_j)|^{s'} \chi_{3Q_{k,i}}(y_i) dy_i \right)^{\frac{1}{s'}} \\
&\leq \mathcal{M}\left(f_1^{s'}\chi_{3Q_{k,i}}, \ldots, f_m^{s'}\chi_{3Q_{k,i}}\right)(x)^{\frac{1}{s'}} \\
&\leq \prod_{j=1}^{m} M\left(|f_j|^{s'}\chi_{3Q_{k,i}}\right)(x)^{\frac{1}{s'}}.
\end{aligned}$$

Next, let

$$A_j := \left(\gamma_0 A_*^{k+1}\right)^{\frac{s'}{m}} m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{-\frac{1}{m}} \left(\fint_{3Q_{k,i}} |f_j(y_j)|^{s'} dy_j \right). \tag{14}$$

We pay attention to the following relationship:

$$\prod_{j=1}^{m} A_j = (\gamma_0 A_*^{k+1})^{s'}. \tag{15}$$

Moreover, by (12) and (14) we have

$$\frac{1}{A_j}\left(\frac{1}{|3Q_{k,i}|} \int_{3Q_{k,i}} |f_j(y_j)|^{s'} dy_j \right) \leq 2^n A_*^{-\frac{s'}{m}}. \tag{16}$$

(15) implies that

$$Q_{k,i} \cap D_{k+1} \subset \left\{ x \in Q_{k,i}; \prod_{j=1}^{m} M(|f_j|^{s'} \chi_{3Q_{k,i}})(x) > \left(\gamma_0 A_*^{k+1}\right)^{s'} \right\}$$
$$\subset \bigcup_{j=1}^{m} \left\{ x \in Q_{k,i}; M(|f_j|^{s'} \chi_{3Q_{k,i}})(x) > A_j \right\}.$$

By the weak-boundedness of M on $L^1(\mathbb{R}^n)$ and (16), we obtain

$$\begin{aligned}
|Q_{k,i} \cap D_{k+1}| &\leq \sum_{j=1}^{m} \left| \left\{ x \in Q_{k,i}; M(|f_j|^{s'} \chi_{3Q_{k,i}})(x) > A_j \right\} \right| \\
&\leq \sum_{j=1}^{m} \frac{3^n}{A_j} \left(\int_{3Q_{k,i}} |f_j(y_j)|^{s'} \, dy_j \right) \\
&= \sum_{j=1}^{m} \frac{3^n}{A_j} \left(\fint_{3Q_{k,i}} |f_j(y_j)|^{s'} \, dy_j \right) |3Q_{k,i}| \\
&\leq \sum_{j=1}^{m} 3^n \cdot 2^n A_*^{-\frac{s'}{m}} \cdot 3^n \, |Q_{k,i}| \\
&= m \cdot 18^n A_*^{-\frac{s'}{m}} \, |Q_{k,i}| \\
&= \frac{1}{2} |Q_{k,i}|.
\end{aligned}$$

A similar argument yields

$$|Q_0| \leq 2|E_0|.$$

Therefore we obtain (13). Let

$$\mathcal{D}_0(Q_0) = \left\{ Q \in \mathcal{D}(Q_0); m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \leq \gamma_0 A_* \right\},$$

and

$$\mathcal{D}_{k,i}(Q_0) = \left\{ Q \in \mathcal{D}(Q_0); Q \subset Q_{k,i}, \gamma_0 A_*^k < m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \leq \gamma_0 A_*^{k+1} \right\}.$$

Then these recover $\mathcal{D}(Q_0)$:

$$\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \left(\bigcup_{k,i} \mathcal{D}_{k,i}(Q_0) \right). \tag{17}$$

We distinguish two cases:

Case 1. $q > 1$.
By duality, we have

$$\left(\int_{Q_0} \left|I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)(x)\right|^q (w_1\cdots w_m)(x)^q dx\right)^{\frac{1}{q}} = \sup_{\|g\|_{L^{q'}(Q_0)}=1} \left\|I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)(w_1\cdots w_m)\cdot g\right\|_{L^1(Q_0)}.$$

So let $g \in L^{q'}(Q_0)$, $g(x) \geq 0$ a.e. $x \in Q_0$, $\mathrm{supp}(g) \subset Q_0$ and $\|g\|_{L^{q'}(Q_0)} = 1$. Moreover let

$$I = \sum_{Q\in\mathcal{D}_0(Q_0)} l(Q)^\alpha \cdot m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \int_Q (w_1\cdots w_m)(x)g(x)dx \tag{18}$$

and

$$II_{k,j} = \sum_{Q\in\mathcal{D}_{k,j}(Q_0)} l(Q)^\alpha \cdot m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \int_Q (w_1\cdots w_m)(x)g(x)dx. \tag{19}$$

By (17), (18) and (19) we have

$$\begin{aligned}
&\int_{Q_0} \left|I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)(x)\right| (w_1\cdots w_m)(x)g(x)dx \\
&\leq C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \sum_{Q\in\mathcal{D}(Q_0)} l(Q)^\alpha m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \int_Q (w_1\cdots w_m)(x)g(x)dx \\
&= C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left(\sum_{Q\in\mathcal{D}_0(Q_0)} + \sum_{k,i}\sum_{Q\in\mathcal{D}_{k,i}(Q_0)}\right) \\
&\qquad \times l(Q)^\alpha \cdot m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} \int_Q (w_1\cdots w_m)(x)g(x)dx \\
&= C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left(I + \sum_{k,i} II_{k,i}\right).
\end{aligned}$$

Next, we estimate $II_{k,i}$. By (12) and Hölder's inequality, we have

$$\begin{aligned}
II_{k,i} &\leq \gamma_0 A_*^{k+1} \sum_{Q\in\mathcal{D}_{k,i}(Q_0)} l(Q)^\alpha \int_Q (w_1\cdots w_m)(y)g(y)dy \\
&\leq \gamma_0 A_*^{k+1} l(Q_{k,i})^\alpha \int_{Q_{k,i}} (w_1\cdots w_m)(y)g(y)dy
\end{aligned}$$

$$
\begin{aligned}
&\le A_* m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} l(Q_{k,i})^{\alpha}\int_{Q_{k,i}}(w_1\cdots w_m)(x)g(x)dx\\
&\le A_*\left|Q_{k,i}\right| m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} l(Q_{k,i})^{\alpha}\left(⨍_{Q_{k,i}}(w_1\cdots w_m)(y)^{aq}dy\right)^{\frac{1}{aq}}\\
&\quad\times\left(⨍_{Q_{k,i}}|g(y)|^{(aq)'}dy\right)^{\frac{1}{(aq)'}}.
\end{aligned}
$$

By virtue of (13) we obtain

$$
\begin{aligned}
II_{k,i} &\le 2A_*\left|E_{k,i}\right| m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}} l(Q_{k,i})^{\alpha}\left(⨍_{Q_{k,i}}(w_1\cdots w_m)(y)^{aq}dy\right)^{\frac{1}{aq}}\\
&\quad\times\left(⨍_{Q_{k,i}}|g(y)|^{(aq)'}dy\right)^{\frac{1}{(aq)'}}\\
&\le 2A_*\int_{E_{k,i}} l(Q_{k,i})^{\alpha} m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}}\left(⨍_{Q_{k,i}}(w_1\cdots w_m)(y)^{aq}dy\right)^{\frac{1}{aq}}\\
&\quad\times\left(⨍_{Q_{k,i}}|g(y)|^{(aq)'}dy\right)^{\frac{1}{(aq)'}}dx\\
&\le 2A_*\int_{E_{k,i}}\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)M\left[g^{(aq)'}\right](x)^{\frac{1}{(aq)'}}dx,
\end{aligned}
$$

where for the last inequality, we used (4):

$$
\tilde{M}^{aq}_{\alpha,s}(\vec{f},w_1\cdots w_m)(x)=\sup_{Q\ni x} l(Q)^{\alpha}\cdot m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}}\left(⨍_{Q}(w_1\cdots w_m)(y)^{aq}dy\right)^{\frac{1}{aq}}.
$$

Similarly, we obtain

$$
I\le 2A_*\int_{E_0}\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)M\left[g^{(aq)'}\right](x)^{\frac{1}{(aq)'}}dx.
$$

Therefore by the $L^{(aq)'}$-boundedness of the Hardy-Littlewood maximal function M, we have

$$
\begin{aligned}
&I+\sum_{k,i} II_{k,i}\\
&\le 2A_*\int_{Q_0}\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)M\left[g^{(aq)'}\right](x)^{\frac{1}{(aq)'}}dx\\
&\le 2A_*\left(\int_{Q_0}\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)^q dx\right)^{\frac{1}{q}}\left(\int_{Q_0}M\left[g^{(aq)'}\right](x)^{\frac{q'}{(aq)'}}dx\right)^{\frac{1}{q'}}\\
&\le 2A_*C\left(\int_{Q_0}\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},(w_1\cdots w_m)\right)(x)^q dx\right)^{\frac{1}{q}},
\end{aligned}
$$

where we used $\|g\|_{L^{q'}(Q_0)} = 1$ as well for the last line. Therefore we have

$$\begin{aligned}&\left\|I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)(w_1\cdots w_m)\right\|_{L^q(Q_0)}\\ &\le C\,\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\left\|\tilde{M}^{aq}_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)\right\|_{L^q(Q_0)}.\end{aligned}$$

By Lemma 3 and the condition (3), we have the following:

$$\tilde{M}^{aq}_{\alpha,s}(\vec{f},w_1\cdots w_m)(x)\le C\left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}}\mathcal{M}_{\alpha,\frac{\vec{P}}{a}}(f_1w_1,\dots,f_mw_m)(x).$$

By Lemma 5, we have the following inequality;

$$\left\|\mathcal{M}_{\alpha,\frac{\vec{P}}{a}}(f_1w_1,\dots,f_mw_m)\right\|_{\mathcal{M}^{q_0}_q}\le C\,\|(f_1w_1,\dots,f_mw_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.$$

Hence, when $q>1$, we have the desired inequality:

$$\begin{aligned}&\left\|I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)w_1\cdots w_m\right\|_{\mathcal{M}^{q_0}_q}\\ &\le C\left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}}\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\|(f_1w_1,\dots,f_mw_m)\|_{\mathcal{M}^{p_0}_{\vec{P}}}.\end{aligned}$$

Case 2. $0<q\le 1$.

Let

$$L:=\left(\sum_{Q\in\mathcal{D}(Q_0)}l(Q)^{\alpha}m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}}\chi_Q(x)\right)^{q}.$$

Since $0<q\le 1$, by (17), we have the following

$$\begin{aligned}L&\le\sum_{Q\in\mathcal{D}(Q_0)}l(Q)^{\alpha q}m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\chi_Q(x)\\ &=\left(\sum_{Q\in\mathcal{D}_0(Q_0)}+\sum_{k,i}\sum_{Q\in\mathcal{D}_{k,i}(Q_0)}\right)l(Q)^{\alpha q}m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\chi_Q(x).\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int_{Q_0}\left|I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)(x)\right|^q(w_1\cdots w_m)(x)^q dx\\ &\le C\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\left(\sum_{Q\in\mathcal{D}_0(Q_0)}+\sum_{k,i}\sum_{Q\in\mathcal{D}_{k,i}(Q_0)}\right)l(Q)^{\alpha q}m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\int_Q(w_1\cdots w_m)(x)^q dx\\ &=C\,\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})}\left(I+\sum_{k,i}II_{k,i}\right),\end{aligned}$$

where

$$I=\sum_{Q\in\mathcal{D}_0(Q_0)} l(Q)^{\alpha q} m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\int_Q (w_1\cdots w_m)(y)^q dy$$

and

$$II_{k,i}=\sum_{Q\in\mathcal{D}_{k,i}(Q_0)} l(Q)^{\alpha q} m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\int_Q (w_1\cdots w_m)(y)^q dy.$$

We estimate $II_{k,i}$. By (12) and Hölder's inequality, we have

$$\begin{aligned}
II_{k,i}&=\sum_{Q\in\mathcal{D}_{k,i}(Q_0)} l(Q)^{\alpha q} m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\int_Q (w_1\cdots w_m)(y)^q dy\\
&\le l(Q_{k,i})^{\alpha q}\left(\gamma_0 A_*^{k+1}\right)^q\sum_{Q\in\mathcal{D}_{k,i}(Q_0)}\int_Q (w_1\cdots w_m)(y)^q dy\\
&\le l(Q_{k,i})^{\alpha q}\left(\gamma_0 A_*^{k+1}\right)^q\int_{Q_{k,i}} (w_1\cdots w_m)(y)^q dy\\
&\le A_*^q\,|Q_{k,i}|\,l(Q_{k,i})^{\alpha q} m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\left(⨍_{Q_{k,i}} (w_1\cdots w_m)(y)^q dy\right).
\end{aligned}$$

By virtue of (13) we have

$$\begin{aligned}
II_{k,i}&\le 2A_*^q\,|E_{k,i}|\,l(Q_{k,i})^{\alpha q} m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{q}{s'}}\left(⨍_{Q_{k,i}} (w_1\cdots w_m)(y)^q dy\right)\\
&=2A_*^q\int_{E_{k,i}}\left(l(Q_{k,i})^{\alpha} m_{3Q_{k,i}}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}}\left(⨍_{Q_{k,i}} (w_1\cdots w_m)(y)^q dy\right)^{\frac{1}{q}}\right)^q dx \qquad (20)\\
&\le 2A_*^q\int_{E_{k,i}}\tilde{M}^q_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)^q dx,
\end{aligned}$$

where for the last line we used (4):

$$\tilde{M}^q_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)=\sup_{Q\ni x} l(Q)^{\alpha} m_{3Q}\left(\vec{f}^{s'}\right)^{\frac{1}{s'}}\left(⨍_Q (w_1\cdots w_m)(y)^q dy\right)^{\frac{1}{q}}.$$

Similarly, we have

$$I\le 2A_*^q\int_{E_0}\tilde{M}^q_{\alpha,s}\left(\vec{f},w_1\cdots w_m\right)(x)^q dx. \qquad (21)$$

(20) and (21) give us the following inequality:

$$I+\sum_{k,i} II_{k,i}\le C\int_{Q_0}\tilde{M}^q_{\alpha,s}(\vec{f},w_1\cdots w_m)(x)^q dx.$$

Hence we obtain the following inequality:

$$\left\| I_{\Omega_*,\alpha,m}(f_1^0,\dots,f_m^0)\, w_1\cdots w_m \right\|_{L^q(Q_0)} \le C\,\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\| \tilde{M}^q_{\alpha,s}\left(\vec{f}, w_1\cdots w_m\right)\right\|_{L^q(Q_0)}.$$

By Lemma 3 for $a > 1$ and the condition (3), we obtain the following pointwise inequality:

$$\tilde{M}^q_{\alpha,s}\left(\vec{f}, w_1\cdots w_m\right)(x) \le C\left[\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{aq_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}} \mathcal{M}_{\alpha,\frac{\vec{P}}{a}}\left(f_1w_1,\dots,f_mw_m\right)(x).$$

By Lemma 5, for $0 < q \le 1$, we obtain the desired inequality. ∎

Lastly we prove Theorem 8.

Proof of Theorem 8. By $q_0 < r_0$, we have $p_0 < \frac{n}{\alpha}$. Moreover, by $\frac{s'}{p_j} < 1$, we have $\frac{s'}{p} < m$. Hence we obtain $1 < s' < mp \le mp_0 < \frac{mn}{\alpha}$. This implies that $0 \le \alpha s' < mn$. By Lemma 8, we obtain the inequality.

$$\left\|\mathcal{M}_{\Omega_*,\alpha,m}\left(\vec{f}\right) v\right\|_{\mathcal{M}^{q_0}_q} \le C\,\|\Omega_*\|_{L^s(\mathbb{S}_{m,n})} \left\|\mathcal{M}_{\alpha s',m}\left(f_1^{s'},\dots,f_m^{s'}\right)^{\frac{1}{s'}} v\right\|_{\mathcal{M}^{q_0}_q}. \tag{22}$$

Meanwhile, we have

$$\left\|\mathcal{M}_{\alpha s',m}\left(f_1^{s'},\dots,f_m^{s'}\right)^{\frac{1}{s'}} v\right\|_{\mathcal{M}^{q_0}_q} = \left\|\mathcal{M}_{\alpha s',m}\left(f_1^{s'},\dots,f_m^{s'}\right) v^{s'}\right\|^{\frac{1}{s'}}_{\mathcal{M}^{q_0/s'}_{q/s'}}. \tag{23}$$

Moreover since

$$\frac{1}{q_0/s'} = \frac{1}{p_0/s'} + \frac{1}{r_0/s'} - \frac{\alpha s'}{n}, \quad \frac{q/s'}{q_0/s'} = \frac{p/s'}{p_0/s'} \quad \text{and} \quad \frac{n}{\alpha s'} < \frac{r_0}{s'},$$

by Lemma 12, we have

$$\begin{aligned}
&\left\|\mathcal{M}_{\alpha s',m}\left(f_1^{s'},\dots,f_m^{s'}\right) v^{s'}\right\|^{\frac{1}{s'}}_{\mathcal{M}^{q_0/s'}_{q/s'}} \\
&\le C\left[v^{s'},\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{q_0}{s'},\frac{r_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}} \left\|\left(|f_1|^{s'}w_1^{s'},\dots,|f_m|^{s'}w_m^{s'}\right)\right\|^{\frac{1}{s'}}_{\mathcal{M}^{p_0/s'}_{\vec{P}/s'}} \\
&= C\left[v^{s'},\vec{w}^{s'}\right]^{\frac{1}{s'}}_{\frac{q_0}{s'},\frac{r_0}{s'},\frac{q}{s'},\frac{\vec{P}}{s'}} \left\|(f_1w_1,\dots,f_mw_m)\right\|_{\mathcal{M}^{p_0}_{\vec{P}}}.
\end{aligned} \tag{24}$$

If we combine (22), (23) and (24), then we have the desired inequality. ∎

5. The Prototypical Example

In [11], we gave an example concerning the multi-Morrey quantity. Here we give another example of a function which does not belong to the product on m Morrey spaces but multi-Morrey quantity is finite.

Example 1. Let $n = 1$, $m = 2$, $0 < p_0 < \infty$, $1 < p_1, p_2 < \infty$ and $\frac{1}{p_0} = \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)$. Moreover we take $-\frac{1}{p_1} < \theta_1 \leq 0$ and $-\frac{1}{p_2} < \theta_2 \leq 0$:

$$\theta_1 + \theta_2 = -\frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right), \tag{25}$$

$$\theta_1 \neq -\frac{1}{2p_1} \quad \text{or} \quad \theta_2 \neq -\frac{1}{2p_2}. \tag{26}$$

Then we have

$$\left(|x|^{\theta_1}, |x|^{\theta_2}\right) \in \mathcal{M}^{p_0}_{(p_1,p_2)}(\mathbb{R}) \backslash \left(\mathcal{M}^{2p_1}_{p_1}(\mathbb{R}) \times \mathcal{M}^{2p_2}_{p_2}(\mathbb{R})\right).$$

In fact, by (26), we have $\left(|x|^{\theta_1}, |x|^{\theta_2}\right) \notin \left(\mathcal{M}^{2p_1}_{p_1}(\mathbb{R}) \times \mathcal{M}^{2p_2}_{p_2}(\mathbb{R})\right)$. Let $f_1(x) = |x|^{\theta_1}$ and $f_2(x) = |x|^{\theta_2}$. Then we have

$$\begin{aligned}
&\|(f_1, f_2)\|_{\mathcal{M}^{p_0}_{(p_1,p_2)}} \\
&= \sup_{I=(a,b)} (b-a)^{\frac{1}{p_0}} \left(\frac{1}{b-a}\int_a^b |f_1(y_1)|^{p_1}\, dy_1\right)^{\frac{1}{p_1}} \left(\frac{1}{b-a}\int_a^b |f_2(y_2)|^{p_2}\, dy_2\right)^{\frac{1}{p_2}} \\
&= \sup_{I=(a,b)} (b-a)^{-\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(\int_a^b |x|^{\theta_1 p_1}\, dx\right)^{\frac{1}{p_1}} \left(\int_a^b |x|^{\theta_2 p_2}\, dx\right)^{\frac{1}{p_2}}.
\end{aligned}$$

By virtue of symmetry, it suffices to distinguish two cases as we shall do below.

Case 1. $0 < a < b$. If we take $0 < t = \frac{a}{b} < 1$, then we have

$$\begin{aligned}
&(b-a)^{-\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(\int_a^b |x|^{\theta_1 p_1}\, dx\right)^{\frac{1}{p_1}} \left(\int_a^b |x|^{\theta_2 p_2}\, dx\right)^{\frac{1}{p_2}} \\
&= \left(\frac{1}{1+\theta_1 p_1}\right)^{\frac{1}{p_1}} \left(\frac{1}{1+\theta_2 p_2}\right)^{\frac{1}{p_2}} \\
&\quad \cdot \left(1-\frac{a}{b}\right)^{-\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(1-\left(\frac{a}{b}\right)^{\theta_1 p_1+1}\right)^{\frac{1}{p_1}} \left(1-\left(\frac{a}{b}\right)^{\theta_2 p_2+1}\right)^{\frac{1}{p_2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{1+\theta_1 p_1}\right)^{\frac{1}{p_1}} \left(\frac{1}{1+\theta_2 p_2}\right)^{\frac{1}{p_2}} \\
&\quad \cdot (1-t)^{-\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(1-t^{\theta_1 p_1+1}\right)^{\frac{1}{p_1}} \left(1-t^{\theta_1 p_2+1}\right)^{\frac{1}{p_2}} \\
&\le \left(\frac{1}{1+\theta_1 p_1}\right)^{\frac{1}{p_1}} \left(\frac{1}{1+\theta_2 p_2}\right)^{\frac{1}{p_2}} \\
&\quad \cdot (1-t)^{-\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} (1-t)^{\theta_1+\frac{1}{p_1}} (1-t)^{\theta_2+\frac{1}{p_2}}
\end{aligned}
$$

where we used the condition $\theta_1, \theta_2 \le 0$ for the last inequality. Hence this quantity is bounded by

$$
\left(\frac{1}{1+\theta_1 p_1}\right)^{\frac{1}{p_1}} \left(\frac{1}{1+\theta_2 p_2}\right)^{\frac{1}{p_2}}.
$$

Case 2. $a < 0 < b$. If we take $0 < t = -\frac{a}{b}$, then we have

$$
\begin{aligned}
&(b-a)^{-\frac{1}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(\int_a^b |x|^{\theta_1 p_1}\, dx\right)^{\frac{1}{p_1}} \left(\int_a^b |x|^{\theta_2 p_2}\, dx\right)^{\frac{1}{p_2}} \\
&= \left(\frac{1}{1+\theta_1 p_1}\right)^{\frac{1}{p_1}} \left(\frac{1}{1+\theta_2 p_2}\right)^{\frac{1}{p_2}} \cdot (1+t)^{\frac{1}{p_0}-\frac{1}{p_1}-\frac{1}{p_2}} \left(1+t^{p_1\theta_1+1}\right)^{\frac{1}{p_1}} \left(1+t^{p_2\theta_2+1}\right)^{\frac{1}{p_2}}.
\end{aligned}
$$

When $0 < t < 1$, the above quantity is bounded by a constant. If $t \ge 1$, then we have

$$
\begin{aligned}
&(1+t)^{\frac{1}{p_0}-\frac{1}{p_1}-\frac{1}{p_2}} \left(1+t^{p_1\theta_1+1}\right)^{\frac{1}{p_1}} \left(1+t^{p_2\theta_2+1}\right)^{\frac{1}{p_2}} \\
&< t^{\frac{1}{p_0}-\frac{1}{p_1}-\frac{1}{p_2}} \left(1+t^{p_1\theta_1+1}\right)^{\frac{1}{p_1}} \left(1+t^{p_2\theta_2+1}\right)^{\frac{1}{p_2}} \\
&= \left(\frac{1+t^{p_1\theta_1+1}}{t^{p_1\theta_1+1}}\right)^{\frac{1}{p_1}} \left(\frac{1+t^{p_2\theta_2+1}}{t^{p_2\theta_2+1}}\right)^{\frac{1}{p_2}} \\
&= \left(1+\frac{1}{t^{\theta_1\cdot p_1+1}}\right)^{\frac{1}{p_1}} \left(1+\frac{1}{t^{\theta_2\cdot p_2+1}}\right)^{\frac{1}{p_2}} \le 2^{\frac{1}{p_1}+\frac{1}{p_2}} < \infty.
\end{aligned}
$$

Acknowledgments

The author is grateful to the anonymous referee for his/her having checked the manuscript carefully. The author would like to thank Professor E. Sato and Professor H. Gunawan for the helpful suggestions which essentially improved this paper.

References

1. D. Adams, A note on Riesz potentials, *Duke Math. J.*, **42** (1975), 765-778.

2. D. Adams and L. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften, Fundamental Principles of Mathematical Sciences, Vol. 314, Springer-Verlag, Berlin, 1996.

3. X. Chen and Q. Xue, Weighted estimates for a class of multilinear fractional type operators, *J. Math. Anal. Appl.*, **362** (2010), 355-373.

4. F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, *Rend. Math. Appl.*, **7** (1987), 273-279.

5. Y. Ding and S. Lu, Weighted norm inequalities for fractional integral operators with rough kernel, *Can. J. Math.*, **50** (1998), 29-39.

6. J. Duoandikoetxea, *Fourier Analysis*, Grad. Studies in Math. 29. Amer. Math. Soc., Providence, RI, 2001.

7. L. Grafakos, *Modern Fourier Analysis*, Second Edition, Graduate Texts in Math., No. 250, Springer, New York, 2008.

8. H. Gunawan and Eridani, Fractional integrals and generalized Olsen inequalities, *Kyungpook Math. J.*, **49** (2009), 31-39.

9. H. Gunawan, Y. Sawano and I. Sihwaningrum, Fractional integral operators in nonhomogeneous spaces, *Bull. Aust. Math. Soc.*, **80** (2009), 324-334.

10. T. Iida, A characterization of a multiple weights class, *Tokyo J. Math.*, **35** (2012), 375-383.

11. T. Iida, E. Sato, Y. Sawano and H. Tanaka, Sharp bounds for multilinear fractional integral operators on Morrey type spaces, *Positivity*, **16** (2012), 339-358.

12. T. Iida, E. Sato, Y. Sawano and H. Tanaka, Weighted norm inequalities for multilinear fractional operators on Morrey spaces, *Studia Math.*, **205** (2011), 139-170.

13. Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.*, **282** (2009), 219-231.

14. A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv in Math.*, **220** (2009), 1222-1264.

15. S. Lu, Y. Ding and D. Yan, *Singular Integrals and Related Topics*, World Scientific publishing, Singapore, 2007.

16. K. Moen, Weighted inequalities for multilinear fractional integral operators, *Collect. Math.*, **60** (2009), 213-238.

17. B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. Amer. Math. Soc.*, **192** (1974), 261-274.

18. P. Olsen, Fractional integration, Morrey spaces and Schrödinger equation, *Comm. Partial Differential Equations*, **20** (1995), 2005-2055.

19. J. Peetre, On the theory of $\mathcal{L}_{p,\lambda}$ spaces, *J. Funct. Anal.*, **4** (1969), 71-87.

20. E. L. Persson and N. Samko, Weighted Hardy and potential operators in the generalized Morrey spaces, *J. Math. Anal. Appl.*, **377** (2011), 792-806.

21. N. Samko, Weighted Hardy and singular operators in Morrey spaces, *J. Math. Anal. Appl.*, **350** (2009), 56-72.

22. N. Samko, Weighted Hardy and potential operators in Morrey spaces, *J. Funct. Spaces Appl.*, (2012), Vol. 2012, doi: 10.1155/2012/678171.

23. N. Samko, On a Muckenhoupt-Type Condition for Morrey spaces, *Mediterranean J. Math.*, 2012, Doi: 10.1007/s00009-012-0208-2.

24. Y. Sawano, S. Sugano and H. Tanaka, Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces, *Trans. Amer. Math. Soc.*, **363** (2011), 6481-6503

25. Y. Sawano, S. Sugano and H. Tanaka, *A Note on Generalized Fractional Integral Operators on Generalized Morrey Spaces*, Boundary Value Problems, Vol. 2009, (2009), Article ID 835865, 18 pp.

26. Y. Sawano, S. Sugano and H. Tanaka, *Olsen's Inequality and Its Applications to Schrödinger Equations*, RIMS Kôkyûroku Bessatsu, B26, Harmonic Analysis and Nonlinear Partial Differential Equations, 51-80.

27. Y. Sawano, S. Sugano and H. Tanaka, Orlicz-Morrey spaces and fractional operators, *Potential Analysis*, **36** (2012), 517-556.

28. S. Sugano, Some inequalities for generalized fractional integral operators on generalized Morrey spaces, *Math. Inequal. Appl.*, **14** (2011), 849-865.

29. H. Tanaka, Morrey spaces and fractional operators, *J. Aust. Math. Soc.*, **88** (2010), 247-259.

Takeshi Iida
Department of General Education
Fukushima National College of Technology
Fukushima, 970-8034
Japan
E-mail: tiida@fukushima-nct.ac.jp