TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 1, pp. 27-37, February 2014
DOI: 10.11650/tjm.18.2014.2747
This paper is available online at http://journal.taiwanmathsoc.org.tw

MEROMORPHIC SOLUTIONS OF DIFFERENCE EQUATION $f(z+1) = R \circ f(z)$

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Abstract. In this paper, we investigate the solutions of difference equation $f(z+1) = R \circ f(z)$ by utilizing Nevanlinna theory, where $R(z)$ is a rational function. And we also research the quantity of zeroes, poles, fixed points, and Borel exceptional values of the solutions.

1. Introduction and Main Results

In this paper, a meromorphic function always means that it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see[1, 2, 3, 4]):

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots .$$

And we denote any quantity by $S(r,f)$ satisfying

$$S(r,f) = o\{T(r,f)\}, \text{ as } r \to \infty,$$

possibly outside of a set E with finite linear measure, not necessarily the same at each occurrence. We use $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$ respectively. We also use $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$, which is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N(r, \frac{1}{f-z})}{\log r}.$$

Yanagihara [5] proved the following theorem with purpose to investigate the solutions of non-linear difference equation $y(x+1) = R(x, y(x))$.

Received December 14, 2012, accepted June 21, 2013.
Communicated by Der-Chen Chang.
2010 *Mathematics Subject Classification*: 30D35, 34M10.
Key words and phrases: Uniqueness, Difference equation, Order.
The research was supported by the Fundamental Research Funds for the Central Universities (No. 2011QNA25).

Theorem A. [5]. *Any nontrivial meromorphic solution* $y(z)$ *of equation*

$$f(z+1) = R \circ f(z)$$

is transcendental unless $\deg_R = 1$.

The author also said that the equation $y(x+1) = R(x, y(x))$ may have rational solutions. E.g.,

$$y(x+1) = \frac{(x^4+1)(-2y^3(x)+y(x)+2x^6+2x-1)}{y^2(x)+1}$$

and

$$y(x+1) = \frac{(y^3(x)+2x^5+x^4)}{x^4}$$

have the solution $y(x) = x^2$. This makes natural questions to ask that what can be said to the solutions of equation $f(z+1) = R \circ f(z)$ provided $\deg_R = 1$, and can any solution be $y(x) = x^2$ as in the examples above too? In this paper, we give a negative answer to the questions and obtain the following theorem.

Theorem 1. *Let* $R(z)$ *be a non-constant rational function. For the following difference equation*

$$f(z+1) = R \circ f(z), \tag{1}$$

(1) suppose it admits a non-constant rational solution $f(z)$*, then both* $R(z)$ *and* $f(z)$ *are fractional linear functions;*
(2) suppose it admits a transcendental meromorphic function $f(z)$ *of finite order* $\sigma(f)$*, then* $R(z)$ *is a fractional linear function, and it is denoted by*

$$R(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$*, furthermore:*
(2.1) if $bc \neq 0$*, then* $\lambda(f) = \lambda(\frac{1}{f}) = \tau(f) = \sigma(f)$*;*
(2.2) if $R \neq id$ *and* $\sigma(f) > 0$*, then*
(2.2.1) $f(z)$ *has at most one finite Borel exceptional value provided* $(d-a)^2 + 4b = 0$ *when* $c \neq 0$*;*
(2.2.2) if $f(z)$ *has Borel exceptional value* ∞*, then* $f(z)$ *has at most one finite Borel exceptional value* $\frac{b}{1-a}$.

Example 1. Equation $f(z+1) = \frac{1}{2-z} \circ f(z)$ admits a fractional linear solution $\frac{z-1}{z}$.

Example 1 shows that the fractional linear solution does exist in (1) of Theorem 1.

Example 2. Equation $f(z+1) = (2-z) \circ f(z)$ admits a solution $e^{\pi i z} + 1$, which satisfies $\lambda(\frac{1}{f}) < \sigma(f)$.

Example 3. Equation $f(z+1) = \frac{-z}{z+1} \circ f(z)$ admits a solution $\frac{-2e^{\pi iz}}{e^{\pi iz}-1}$, which satisfies $\lambda(f) < \sigma(f)$ and has two finite Borel exceptional values $0, -2$.

Examples 2-3 show that the condition $bc \neq 0$ is necessary in (2.1) of Theorem 1. Example 3 also shows that the conclusion may be not valid if $(d-a)^2 + 4b \neq 0$ when $c \neq 0$ in (2.2.1) of Theorem 1. And Example 2 shows the case that $f(z)$ has Borel exceptional value ∞ and $\frac{b}{1-a}$ may happen in (2.2.2) of Theorem 1.

In addition, comparing with many papers [6, 7] researched complex difference Riccati equation, there is only few paper [8] dealing with the properties of solutions of complex difference Riccati equation, thus we put our effort on it. Take paper [8] for example, the authors obtained the following theorem.

Theorem B. [8]. *Let $\delta = \pm 1$ be a constant and $A(z) = \frac{m(z)}{n(z)}$ be an irreducible non-constant rational function, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z) = m$ and $\deg n(z) = n$. If $f(z)$ is a transcendental finite order meromorphic solution of*

$$f(z+1) = \frac{A(z) + \delta f(z)}{\delta - f(z)},$$

then,
(i) if $\sigma(f) > 0$, then f has at most one Borel exceptional value;
(ii) $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$;
(iii) if $A(z) \not\equiv -z^2 - z + 1$, then the exponent of convergence of fixed points of f satisfies $\tau(f) = \sigma(f)$.

In this paper, we consider the more general case and obtain the following theorem.

Theorem 2. *Let $b(z), c(z)$ be two non-constant rational functions. Suppose the following difference equation*

$$f(z+1) = \frac{af(z) + b(z)}{c(z)f(z) + d} \tag{2}$$

admits a transcendental meromorphic function $f(z)$ of finite order, then
(i) $\lambda(f) = \lambda(\frac{1}{f}) = \sigma(f)$;
(ii) $\tau(f) = \sigma(f)$ provided $(zc(z) + d)(z+1) - az - b(z) \not\equiv 0$.
Furthermore, if $\frac{b(z)}{c(z)}$ is not any constant and $\sigma(f) > 0$, then $f(z)$ has at most one Borel exceptional value.

2. Some Lemmas

To prove our results, we need some lemmas as follows.

Lemma 1. (see [3]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane and*

$$R(f) = \frac{p(f)}{q(f)},$$

where $p(f) = \sum_{k=0}^{p} a_k f^k$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in $f(z)$. If the coefficients a_k, b_j are small functions of $f(z)$ and $a_k(z) \not\equiv 0, b_j(z) \not\equiv 0$, then

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2. (see [9]). *Let $c_1, \ldots c_n$ be non-zero constants and suppose that $f(z)$ is a non-rational meromorphic solution of a difference equation of the form*

$$\Pi_{i=1}^{n} f(z + c_i) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_p(z)f^p(z)}{b_0(z) + b_1(z)f(z) + \cdots + b_t(z)f^t(z)} \tag{3}$$

with meromorphic coefficients $a_i(z), b_j(z)$ of growth $S(r, f)$ such that $a_p(z), b_t(z) \not\equiv 0$. If

$$\max\{\lambda(f), \lambda(\frac{1}{f})\} < \sigma(f),$$

then equation (3) is form of

$$\Pi_{i=1}^{n} f(z + c_i) = c(z) f^k(z),$$

where $c(z)$ is meromorphic, $T(r, c) = S(r, f)$ and $k \in Z$.

Lemma 3. (see [10]). *Let $w(z)$ be a transcendental meromorphic solution of finite order of difference equation*

$$P(z, w) = 0,$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a \in S(r, w)$, then

$$m(r, \frac{1}{w - a}) = S(r, w).$$

Lemma 4. (see [10]). *Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f), P(z, f), Q(z, f)$ are difference polynomials in $f(z)$ such that the total degree of $H(z, f)$ in $f(z)$ and its shifts is n and that the corresponding total degree of $Q(z, f)$ is at most n. If $H(z, f)$ just contains one term of maximal total degree, then for any $\varepsilon > 0$, holds

$$m(r, P(z, f)) = O(r^{\rho - 1 + \varepsilon}) + S(r, f)$$

possible outside of an exceptional set of finite logarithmic measure.

Lemma 5. (see [11]). *Let $f(z)$ be a meromorphic function with finite order σ and η be a nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

3. The Proofs

3.1. Proof of Theorem 1.

(1) Suppose Equation (1) admits a non-constant rational solution $f(z)$. Then, by Lemma 1 , we obtain

$$T(r, f(z+1)) = \deg_f \log r + O(1) = T(r, R \circ f(z)) = \deg_R \deg_f \log r + O(1).$$

Thus we get $\deg_R \leq 1$, and then $R(z)$ is a fractional linear function. We divide the proof into two distinguish cases as follows.

Case 1.1. $c = 0$, we assume $d = 1$ without loss of generality, then Equation (1) becomes

$$f(z+1) = af(z) + b. \tag{4}$$

We suppose that $f(z)$ has a pole z_0, then by Equation (4), we obtain that $z_0 + 1, z_0 + 2, \cdots$ are also poles, which means $f(z)$ is transcendental. Then we obtain a contradiction. Thus $f(z)$ is a polynomial. Noting the following fact that

$$a = \frac{f(z+1) - b}{f(z)} \to 1, \text{ as } z \to \infty,$$

we obtain $a = 1$, and then Equation (1) becomes

$$f(z+1) = f(z) + b, \text{ i.e., } f'(z+1) = f'(z).$$

Thus $f'(z)$ is a constant otherwise it is a non-constant period function, i.e., it is a transcendental meromorphic function, which is a contradiction that f is a non-constant rational solution. So we obtain that both $f(z)$ and R are linear functions.

Case 1.2. $c \neq 0$, we assume $c = 1$ without loss of generality, then Equation (1) becomes

$$f(z+1) - a = \frac{b - ad}{f(z) + d}. \tag{5}$$

Let $A = b - ad, (\neq 0)$, $f(z) = \frac{m(z)}{n(z)}$ and $m = \deg_{m(z)}, n = \deg_{n(z)}$, where $m(z), n(z)$ are two mutually prime polynomials. If $m > n$, then Equation (5) implies

$$o(1) = \frac{b - ad}{f(z) + d} = f(z+1) - a \to \infty, \text{ as } z \to \infty,$$

which is impossible. Thus $m \leq n$. Substituting $f(z) = \frac{m(z)}{n(z)}$ into Equation (5), we obtain

(6) $$(m(z+1) - an(z+1))(m(z) + dn(z)) = An(z)n(z+1).$$

Since $m(z), n(z)$ are two mutually prime polynomials, we obtain that $m(z)+dn(z), n(z)$ are two mutually prime polynomials. In the similar way, we obtain that $m(z+1), n(z+1)$ are two mutually prime polynomials, and then $m(z+1) - an(z+1), n(z+1)$ are two mutually prime polynomials. Thus by Equation (6), we obtain

$$n(z)|m(z+1) - an(z+1) \quad \text{and} \quad n(z+1)|m(z) + dn(z).$$

Noting $m \leq n$, then $\exists$ a constant C such that

(7) $$m(z+1) - an(z+1) = Cn(z) \quad \text{and} \quad C(m(z) + dn(z)) = An(z+1).$$

It is obvious that $C \neq 0$. By eliminating $m(z)$ in Equation (7), we obtain that

(8) $$C^2 n(z) + (aC + Cd)n(z+1) = An(z+2).$$

Rewriting Equation (8) as the following form

$$C^2 + (aC + Cd) \leftarrow C^2 + \frac{(aC + Cd)n(z+1)}{n(z)} = \frac{An(z+2)}{n(z)} \to A, \text{ as } z \to \infty,$$

we obtain $A = C^2 + aC + Cd$. Then Equation (8) becomes

(9) $$C^2(n(z+2) - n(z)) + (aC + Cd)(n(z+2) - n(z+1)) = 0.$$

Set $g(z) = n(z+1) - n(z)$, then Equation (9) becomes

(10) $$C^2(g(z+1) + g(z)) + (aC + Cd)g(z+1) = 0.$$

From Equation (10), we obtain $2C^2 + (aC + Cd) = 0$ via a similar method. Thus Equation (10) becomes $g(z+1) = g(z)$, then $g(z) = n(z+1) - n(z)$ is a constant, i.e., $n(z)$ is a linear function. Noting $m \leq n$ once again, we obtain that $f(z)$ is a fractional linear function.

(2) Suppose Equation (1) admits a transcendental meromorphic function $f(z)$ of finite order. Then, by Lemma 5, we obtain that

$$T(r, f(z+1)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) = \deg_R T(r, f) + S(r, f).$$

Thus we get $\deg_R \leq 1$, then $R(z)$ is a fractional linear function.
(2.1) We assume $c = 1$ without loss of generality, and rewrite Equation (1) as the following form

(11) $$f(z)f(z+1) = af(z) + b - df(z+1).$$

By Equation (11) and Lemma 4, we obtain

$$m(r, f) = S(r, f).$$

Thus

$$N(r, f) = T(r, f) + S(r, f),$$

and then $\lambda(\frac{1}{f}) = \sigma(f)$. Noting $b \neq 0$, by Equation (11) and Lemma 3, we obtain

$$m(r, \frac{1}{f}) = S(r, f).$$

Thus

$$N(r, \frac{1}{f}) = T(r, f) + S(r, f),$$

and then $\lambda(f) = \sigma(f)$. Setting $f(z) = y(z) + z$ and substituting it into Equation (11), we obtain

$$T(r, f) = T(r, y) + O(\log r)$$

and

$$\begin{aligned} P(z, y) : &= y(z)y(z+1) + y(z)(z+1-a) + y(z+1)(z+d) \\ &+ (z+1)(z+d) - az - b = 0. \end{aligned} \tag{12}$$

Since $P(z, 0) = (z+1)(z+d) - az - b \not\equiv 0$, from Equation (12) and Lemma 3, we obtain

$$m(r, \frac{1}{y}) = S(r, y).$$

Thus

$$N(r, \frac{1}{y}) = T(r, y) + S(r, y) = T(r, f) + S(r, f),$$

and then $\tau(f) = \sigma(f)$.

(2.2.1.) Suppose that $f(z)$ has two finite Borel exception values $A, B, (A \neq B)$. Set

$$g(z) = \frac{f(z) - A}{f(z) - B}. \tag{13}$$

Then $T(r, f) = T(r, g) + O(1)$ and

$$\lambda(g) = \lambda(f - A) < \sigma(g), \quad \lambda(\frac{1}{g}) = \lambda(f - B) < \sigma(g).$$

From Equation (13), we get

$$f(z) = \frac{A - Bg(z)}{1 - g(z)}. \tag{14}$$

We consider two cases as follows.
Case 2.2.1.1. $c=0$, we assume $d=1$ without loss of generality again. Substituting Equation(14) into Equation (4), we obtain

$$g(z+1)=\frac{A-aA-b+(aB-A+b)g(z)}{B-aA-b+(aB+b-B)g(z)}. \tag{15}$$

It is obvious that $B-aA-b, aB+b-B$ can not be zero synchronously. From Lemma 2, we obtain

$$g(z+1)=c(z)g^k(z), \tag{16}$$

where $c(z)$ is meromorphic, $T(r,c)=S(r,g)$ and $k\in Z$. From Lemma 5, we obtain $k=1$. And substituting $g(z+1)=c(z)g(z)$ into Equation (15), we get

$$c(z)g^2(aB+b-B)=A-aA-b+(aB-A+b-c(z)(B-aA-b))g.$$

Thus we get

$$aB+b-B=A-aA-b=0.$$

It implies that $A=B=\frac{b}{1-a}$ or $R=id$, which is a contradiction.
Case 2.2.1.2. $c\neq 0$, we assume $c=1$. Substituting Equation(14) into Equation (5) and using the similar method in Case 2.2.1.1, we get

$$g(z+1)=\frac{A^2+Ad-Aa-b-(AB+Ad-Ba-b)g(z)}{AB+Bd-Aa-b-(B^2+Bd-Ba-b)g(z)}$$

and

$$B^2+Bd-Ba-b=A^2+Ad-Aa-b=0. \tag{17}$$

But Equation(17) implies that $A=B$ provided $(d-a)^2+4b=0$, which is a contraction.
(2.2.2.) For the case, $f(z)$ has Borel exceptional value ∞ and one finite Borel exceptional value A, we set $g(z)=f(z)-A$, then $T(r,g)=T(r,f)+O(1)$ and $\lambda(g)<\sigma(g), \lambda(\frac{1}{g})<\sigma(g)$. We consider two following cases.
Case 2.2.2.1. $c\neq 0$, we assume $c=1$. Using the similar method in Case 2.2.1.1, we get

$$g(z+1)=\frac{Aa-Ad-A^2+b+(a-A)g(z)}{A+d+g(z)}=c(z)g(z),$$

where $c(z)$ is meromorphic such that $T(r,c)=S(r,g)$. It is impossible obviously.
Case 2.2.2.2. $c=0$, we assume $d=1$. Using the similar method in Case 2.2.1.1, we get

$$g(z+1)=ag(z)+Aa-A+b=c(z)g(z),$$

where $c(z)$ is meromorphic such that $T(r,c) = S(r,g)$. Thus $Aa - A + b = 0$, which means $A = \frac{b}{1-a}$ provided $R \neq id$. The proof of Theorem 1 is completed. ■

3.2. Proof of Theorem 2.

We rewrite Equation (2) as the following form

$$c(z)f(z)f(z+1) = af(z) + b(z) - df(z+1). \tag{18}$$

From Equation (18) and Lemma 4, noting $c(z) \not\equiv 0$, we obtain

$$m(r,f) = S(r,f).$$

Thus

$$N(r,f) = T(r,f) + S(r,f),$$

and then $\lambda(\frac{1}{f}) = \sigma(f)$. Noting $b(z) \not\equiv 0$, From Equation (18) and Lemma 3, we obtain

$$m(r,\frac{1}{f}) = S(r,f).$$

Thus

$$N(r,\frac{1}{f}) = T(r,f) + S(r,f),$$

and then $\lambda(\frac{1}{f}) = \sigma(f)$. Setting $f(z) = y(z) + z$ and substituting it into Equation (18), we obtain

$$T(r,f) = T(r,y) + O(\log r)$$

and

$$\begin{aligned} P(z,y) := {} & c(z)y(z)y(z+1) + y(z)(zc(z) + c(z) - a) + y(z+1)(zc(z) + d) \\ & + (zc(z) + d)(z+1) - az - b(z) = 0. \end{aligned} \tag{19}$$

Since $P(z,0) = (zc(z) + d)(z+1) - az - b(z) \not\equiv 0$, From Equation (19) and Lemma 3, we obtain

$$m(r,\frac{1}{y}) = S(r,y).$$

Thus

$$N(r,\frac{1}{y}) = T(r,y) + S(r,y) = T(r,f) + S(r,f),$$

and then $\tau(f) = \sigma(f)$. Suppose $f(z)$ has two finite Borel exception values $A, B, (A \neq B)$. Set

$$g(z) = \frac{f(z) - A}{f(z) - B}, \text{ i.e., } f(z) = \frac{A - Bg(z)}{1 - g(z)},$$

and substitute it into Equation (2), we obtain

$$\lambda(g) = \lambda(f(z) - A) < \sigma(g),\ \lambda(\frac{1}{g}) = \lambda(f(z) - B) < \sigma(g)$$

and

$$g(z+1) = \frac{A^2c(z) + Ad - Aa - b(z) - (ABc(z) + Ad - Ba - b(z))g(z)}{ABc(z) + Bd - Aa - b(z) - (B^2c(z) + Bd - Ba - b(z))g(z)}. \tag{20}$$

From Equation (20) and Lemma 2, we obtain $B^2c(z) - b(z) = 0, A^2c(z) - b(z) = 0$ in the similar way, which contradict our condition that $\frac{b(z)}{c(z)}$ is not any constant. If $f(z)$ has one finite Borel exception value A and ∞, then set $g(z) = f(z) - A$ and substitute it into Equation (2), we obtain

$$g(z+1) = \frac{Aa + b(z) - A^2c(z) - Ad + (a - Ac(z))g(z)}{Ac(z) + d + c(z)g(z)}. \tag{21}$$

From Equation (21) and Lemma 2, we obtain $c(z) = 0$ in the similar way, which is also a contradiction. The proof of Theorem 2 is completed. ■

Acknowledgment

The author would like to thank the referee for his/her comments and suggestions.

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