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QUASI-PERIODIC SOLUTIONS OF 1D NONLINEAR SCHRÖDINGER EQUATION WITH A MULTIPLICATIVE POTENTIAL

Xiufang Ren

Abstract. This paper deals with one-dimensional (1D) nonlinear Schrödinger equation with a multiplicative potential, subject to Dirichlet boundary conditions. It is proved that for each prescribed integer $b > 1$, the equation admits small-amplitude quasi-periodic solutions, whose b-dimensional frequencies are small dilation of a given Diophantine vector. The proof is based on a modified infinite-dimensional KAM theory.

1. Introduction and Statement of the Theorem

The aim in the present paper is to prove the existence of quasi-periodic solutions, whose frequencies are small dilation of a given Diophantine vector ω^*, with dilation factor λ, i.e.,

$$\omega = \lambda\omega^*,\ \lambda \approx 1,\ \lambda \in \mathbb{R}, \tag{1.1}$$

of the one-dimensional (1D) nonlinear Schrödinger equation

$$iu_t - u_{xx} + V(x)u + |u|^2u + f(|u|^2)u = 0, \quad t \in \mathbb{R},\ x \in [0,\pi], \tag{1.2}$$

subject to Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,\pi), \tag{1.3}$$

where $V(x)$ is real analytic on $[0,\pi]$, f is real analytic near $u = 0$, with $f(0) = f'(0) = 0$.

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Before describing our main result in detail, we present some literature on quasi-periodic solutions for Hamiltonian PDEs.

In the 90's, the finite-dimensional KAM theory (see for example the works of Bourgain [5], Eliasson [9], Li-Yi [18], Xu-You [23], You [24]) has been successfully extended to infinite dimensions, dealing with certain classes of partial differential equations carrying a Hamiltonian structure. In particular, the existence problem of quasi-periodic solutions for Hamiltonian PDEs has received considerable attention in the last twenty years. The main difficulty is the presence of arbitrarily 'small divisors' in the series expansion of the solutions. The first pioneering existence results of quasi-periodic solutions have been proved by Kuksin [15] and Wayne [26] for 1D analytic PDEs on an interval with Dirichlet boundary conditions, through suitable extensions of KAM theory. In this case, the eigenvalues of the Laplacian are simple(see also Kuksin [16], Kuksin-Pöschel [17], Pöschel [21]).

Later on, many authors developed different versions of infinite-dimensional KAM theorems, by studying 1D nonlinear Schrödinger or wave equations, with constant, parameterized or multiplicative potentials, subject to Dirichlet or periodic boundary conditions. Concretely, they proved the existence of small-amplitude quasi-periodic solutions or the persistence of lower dimensional invariant tori for these Hamiltonian PDEs(see the works of Bambusi-Graffi [1], Geng-Yi [14], Liang-You [19], Liu-Yuan [20], Yuan [25]). In addition, there are also some existence results of quasi-periodic solutions concerning higher dimensional Schrödinger or wave equations, see Bourgain [6, 7], Eliasson-Kuksin [10], Geng-You [12], Wang [27, 28].

For finite-dimensional Hamiltonian systems, we quote the results of Bourgain [5] and Eliasson [9] as follows.

Eliasson [9] addressed a revised finite-dimensional KAM theorem, considering a real analytic function

$$h(y,z) = h_0(y) + \langle \Omega(y), y' \rangle + O^3(z),$$

on certain open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and proved the existence of an invariant n-torus Γ in a neighborhood of $\{y = y_0, z = 0\}$, under the non-degenerate conditions

$$\begin{aligned} &\det(D\omega(y)) \neq 0, \\ &\langle l, \Omega(y) - \omega(y)(D\omega(y))^{-1} D\Omega(y) \rangle \neq 0, \forall y \in \mathbb{R}^n,\ l \in \mathbb{Z}^m \backslash 0,\ |l| \leq 3, \end{aligned} \tag{1.4}$$

and the Melnikov's non-resonance conditions

$$|\langle k, \omega(y) \rangle + \langle l, \Omega(y) \rangle| \geq K^{-1}(|k| + |l|)^{-\tau},\ \ \forall (k,l) \in \mathbb{Z}^n \times \mathbb{Z}^m \setminus 0,\ |l| \leq 3,$$

where $\tau > n-1$. Moreover, the tangential frequency vector $\widetilde{\omega}$ of Γ is of the form

$$\widetilde{\omega} = t\omega(y_0),\ \omega(y) = Dh_0(y),\ t \in \mathbb{R},\ t \approx 1. \tag{1.5}$$

In addition, Bourgain [5] used the KAM-method and the Nash-Moser type methods to show the persistence of the invariant torus $\mathbb{T}^b \times \{0\} \times \{0\}$ in $\mathbb{R}^{2b} \times \mathbb{R}^{2r}$-phase space for a real analytic Hamiltonian function

$$\begin{aligned} H &= H(I, \theta, y) = H(I_1, \cdots, I_b, \theta_1, \cdots, \theta_b, y_1, \cdots, y_r) \\ &= \langle \lambda_0, I \rangle + \sum_{s=1}^{r} \mu_s |y_s|^2 + |I|^2 + \varepsilon H_1(I, \theta, y), \end{aligned}$$

under the first Melnikov's non-resonance condition

$$\langle \lambda_0, k \rangle - \mu_s \neq 0, \ \ \forall k \in \mathbb{Z}^b, \ s = 1, 2, \cdots, r.$$

Moreover, the perturbed frequency vector λ can be taken of the form

$$\lambda = t\lambda_0, \ t \in \mathbb{R}, \ t \approx 1. \tag{1.6}$$

At the same time, he raised an open problem that if this result can be generalized to an infinite-dimensional case in the Nash-Moser setting, only under the first Melnikov's non-resonance condition.

It is worth mentioning that, in the last two years, Berti-Biasco [2], Geng-Ren [13] have extended the above finite-dimensional results to infinite dimensions, dealing with nonlinear wave equations with constant potentials in the KAM setting. In addition, Ren [22] has proved the same result for 1D nonlinear Schrödinger equation with a Fourier multiplier via infinite-dimensional KAM theory. In the present paper, we restrict our attention to the multiplicative potential case (see (1.2)), and we are aiming to partially provide a positive answer to the open problem raised by Bourgain [5]. Actually, we will show the existence of quasi-periodic solutions with tangential frequencies (1.1) under conditions (1.3) and some non-degenerate conditions in the KAM setting. For a result concerning higher dimensional Schrödinger equation in the Nash-Moser setting in this direction, see Berti-Bolle [3].

The main difficulty in this manuscript consists in conducting the measure estimates. We avoid constructing a translation transformation in each KAM step adopted in Geng-Ren [13] and Ren [22], but apply Fubini's theorem (see Pöschel [21, Lemma 5]) and some techniques in Berti-Biasco [2] to simplify the estimation process. In addition, the equation considered in this manuscript is more general than that of Ren [22], since the Fourier multiplier in Ren [22] is artificial to some extent.

We outline the main steps in this manuscript as follows. Firstly, we deduce the Birkhoff normal form up to order four. This can be realized by the results of Du-Yuan [8], Kuksin-Pöschel [17], Yuan [25]. Secondly, we conduct one step of KAM iteration, which is similar to that of Ren [22]. Thirdly, after operating infinitely many KAM iterations, we will prove the existence of positive measure Cantor-like parameter set of b-dimensional amplitude ξ (see Proposition 6.1), by means of Fubini's theorem.

Finally, according to the proof process of Proposition 6.1, we obtain the existence of positive measure Cantor-like 1D parameter set of dilation factor λ(see Proposition 6.2), by means of some techniques in Berti-Biasco [2] and the cut-off procedure in Berti-Bolle [4].

Now, we are in a position to state our main result.

Theorem. *Consider 1D nonlinear Schrödinger equation* (1.2) *with Dirichlet boundary conditions* (1.3), *choose a fixed b-index integer set* $J_b := \{i_1, \cdots, i_b\}, b > 1$, *satisfying* $i_1 < \cdots < i_b$, *and* i_1 *is large enough, then for any* $\tau > 3b+4$, *sufficiently small* $\varepsilon > 0$, *and* $\rho_* \in (0,1)$, *there exists positive-measure Cantor-like subset* $\widetilde{\mathcal{O}} \subset [\varepsilon^2\rho_*, 2\varepsilon^2\rho_*]^b$, *such that for each* $\xi \in \widetilde{\mathcal{O}}$, (1.2) *has a small-amplitude real analytic quasi-periodic solution*

$$u(t,x) = \sum_{j=1}^{b} \sqrt{\xi_j} e^{i\omega_j t}(\sin i_j x - \frac{\cos i_j x}{2i_j}\int_0^x V(s)\,\mathrm{d}s + O(\frac{1}{i_j^2})) + o(|\xi|^{\frac{1}{2}}),$$

with Diophantine frequency $\omega = \lambda\omega^*, \lambda \approx 1, \omega^* \in \mathcal{D}_{\varepsilon^2\rho_*,\tau}(\mathit{see\ Remark}\ 3.4)$.

Remark 1.1. The assumption that i_1 should be large enough is necessary, see details in Proposition 3.2.

2. Relevant Notations

For given b vectors in $\mathbb{Z}^+$, say $\{i_1, \cdots, i_b\} := J_b$, denote $\mathbb{Z}_1 = \mathbb{Z}^+ \setminus J_b$. For given $\rho > 0$, let ℓ^ρ be the Banach space of bi-infinite, complex valued sequences $z = (\cdots, z_n, \cdots)_{n\in\mathbb{Z}_1}$(its complex conjugate $\bar{z} = (\cdots, \bar{z}_n, \cdots)_{n\in\mathbb{Z}_1}$, $\bar{z}_n \in \mathbb{C}$), endowed with the finite weighted norm

$$\|z\|_\rho = \sum_{n\in\mathbb{Z}_1} |z_n| e^{n\rho}.$$

We define the complex neighborhood of $\mathbb{T}^b \times \{y=0\} \times \{z=0\} \times \{\bar{z}=0\}$ in $\mathbb{T}^b \times \mathbb{R}^b \times \ell^\rho \times \ell^\rho$ as

$$D(r,s) = \{(x,y,z,\bar{z}) : |\Im x| < r, |y| < s^2, \|z\|_\rho < s, \|\bar{z}\|_\rho < s\},$$

where $|\cdot|$ denotes the sup-norm of complex vectors. Let $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n\in\mathbb{Z}_1}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n\in\mathbb{Z}_1}$, $\alpha_n, \beta_n \in \mathbb{N}$, with finitely many non-zero components of positive integers. The product $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$. For any given real analytic function

$$F(x,y,z,\bar{z}) = \sum_{\alpha,\beta} F_{\alpha\beta}(x,y) z^\alpha \bar{z}^\beta,$$

which depends on a parameter $\xi \in \mathcal{O}$ Whitney smoothly, we define its weighted norm as

$$
\begin{aligned}
\|F\|_{D(r,s),\mathcal{O}} &\equiv \sup_{\substack{\|z\|_\rho<s\\ \|\bar z\|_\rho<s}} \sum_{\alpha,\beta} \|F_{\alpha\beta}\||z^\alpha||\bar z^\beta|,\\
F_{\alpha\beta} &= \sum_{k\in\mathbb{Z}^b, l\in\mathbb{N}^b} F_{kl\alpha\beta}(\xi) y^l e^{i\langle k,x\rangle},\\
\|F_{\alpha\beta}\| &\equiv \sum_{k,l} |F_{kl\alpha\beta}|_{\mathcal{O}} s^{2|l|} e^{|k|r},\ |F_{kl\alpha\beta}|_{\mathcal{O}} = \sup_{\xi\in\mathcal{O}} \max_{|p|\le 1} \left|\frac{\partial^p F_{kl\alpha\beta}(\xi)}{\partial \xi^p}\right|
\end{aligned}
\tag{2.1}
$$

($\langle\cdot,\cdot\rangle$ being the standard inner product in $\mathbb{C}^b$).

The weighted norm of the Hamiltonian vector field

$$X_F = (F_y, -F_x, \{iF_{z_n}\}_{n\in\mathbb{Z}_1}, \{-iF_{\bar z_n}\}_{n\in\mathbb{Z}_1})$$

associated with F is defined as [1]

$$
\begin{aligned}
\|X_F\|_{D(r,s),\mathcal{O}} \equiv\ & \|F_y\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_x\|_{D(r,s),\mathcal{O}}\\
&+\frac{1}{s}\Big(\sum_{n\in\mathbb{Z}_1} \|F_{z_n}\|_{D(r,s),\mathcal{O}} e^{n\rho} + \sum_{n\in\mathbb{Z}_1} \|F_{\bar z_n}\|_{D(r,s),\mathcal{O}} e^{n\rho}\Big).
\end{aligned}
\tag{2.2}
$$

For any real analytic functions F and G, define the poisson bracket

$$\{F,G\} = \langle \frac{\partial F}{\partial x}, \frac{\partial G}{\partial y}\rangle - \langle \frac{\partial F}{\partial y}, \frac{\partial G}{\partial x}\rangle + i\sum_n \Big(\frac{\partial F}{\partial z_n}\frac{\partial G}{\partial \bar z_n} - \frac{\partial F}{\partial \bar z_n}\frac{\partial G}{\partial z_n}\Big).$$

For given $l \in \mathbb{Z}^\infty$, define the norms

$$|l| = \sum_{j\ge 1} |l_j|, \quad \langle l\rangle = \max\{1, |\sum_{j\ge 1} j\, l_j|\}.$$

3. Normal Forms

The aim of this section is to investigate the following equation

$$iu_t + Lu + |u|^2u + f(|u|^2)u = 0,\ L = -\frac{d^2}{dx^2} + V(x), \quad t\in\mathbb{R},\ x\in[0,\pi],$$

with Dirichlet boundary conditions (1.3). We can rewrite it as the Hamiltonian equation

$$u_t = 2i\frac{\partial H}{\partial \bar u},$$

[1] The norm $\|\cdot\|_{D(r,s),\mathcal{O}}$ for scalar functions is defined in (2.1). The vector function $G: D(r,s)\times\mathcal{O} \to \mathbb{C}^m$, ($m<\infty$) is similarly defined as $\|G\|_{D(r,s),\mathcal{O}} = \sum_{i=1}^m \|G_i\|_{D(r,s),\mathcal{O}}$.

where

$$H = \frac{1}{2}\int_0^{\pi} (|u_x|^2 + V(x)|u|^2)\,\mathrm{d}x + \frac{1}{4}\int_0^{\pi} |u|^4\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} g(|u|^2)\,\mathrm{d}x$$

and $g = \int_0 f\mathrm{d}z$. As is well known that

$$\mu_j = j^2 + \frac{1}{\pi}\int_0^{\pi} V(x)\,\mathrm{d}x + O(\frac{1}{j^2})$$

are the eigenvalues of the associated Sturm-Liouville operator L on the interval $[0,\pi]$, and the eigenfunctions are

$$\phi_j(x) = \kappa_j^{-1}\Big(\sin jx - \frac{\cos jx}{2j}\int_0^{x} V(s)\,\mathrm{d}s + O(\frac{1}{j^2})\Big), j \in \mathbb{Z}^+,\ \forall x \in [0,\pi], \tag{3.1}$$

with $||\phi_j(x)||_{L^2_{[0,\pi]}} = 1$.

Proposition 3.1. $\kappa_j^2 = \frac{\pi}{2} + O(\frac{1}{j^2})$, *and*

$$\kappa_i^2\kappa_j^2 \int_0^{\pi} \phi_i^2(x)\phi_j^2(x)\,\mathrm{d}x = \begin{cases} \frac{\kappa_i^2}{2} + O(\frac{1}{j^2}) + O(\frac{1}{ij|i-j|}), & i \neq j, \\ \frac{\kappa_i^2}{2} + \frac{\pi}{8} + O(\frac{1}{i^2}), & i = j. \end{cases}$$

The details can be found in Yuan [25, Lemma 3.2].

Let $u(t,x) = \sum_{j\in\mathbb{Z}^+} q_j(t)\phi_j(x)$, then associated with the symplectic structure $\frac{i}{2}\sum_{j\geq 1} dq_j \wedge d\overline{q}_j$, we obtain the equations

$$\dot{q_j} = 2i\frac{\partial H}{\partial \overline{q}_j}, \quad j \geq 1,$$

and the corresponding Hamiltonian is

$$H = \Lambda + G + Q = \frac{1}{2}\sum_{j\in\mathbb{Z}^+} \mu_j|q_j|^2 + \frac{1}{4}\int_0^{\pi} |u|^4\,\mathrm{d}x + \frac{1}{2}\int_0^{\pi} g(|u|^2)\,\mathrm{d}x,$$

$$\Lambda = \frac{1}{2}\sum_{j\in\mathbb{Z}^+} \mu_j|q_j|^2, Q = \frac{1}{2}\int_0^{\pi} g(|u|^2)\,\mathrm{d}x,$$

$$G = \frac{1}{4}\int_0^{\pi} |u|^4\,\mathrm{d}x = \frac{1}{4}\sum_{i,j,k,l} G_{ijkl}q_iq_j\overline{q}_k\overline{q}_l, \tag{3.2}$$

$$G_{ijkl} = \int_0^{\pi} \phi_i(x)\phi_j(x)\phi_k(x)\phi_l(x)\,\mathrm{d}x.$$

Let $q = (\tilde{q}, \hat{q}) \in \ell^p$, where $\tilde{q} = (q_j)_{j\in J_b}, \hat{q} = (q_j)_{j\in\mathbb{Z}_1}$, then we have the following

Lemma 3.1. *The gradient $(G+Q)_{\overline{q}}$ is a real analytic map from a neighborhood of the origin of ℓ^ρ into ℓ^ρ, with*

$$\|(G+Q)_{\overline{q}}\|_\rho = O(\|q\|_\rho^3).$$

Lemma 3.2. *There exists a real analytic symplectic change of coordinates Γ in a neighborhood of the origin in ℓ^ρ which takes the Hamiltonian $H=\Lambda+G+Q$ into its partial Birkhoff normal form up to order four, that is*

$$H\circ\Gamma = \Lambda+\overline{G}+\widehat{G}+K,$$

such that $X_{\overline{G}}$, X_K are real analytic in a neighborhood of the origin in ℓ^ρ, where

$$\begin{aligned}
\overline{G} &= \frac{1}{2}\sum_{\text{one of}\{i,j\}\in J_b}\overline{G_{ij}}|q_i|^2|q_j|^2,\ |\widehat{G}| = O(\|\hat{q}\|_\rho^4),\ |K| = O(\|q\|_\rho^6),\\
\overline{G_{ij}} &= \frac{4-\delta_{ij}}{4\pi}+O(\frac{1}{i^2})+O(\frac{1}{j^2})+O(\frac{1}{ij|i-j|}),\ i,j\in J_b, \qquad (3.3)\\
\overline{G_{ij}} &= \frac{1}{\pi}+O(\frac{1}{i^2})+O(\frac{1}{ij|i-j|})\ ,i\in J_b, j\in\mathbb{Z}_1.
\end{aligned}$$

The above two lemmata can be proved by means of Proposition 3.1, see details in Du-Yuan [8], Kuksin-Pöschel [17].

By the same argument as that of Ren [22], we can introduce a transformation

$$\begin{cases} \tilde{q}_j = \sqrt{2(\xi_j+y_j)}e^{-ix_j}, & j\in J_b,\\ \hat{q}_j = \sqrt{2}z_j, & j\in\mathbb{Z}_1,\end{cases}$$

where $\xi\in[\rho_*,2\rho_*]^b$, $\rho_*\in(0,1)$(will appear in Section 4) to obtain the new symplectic structure $\sum_{j\in J_b} dx_j\wedge dy_j + i\sum_{j\in\mathbb{Z}_1} dz_j\wedge d\overline{z}_j = \frac{i}{2}\sum_{j\geq 1} dq_j\wedge d\overline{q}_j$. Meanwhile, the Hamiltonian is transformed into the following

$$H=\Lambda+\overline{G}+\widehat{G}+K = \langle\omega(\xi),y\rangle+\sum_{j\in\mathbb{Z}_1}\Omega_j(\xi)z_j\overline{z}_j+\frac{1}{2}\langle Ay,y\rangle+\langle By,Z\rangle+\widehat{R},$$

with $\widehat{R}=\widehat{G}+K,\ \omega(\xi)=\alpha+A\xi, \Omega(\xi)=\beta+B\xi,$ where

$$\begin{aligned}
\alpha &= (\mu_{i_1},\cdots,\mu_{i_b}), \beta=(\mu_j)_{j\in\mathbb{Z}_1},\\
A &= (\overline{G_{ij}})_{i,j\in J_b},\ B=(\overline{G_{ji}})_{j\in J_b, i\in\mathbb{Z}_1},\\
|\widehat{R}| &= O(|\xi|^3)+O(|y|^3)+O(|\xi|^2|y|)+O(|\xi||y|^2)+O(\|z\|_\rho^4) \qquad (3.4)\\
&\quad +O(|\xi|^{\frac{1}{2}}\|z\|_\rho^5)+O(|\xi|\|z\|_\rho^4)+O(|y|\|z\|_\rho^4)+O(|\xi|^{\frac{3}{2}}\|z\|_\rho^3)\\
&\quad +O(|\xi|^2\|z\|_\rho^2)+O(|y|^2\|z\|_\rho^2)+O(|\xi||y|\|z\|_\rho^2)+O(|\xi|^{\frac{5}{2}}\|z\|_\rho).
\end{aligned}$$

Setting $Z = (|z_{n_1}|^2, |z_{n_2}|^2, ...), n_j \in \mathbb{Z}_1, j \geq 1$, and conducting the same rescaling process as that of Ren [22], we obtain the Hamiltonian

$$\begin{aligned} \widetilde{H}(x, y, z, \overline{z}, \xi) &= \varepsilon^{-5} H(x, \varepsilon^3 y, \varepsilon^{\frac{3}{2}} z, \varepsilon^{\frac{3}{2}} \overline{z}, \varepsilon^2 \xi) \\ &= \langle \varepsilon^{-2}\alpha + A\xi, y\rangle + \langle \varepsilon^{-2}\beta + B\xi, z\overline{z}\rangle + \frac{\varepsilon}{2}\langle Ay, y\rangle + \frac{\varepsilon}{2}\langle By, Z\rangle \\ &\quad +\varepsilon O(|\xi|^3) + \varepsilon^4 O(|y|^3) + \varepsilon^2 O(|\xi|^2|y|) + \varepsilon^3 O(|\xi||y|^2) \\ &\quad +\varepsilon O(||z||_\rho^4) + \varepsilon^{\frac{7}{2}} O(|\xi|^{\frac{1}{2}}\|z\|_\rho^5) + \varepsilon^3 O(|\xi|\|z\|_\rho^4) \\ &\quad +\varepsilon^4 O(|y|\|z\|_\rho^4) + \varepsilon^{\frac{5}{2}} O(|\xi|^{\frac{3}{2}}\|z\|_\rho^3) + \varepsilon^2 O(|\xi|^2\|z\|_\rho^2) \\ &\quad +\varepsilon^4 O(|y|^2\|z\|_\rho^2) + \varepsilon^3 O(|\xi||y|\|z\|_\rho^2) + \varepsilon^{\frac{3}{2}} O(|\xi|^{\frac{5}{2}}\|z\|_\rho \\ &:= \langle \widetilde{\omega}(\xi), y\rangle + \langle \widetilde{\Omega}(\xi), z\overline{z}\rangle + \widetilde{P} := \widetilde{N} + \widetilde{P}, \end{aligned} \tag{3.5}$$

which serves as our new starting point, and depends on one real parameter ξ varying in a compact set $\mathcal{O} \subset \mathbb{R}^b$($\mathcal{O}$ will be specified in Section 4), where $\widetilde{\omega}(\xi) = \varepsilon^{-2}\alpha + A\xi, \widetilde{\Omega}(\xi) = \varepsilon^{-2}\beta + B\xi$. For simplicity, we still denote $\widetilde{H}$ by H, $\widetilde{\omega}$ by ω, $\widetilde{\Omega}$ by Ω, $\widetilde{N}$ by N, $\widetilde{P}$ by P.

Remark 3.1. (3.5) is almost the same as Ren [22, (7)], however, the term $\langle By, Z\rangle$ can not be eliminated in this context, which will add to the hardship of the measure estimates.

To proceed we formulate the essential properties of the new Hamiltonian, which are relevant for our argument.

$(A1)$ *Regularity of the perturbation:* The perturbation P is *regular* in the sense that $\|X_P\|_{D(r,s),\mathcal{O}} < \infty$.

$(A2)$ *Non-degeneracy of tangential frequencies:* $\omega : \xi \mapsto \omega(\xi)$ is non-degenerate in the sense that $\det(D\omega(\xi)) \neq 0, \ \forall \xi \in \mathcal{O}$.

$(A3)$ *Asymptotic condition of normal frequencies:* $\forall j \in \mathbb{Z}_1, \Omega_j(\xi) = \bar{\Omega}_j + \tilde{\Omega}_j$, where

$$\bar{\Omega}_j = \varepsilon^{-2}(j^2 + \frac{1}{\pi}\int_0^\pi V(x)\,\mathrm{d}x + O(\frac{1}{j^2})) + O(\rho_*^b), \ |\tilde{\Omega}_j|_{\mathcal{O}} = O(\varepsilon).$$

$(A4)$ *Melnikov's non-resonance conditions:* For fixed $\tau > 3b+4$, and for $\gamma > 0$ small enough, we assume that $(\omega, \Omega) \in DC(\gamma)$, i.e.,

$$|\langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle| \geq \frac{\gamma\langle l\rangle}{|k|^\tau}, \ \ \forall (k, l) \in \mathbb{Z}^b \times \mathbb{Z}^\infty, k \neq 0,$$

where $\xi \in \mathcal{O}, |k| \leq K, |\sum_{n\in\mathbb{Z}_1}(\alpha_n - \beta_n)n| \leq K, \ 0 \leq |l| \leq 2.$

Remark 3.2. ρ_* in (A3) and γ, K in (A4) will be specified in Section 4.

Remark 3.3. We say that $N \in NF(r,s,\underline{M}) \bigcap DC(\gamma)$, if $N = \langle \omega, y\rangle + \langle \Omega, z\overline{z}\rangle$ is defined on $D(r,s) \times \mathcal{O}$, $(\omega,\Omega) \in DC(\gamma)$, and for some $\underline{M} := (M_1, M_2) \in (\mathbb{R}^+)^2$, for all $\xi \in \mathcal{O}$, we have the following estimates

$$(3.6)\qquad \begin{aligned} &(C1)\ \ |\omega|_{\mathcal{O}} = \sup |\omega(\xi)|_{\mathcal{O}} \le M_1 < \infty; \\ &(C2)\ \ |(D\omega)^{-1}|_{\mathcal{O}} = \sup |(D\omega(\xi))^{-1}|_{\mathcal{O}} \le M_2 < \infty. \end{aligned}$$

See details in Ren [22, Definition 2.1].

Remark 3.4. For $\tau > 3b+4$, we define the set of Diophantine vectors

$$\mathcal{D}_{\gamma,\tau} = \Big\{\omega \in \mathbb{R}^b : |\langle k,\omega\rangle| \ge \frac{\gamma}{|k|^\tau},\ \ \forall k \in \mathbb{Z}^b \setminus 0\Big\}.$$

Proposition 3.2. *For* $A = (\overline{G_{ij}})_{i,j\in J_b}$, *we have* $\det A \neq 0$.

Outline of the proof. For $i,j \in J_b$, *let* $a_{ij} = \overline{G_{ij}} - \frac{4-\delta_{ij}}{4\pi}$, *due to (3.3), there exists a constant* $\tilde{c} > 0$, *such that* $|a_{ij}| \le \frac{\tilde{c}}{i_1^2}$, *which can be small enough if* i_1 *is large enough. Notice that* $4\pi A = 4X - I + \tilde{A}$, *where* I *is the identity matrix and all elements of* X *are 1,* $\tilde{A} = (a_{ij})_{i,j\in J_b}$, *since* $\det(4X - I) \neq 0$, *we know that there exists an elementary transformation* T, *such that* $T(4X - I) = diag(\sigma_1, \cdots, \sigma_b)$. *Putting the same transformation* T *on* $4X - I + \tilde{A}$ *and letting* $T\tilde{A} = \dot{\tilde{A}} = (\dot{a}_{ij})$, *we can choose* i_1 *large enough to make sure that* $|\sigma_j| > \sum_{i=1}^{b} |\dot{a}_{ij}|$, *for any* $1 \le j \le b$. *Hence,* $\det(4X - I + \tilde{A}) \neq 0$, *that is* $\det A \neq 0$.

See details in Du-Yuan [8, Lemma 6.1].

Proposition 3.3. *If* $\|X_F\|_{D(r,s)} < \varepsilon'$, $\|X_G\|_{D(r,s)} < \varepsilon''$, *then*

$$\|X_{\{F,G\}}\|_{D(r-\rho,\eta s)} < c\rho^{-1}\eta^{-2}\varepsilon'\varepsilon'',\ \ \rho > 0,\ \ \eta \ll 1.$$

For the proof, see Geng-You [11, Lemma 7.3].

Remark 3.5. In what follows, we denote the above ε by ε_*. From $(3.4), (3.5)$ and Proposition 3.2, we know that there exist $M_1^* > 0, M_2^* > 0$, such that the frequencies mapping $\xi \mapsto \omega(\xi)$ satisfies conditions $(C1)$ and $(C2)$ with respect to $\underline{M}^* := (M_1^*, M_2^*)$ on $[\rho_*, 2\rho_*]^b$(ρ_* will appear in Section 4).

4. KAM Step

At first, we fix $r, s, \varepsilon_* > 0, \iota \ge 4, \tau > 3b+4$. Initially, we set $\omega_0 = \omega_*, \Omega^0 = \Omega^*, P^0 = P, r_0 = r, s_0 = s, \gamma_0 = \gamma_* = \varepsilon_*^{\frac{1}{20}}, \rho_* = \varepsilon_*^{\frac{1}{25}}, \underline{M}^0 = \underline{M}^*, K_0 = \varepsilon_*^{-\frac{1}{12(2\tau+1)}}$, and restrict (3.5) to $D(r,s)$, restrict ξ to $[\rho_*, 2\rho_*]^b$, such that

$$H_0 = N_0 + P^0, N_0 = \langle \omega_0, y\rangle + \langle \Omega^0, z\overline{z}\rangle \in NF(r_0, s_0, \underline{M}^0) \bigcap DC(\gamma_0),$$

where $\omega_* = \omega(\xi) = \varepsilon_*^{-2}\alpha + A\xi \in \mathcal{D}_{\iota\gamma_*,\tau}$, $\Omega^* = \Omega(\xi) = \varepsilon_*^{-2}\beta + B\xi$, $\xi \in \mathcal{O}_0$,

$$\mathcal{O}_0 = \Big\{\xi \in [\rho_*, 2\rho_*]^b : |\langle k, \omega_0\rangle + \langle l, \Omega^0\rangle| \geq \frac{\gamma_0\langle l\rangle}{|k|^\tau}, 0 < |k| \leq K_0,$$
$$|\sum_{n\in\mathbb{Z}_1}(\alpha_n - \beta_n)n| \leq K_0, 0 \leq |l| \leq 2\Big\} \bigcap \omega_*^{-1}(\mathcal{D}_{\iota\gamma_*,\tau}).$$

Notice that, there exists a positive constant c_*, such that

$$\|X_{P^0}\|_{D(r_0,s_0),\mathcal{O}_0} \leq c_*\varepsilon_* := \varepsilon_0. \tag{4.1}$$

Suppose that after ν^{th} KAM step, we arrive at a Hamiltonian

$$H = H_\nu = N + P(x, y, z, \overline{z}, \xi, \varepsilon), \tag{4.2}$$

$$N = N_\nu = \langle\omega, y\rangle + \langle\Omega, z\overline{z}\rangle \in NF(r, s, \underline{M}) \bigcap DC(\gamma), P = P^\nu,$$

which is real analytic on $D = D_\nu = D(r_\nu, s_\nu)$, for some $r = r_\nu \leq r_0, s = s_\nu \leq s_0$, and depends on $\xi \in \mathcal{O}_\nu$ Whitney smoothly, where $\omega = \omega_\nu(\xi), \Omega = \Omega^\nu(\xi), \underline{M} = \underline{M}^\nu := (M_1^\nu, M_2^\nu), \gamma = \gamma_\nu, K_\nu = 2^\nu K_0$,

$$\mathcal{O}_\nu = \Big\{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu\rangle + \langle l, \Omega^\nu\rangle| \geq \frac{\gamma_\nu\langle l\rangle}{|k|^\tau}, 0 < |k| \leq K_\nu,$$
$$|\sum_{n\in\mathbb{Z}_1}(\alpha_n - \beta_n)n| \leq K_\nu, 0 \leq |l| \leq 2\Big\} \bigcap \omega_*^{-1}(\mathcal{D}_{\iota\gamma_*,\tau}).$$

Suppose also that for some $0 < \varepsilon = \varepsilon_\nu \leq \varepsilon_0$,

$$\|X_P\|_{D,\mathcal{O}} \leq \varepsilon. \tag{4.3}$$

Next, we will look for $F^\nu := F$, defined on $D_+ = D(r_+, s_+) \subset D$, such that the time one map $\Phi_F^1 := \Phi$ of the Hamiltonian vector field X_F defines a map $D_+ \to D$ and transforms H into H_+, where Φ_F^t is the hamiltonian flow of F. Moreover, the new Hamiltonian $H_+ := H \circ \Phi = N_+ + P^+$ satisfies $(A1) - (A4)$ again with respect to $r_+, s_+, \varepsilon_+, \underline{M}^+ := (M_1^+, M_2^+), \gamma_+$, and $N_+ \in NF(r_+, s_+, \underline{M}^+) \bigcap DC(\gamma_+)$. In addition, (4.3) still holds for $P^+, D_+, \mathcal{O}_+, \varepsilon_+$.

4.1. Solving the homological equations.

Let $R = P_0 + P_1 + P_2$, which is the truncation of the Taylor-Fourier series of P up to order 2, let $\widetilde{P} = P - R$, we wish to construct a function $F = F_0 + F_1 + F_2, [F] = 0$, such that

$$\{F, N\} = R - [R]. \tag{4.4}$$

Lemma 4.1. *Consider equation* (4.4), *let* $D_j = D(r_j, s_j) = D(r_{\nu+1} + \frac{j}{4}(r_\nu - r_{\nu+1}), \frac{j}{4}s_\nu)$, $0 < j \leq 4$, *then*

$$\|X_F\|_{D_3,\mathcal{O}} < \cdot\gamma_\nu^{-2}(r_\nu - r_{\nu+1})^{-(2\tau+2)}\|X_P\|_{D_4,\mathcal{O}}.$$

Proof. The proof follows from standard arguments using Cauchy estimate, see Ren [22, Proposition 3.1] and Geng-You [12, Lemma 4.2]. ■

Now, we define $\varepsilon_+ = c\gamma^{-2}(r-r_+)^{-(2\tau+3)}\varepsilon^{\frac{4}{3}}$, $s_+ = \frac{1}{8}\eta s, \eta = \varepsilon^{\frac{1}{3}}$, $D_+ = D(r_+, s_+)$, $D^\nu_{j\eta} = D_{j\eta} = D(r_+ + j\rho, \frac{j}{4}\eta s)$, where $\rho = \frac{1}{4}(r - r_+), 0 < j \leq 4, c$ is a constant that does not depend on the KAM step. It is clear that $D_+ \subset D_{j\eta} \subset D_j \subset D$, and $|P_{0100}|_{\mathcal{O}_+} = O(\varepsilon)$, $|P_{0011}|_{\mathcal{O}_+} = O(\varepsilon)$. For the definitions of P_{0100}, P_{0011}, see Ren [22, Definition 2.1].

4.2. Defining the new Hamiltonian H_+.

Due to (4.4) and the second order Taylor formula, we have

$$\begin{aligned} H_+ &= H \circ \Phi_F^1 = N_+ + P^+, \\ N_+ &= N + [R_0] + [R_2], \\ P^+ &= \int_0^1 \{tR + (1-t)[R], F\} \circ \Phi_F^t \,\mathrm{d}t + \widetilde{P} \circ \Phi_F^1. \end{aligned} \tag{4.5}$$

More precisely, after $(\nu+1)^{th}$ KAM step, we obtain the new Hamiltonian

$$\begin{aligned} H_{\nu+1} &= H \circ \Phi_0 \circ \cdots \circ \Phi_\nu = N_{\nu+1} + P^{\nu+1}, \\ N_{\nu+1} &= \langle \omega_{\nu+1}, y \rangle + \langle \Omega^{\nu+1}, z\overline{z} \rangle, \\ \omega_{\nu+1} &= \omega_\nu + P^\nu_{0100} = \varepsilon_*^{-2}\alpha + A\xi + P^0_{0100} + \cdots + P^\nu_{0100}, \\ \Omega^{\nu+1} &= \Omega^\nu + P^\nu_{0011} = \varepsilon_*^{-2}\beta + B\xi + P^0_{0011} + \cdots + P^\nu_{0011}, \end{aligned} \tag{4.6}$$

where $P^0_{0100} = \varepsilon_*^2 O(|\xi|^2|y|), P^0_{0011} = \varepsilon_*^2 O(|\xi|^2\|z\|^2_\rho)$ (see (3.5)).

4.3. Estimating the new normal form N_+.

Let $D\omega = \frac{d\omega}{d\xi}$, concerning ω_+, we have $N_+ \in NF(r_+, s_+, \underline{M}^+), \underline{M}^+ \equiv (\frac{3}{2}M_1^0, M_2^0)$. In fact,

$$|\omega_+|_{\mathcal{O}_+} \leq \frac{3}{2}|\omega_*|_{[\rho_*,2\rho_*]^b} \leq \frac{3}{2}M_1^0, \ |(D\omega_+)^{-1}|_{\mathcal{O}_+} = |A^{-1}| = |(D\omega_*)^{-1}|_{[\rho_*,2\rho_*]^b} \leq M_2^0.$$

4.4. Estimating the symplectic transformation Φ.

Lemma 4.2. *If*

$$\varepsilon \ll (\gamma^2(r - r_+)^{2\tau+3})^3, \tag{4.7}$$

then we have

$$\Phi_F^1 : D_{1\eta} \times \mathcal{O}_+ \to D, \ -1 \le t \le 1,$$

and for all $\nu \ge 1$, *we also have* $\|D\Phi_F^1 - Id\|_{D_{1\eta},\mathcal{O}_+} < \varepsilon^{\frac{1}{2}}$.

Proof. Lemma 4.1 implies that $\|X_F\|_{D_3,\mathcal{O}_+} < \cdot\gamma^{-2}(r-r_+)^{-(2\tau+2)}\varepsilon := \beta$. Assume that $\beta \ll \eta^2\rho$ (being equivalent to (4.7)), by the proof of Ren [22, Proposition 3.2], we can get $\Phi_F^t : D_{2\eta} \to D_{3\eta}, -1 \le t \le 1$, thus, $\Phi = \Phi_F^1 : D_+ \times \mathcal{O}_+ \to D$ is well defined, and an immediate consequence is $\|D\Phi_F^1 - Id\|_{D_{1\eta},\mathcal{O}_+} \le 2\|D^2F\|_{D_2,\mathcal{O}_+} < \cdot\beta < \varepsilon^{\frac{1}{2}}$. This completes the proof. ■

4.5. Estimating the new perturbation P^+.

Since

$$P^+ = \int_0^1 \{G(t), F\} \circ \Phi_F^t \,\mathrm{d}t + \widetilde{P} \circ \Phi_F^1,$$

where $G(t) = tR + (1-t)[R]$, we have

$$X_{P^+} = \int_0^1 (\Phi_F^t)^* X_{\{G(t),F\}} \,\mathrm{d}t + (\Phi_F^1)^* X_{\widetilde{P}}.$$

It follows from Proposition 3.3 that

$$\|X_{\{G(t),F\}}\|_{D_{2\eta},\mathcal{O}_+} \le \cdot\rho^{-1}\eta^{-2}\|X_R\|_{D_3,\mathcal{O}_+}\|X_F\|_{D_3,\mathcal{O}_+} \le c_1\gamma^{-2}\rho^{-(2\tau+3)}\eta^{-2}\varepsilon^2,$$

$$\|X_{\widetilde{P}}\|_{D_{2\eta},\mathcal{O}_+} \le c_2\eta\|X_P\|_{D,\mathcal{O}_+} \le c_2\eta\varepsilon.$$

Recall that $\rho = \frac{1}{4}(r - r_+), \eta = \varepsilon^{\frac{1}{3}}, \varepsilon_+ = c\gamma^{-2}(r - r_+)^{-(2\tau+3)}\varepsilon^{\frac{4}{3}}$, if (4.7) holds, we have $\|X_{P^+}\|_{D_+,\mathcal{O}_+} \le c\gamma^{-2}(r - r_+)^{-(2\tau+3)}\varepsilon^{\frac{4}{3}} = \varepsilon_+ \ll \varepsilon$, where $c = 2\max\{16c_1, c_2\} > 0$. At this time, $\varepsilon_+ = \varepsilon^{\kappa}$, for some $1 < \kappa < \frac{4}{3}$, thus, $\varepsilon_\nu < \cdot\varepsilon_*^{\kappa^\nu}$.

This completes one step of KAM iterations.

5. Iteration and Convergence

5.1. Iterative lemma.

For any given $r, s, c_*, \varepsilon_*, \iota \ge 4, \tau > 3b + 4$, we define, for all $\nu \ge 1$, the following sequences

$$r_\nu = \frac{r_0}{2}(1 + 2^{-\nu}), \ r_0 = r,$$

$$s_\nu = \frac{1}{8}\eta_{\nu-1}s_{\nu-1} = 2^{-3\nu}(\prod_{j=0}^{\nu-1}\varepsilon_j)^{\frac{1}{3}}s_0, \ s_0 = s,$$

$$\varepsilon_\nu = c\gamma_{\nu-1}^{-2}(r_{\nu-1} - r_\nu)^{-(2\tau+3)}\varepsilon_{\nu-1}^{\frac{4}{3}}, \ \varepsilon_0 = c_*\varepsilon_*,$$

$$
\begin{aligned}
\underline{M}^{\nu} &= (M_1^{\nu}, M_2^{\nu}) \equiv (\frac{3}{2}M_1^0, M_2^0), \underline{M}^0 = \underline{M}^*,\\
\gamma_\nu &= \frac{\gamma_0}{2}(1+2^{-\nu}), \gamma_0 = \gamma_* = \varepsilon_*^{\frac{1}{20}},\\
\rho_* &= \varepsilon_*^{\frac{1}{25}} = \gamma_*^{\frac{4}{5}},\\
K_\nu &= 2^\nu K_0,\ K_0 = \varepsilon_*^{-\frac{1}{12(2\tau+1)}},\\
D_{\nu-1} &= D(r_{\nu-1}, s_{\nu-1}),\\
\widetilde{D}_{\nu-1}^j &= D(r_\nu + \frac{j}{4}(r_{\nu-1}-r_\nu), 2js_\nu), j = 2,\, 3,\\
\mathcal{O}_\nu &= \Big\{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu\rangle + \langle l, \Omega^\nu\rangle| \ge \frac{\gamma_\nu \langle l\rangle}{|k|^\tau},\ 0 < |k| \le K_\nu,\\
&\qquad |\sum_{n\in\mathbb{Z}_1}(\alpha_n - \beta_n)n| \le K_\nu, 0 \le |l| \le 2\Big\} \bigcap \omega_*^{-1}(\mathcal{D}_{\iota\gamma_*,\tau}).
\end{aligned}
$$

Remark 5.1. Due to the definitions of $\varepsilon_\nu, \gamma_\nu, r_\nu,$ we can easily verify that (4.7) holds for all $\nu = 0, 1, 2, \cdots$.

Lemma 5.1. *Suppose that for all $\nu \ge 0, H_\nu = N_\nu + P^\nu$ is given on $D_\nu \times \mathcal{O}_\nu$, which is real analytic in $(x, y, z, \overline{z}) \in D_\nu$, and Whitney smooth in $\xi \in \mathcal{O} \subset [\rho_*, 2\rho_*]^b$, where*

$$
\begin{aligned}
N_\nu &= \langle \omega_\nu(\xi), y\rangle + \sum_{j\in\mathbb{Z}_1} \Omega_j^\nu(\xi) z_j \overline{z}_j \in NF(r_\nu, s_\nu, \underline{M}^\nu) \cap DC(\gamma_\nu),\\
\omega_* &= \omega(\xi) = \varepsilon_*^{-2}\alpha + A\xi,\ \Omega^* = \Omega(\xi) = \varepsilon_*^{-2}\beta + B\xi,\ \xi \in \mathcal{O}_0,\\
&|\omega_*|_{[\rho_*,2\rho_*]^b} \le M_1^0,\ \omega_{\nu+1} = \omega_\nu + P_{0100}^\nu,\ \Omega^{\nu+1} = \Omega^\nu + P_{0011}^\nu,\\
\alpha &= (i_1^2 + \frac{1}{\pi}\int_0^\pi V(x)\,\mathrm{d}x + O(\frac{1}{i_1^2}), \cdots, i_b^2 + \frac{1}{\pi}\int_0^\pi V(x)\,\mathrm{d}x + O(\frac{1}{i_b^2})),\\
\beta &= (j^2 + \frac{1}{\pi}\int_0^\pi V(x)\,\mathrm{d}x + O(\frac{1}{j^2}))_{j\in\mathbb{Z}_1},\\
A &= \Big(\frac{4-\delta_{ij}}{4\pi} + O(\frac{1}{i^2}) + O(\frac{1}{j^2}) + O(\frac{1}{ij|i-j|})\Big)_{i,j\in J_b},\\
B &= \Big(\frac{1}{\pi} + O(\frac{1}{j^2}) + O(\frac{1}{ij|i-j|})\Big)_{j\in\mathbb{Z}_1, i\in J_b},
\end{aligned} \tag{5.1}
$$

and the real analytic functions P^ν satisfy

$$
\|X_{P^\nu}\|_{D_\nu,\mathcal{O}_\nu} \le \varepsilon_\nu,\ \nu \ge 0. \tag{5.2}
$$

Then there exists a symplectic diffeomorphism $\Phi_\nu : \widetilde{D}_\nu^2 \to \widetilde{D}_\nu^3$, such that $H_{\nu+1} = (N_\nu + P^\nu)\circ\Phi_\nu = N_{\nu+1} + P^{\nu+1}$, which is defined on $D_{\nu+1} \times \mathcal{O}_{\nu+1}$, and the same *properties as* (5.1) *and* (5.2) *are satisfied with* $\nu+1$ *in place of* ν*, where* $\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \bigcup_{kl} R_{kl}^{\nu+1}$,

$$R_{kl}^{\nu+1} = \Big\{\xi \in \mathcal{O}_\nu : |\langle k, \omega_{\nu+1}\rangle + \langle l, \Omega^{\nu+1}\rangle| < \frac{\gamma_{\nu+1}\langle l\rangle}{|k|^\tau}, 0 < |k| \le K_{\nu+1}, \\ |\sum_{n\in\mathbb{Z}_1}(\alpha_n - \beta_n)n| \le K_{\nu+1}, 0 \le |l| \le 2\Big\}.$$

Moreover, for all $\nu \ge 0$*, there exists* $1 < \kappa < \frac{4}{3}$*, such that*

$$\|D\Phi_{\nu+1} - Id\|_{D_{\nu+1},\mathcal{O}_{\nu+1}} < \cdot\varepsilon_*^{\frac{1}{2}\kappa^\nu}.$$

5.2. Convergence.

Inductively, we have $\Psi_\nu(\xi) := \Phi_0 \circ \cdots \circ \Phi_{\nu-1} : D_\nu \times \mathcal{O}_\nu \to D_0$, where $\Psi_0(\xi) = id$, such that for all $\nu \ge 1$, $H_\nu = H_0 \circ \Psi_\nu(\xi) = N_\nu + P^\nu$, where

$$N_\nu = \langle \omega_\nu, y\rangle + \sum_{j\in\mathbb{Z}_1} \Omega_j^\nu z_j \overline{z}_j \in NF(r_\nu, s_\nu, \underline{M}^\nu) \bigcap DC(\gamma_\nu).$$

Let $\mathcal{O}_\infty = \bigcap_{\nu\ge 0} \mathcal{O}_\nu$. Then we can use Lemma 5.1 and the argument similar to that of Pöschel [21] to verify that $\omega_\nu, \Omega_j^\nu, P^\nu, \Psi_\nu, D\Psi_\nu$ converge uniformly on $D(\frac{r_0}{2}, 0) \times \mathcal{O}_\infty$ to $\omega_\infty, \Omega_j^\infty, P^\infty, \Psi_\infty, D\Psi_\infty$, respectively. Let $\omega_\infty(\xi) = \lambda\omega_*(\xi)$, for each $\xi \in \mathcal{O}_\infty$, we have $\lambda \in \Lambda_\star \subset [\frac{1}{2}, \frac{3}{2}]$, where $\Lambda_\star$ is a one-dimensional Cantor-like parameter space, whose measure has the same order of magnitude as that of $\omega_\infty(\mathcal{O}_\infty) \bigcap \omega_* \mathbb{R}^+$. At this time,

$$N_\infty = \langle \lambda\omega_*, y\rangle + \sum_{j\in\mathbb{Z}_1} \Omega_j^\infty z_j \overline{z}_j \in NF(\frac{r_0}{2}, 0, \underline{M}^\infty) \bigcap DC(\frac{1}{2}\varepsilon_*^{\frac{1}{20}}),$$

where $\underline{M}^\infty = (\frac{3}{2}M_1^0, M_2^0)$. Let $\phi_{H_0}^t$ be the flow of X_{H_0}, since on $D(\frac{r_0}{2}, 0) \times \mathcal{O}_\infty$, $H_0 \circ \Psi_\nu = H_\nu$, we have

$$\phi_{H_0}^t \circ \Psi_\nu = \Psi_\nu \circ \phi_{H_\nu}^t. \tag{5.3}$$

Then, we can pass the limit on both sides of (5.3) to arrive at

$$\phi_{H_0}^t \circ \Psi_\infty = \Psi_\infty \circ \phi_{H_\infty}^t, \quad \Psi_\infty : D(\frac{r_0}{2}, 0) \times \mathcal{O}_\infty \to D(r, s). \tag{5.4}$$

Since $\varepsilon_\nu < \cdot\varepsilon_*^{\kappa^\nu}, 1 < \kappa < \frac{4}{3}$, we have $\|X_{P^\infty}\|_{D(\frac{r_0}{2},0)\times\mathcal{O}_\infty} \equiv 0$. As a result, on $D(\frac{r_0}{2}, 0)$, for every choice of $\xi \in \mathcal{O}_\infty$, and for all $\lambda \in \Lambda_\star$, we obtain

$$\phi_{H_0}^t \circ \Psi_\infty(\mathbb{T}^b \times \{\lambda\}) = \Psi_\infty \circ \phi_{N_\infty}^t(\mathbb{T}^b \times \{\lambda\}) = \Psi_\infty(\mathbb{T}^b \times \{\lambda\}).$$

Hence, $\Psi_\infty(\mathbb{T}^b \times \{\lambda\})$ is an embedded invariant torus of the original perturbed Hamiltonian system at $\lambda \in \Lambda_\star$. In the following, we will show that the relative Lebesgue measure of $\Lambda_\star$ in $[\frac{1}{2}, \frac{3}{2}]$ is positive, and will prove Theorem in Section 1.

6. Measure Estimates

In Sections 4.2 and 4.3, we have

$$\omega_0(\xi) = \varepsilon_*^{-2}\alpha + A\xi = \omega_*,\ \omega_{\nu+1}(\xi) = \varepsilon_*^{-2}\alpha + A\xi + P_{0100}^0 + \cdots + P_{0100}^\nu, \tag{6.1}$$

$$\Omega^0(\xi) = \varepsilon_*^{-2}\beta + B\xi = \Omega^*,\ \Omega^{\nu+1}(\xi) = \varepsilon_*^{-2}\beta + B\xi + P_{0011}^0 + \cdots + P_{0011}^\nu, \tag{6.2}$$

and have verified that the mapping $\xi \mapsto \omega_{\nu+1}(\xi)$ satisfy conditions $(C1)$ and $(C2)$ with respect to $\underline{M}^{\nu+1}$. Now, we are in a position to prove Theorem in Section 1 by the following two propositions.

Proposition 6.1. *There exists $\iota \geq 4$, such that for every vector $\omega_* \in \mathcal{D}_{\iota\gamma_*,\tau}$, we have*

$$meas([\rho_*, 2\rho_*]^b \setminus \mathcal{O}_\infty) < \gamma_*^{\frac{1}{2}} meas([\rho_*, 2\rho_*]^b),$$

where

$$\mathcal{O}_\infty = \bigcap_{\nu\geq 0} \mathcal{O}_\nu,\ \ \mathcal{O}_\nu = \Big\{\xi \in \mathcal{O}_{\nu-1} : |\langle k, \omega_\nu(\xi)\rangle + \langle l, \Omega^\nu(\xi)\rangle| \geq \frac{\gamma_\nu\langle l\rangle}{|k|^\tau},$$
$$0 < |k| \leq K_\nu,\ |\sum_{n\in\mathbb{Z}_1}(\alpha_n - \beta_n)n| \leq K_\nu, 0 \leq |l| \leq 2\Big\} \bigcap \omega_*^{-1}(\mathcal{D}_{\iota\gamma_*,\tau}). \tag{6.3}$$

Proposition 6.2. *If $0 \notin \omega_*([\rho_*, 2\rho_*]^b)$, we have*

$$meas(\omega_\infty([\rho_*, 2\rho_*]^b \setminus \mathcal{O}_\infty) \bigcap \omega_*\mathbb{R}^+) < \varepsilon_*^{2+\frac{1}{24}}\, meas([\frac{1}{2}, \frac{3}{2}]).$$

Since $\omega_\infty(\xi) = \lambda\omega_*(\xi)$, for each $\xi \in \mathcal{O}_\infty \subset [\rho_*, 2\rho_*]^b, \lambda \in \Lambda_\star \subset [\frac{1}{2}, \frac{3}{2}]$, from (6.1) and (6.2), we have

$$\begin{aligned}
&\langle k, \omega_\infty(\xi) + \langle l, \Omega^\infty(\xi)\rangle \\
=\ & \lambda\langle k + A^{-1}B^T l, \omega_*\rangle + \varepsilon_*^{-2}\langle l, \beta - BA^{-1}\alpha\rangle \\
&+\langle l, -BA^{-1}(P_{0100}^0 + \cdots + P_{0100}^\infty) + P_{0011}^0 + \cdots + P_{0011}^\infty\rangle, \\
=\ & \lambda\langle k + A^{-1}B^T l, \omega_*\rangle + \varepsilon_*^{-2}\langle l, \beta - BA^{-1}\alpha\rangle + o(\varepsilon_*) := h_{kl}(\lambda).
\end{aligned}$$

We can follow Berti-Biasco [2, Remark 8.1] and the cut-off procedure in Berti-Bolle [4] to prove the above propositions. Specifically, we need to consider the following resonance sets:

$$R_{kl}^{\nu+1} = \{\xi\in\mathcal{O}_\nu : |\langle k, \omega_{\nu+1}(\xi)\rangle + \langle l, \Omega^{\nu+1}(\xi)\rangle| < \frac{\gamma_{\nu+1}\langle l\rangle}{|k|^\tau}, 0 < |k| \leq K_{\nu+1}\}, \tag{6.4}$$

(6.5) $$\widetilde{\mathcal{R}}_{kl} = \{\lambda \in [\frac{1}{2}, \frac{3}{2}] : |h_{kl}(\lambda)| < \frac{2\gamma_*\langle l\rangle}{|k|^\tau}, k \neq 0\}.$$

Lemma 6.1. *For* $\xi \in R_{kl}^{\nu+1}, \lambda \in \widetilde{\mathcal{R}}_{kl}$, *we have*

(6.6) $$\langle l\rangle \leq 3\varepsilon_*^2|k|(|\omega_*|+1) := c_0|k|.$$

Proof. Obviously, (6.6) holds for $l = 0$. It is worth pointing out that for $1 \leq |l| \leq 2$, we have $|l| \leq 2\langle l\rangle, \langle l\rangle \leq 2|\langle l, \beta\rangle|$. In fact, we only need to consider the following case

$$|l| = 2, \;\; l = \pm(e_i - e_j), \;\; i > j, \;\; i, j \in \mathbb{Z}_1,$$

where the unit vector $e_j := (\cdots, 0, 1, 0, \cdots)$ with zero components except the j^{th} one. Hence $|\langle l, \beta\rangle| = i^2 - j^2 > 2(i-j) > \frac{1}{2}\langle l\rangle$. Thus, for each $\xi \in R_{kl}^{\nu+1}, \lambda \in \widetilde{\mathcal{R}}_{kl}$, we have

$$\begin{aligned} |k||\omega_{\nu+1}(\xi)| &\geq \varepsilon_*^{-2}|\langle l, \beta\rangle| - |l| \cdot |\Omega^{\nu+1}(\xi) - \varepsilon_*^{-2}\beta| - \gamma_*\langle l\rangle, \\ |k||\omega_\infty(\xi)| &\geq \varepsilon_*^{-2}|\langle l, \beta\rangle| - |l| \cdot |\Omega^\infty(\xi) - \varepsilon_*^{-2}\beta| - 2\gamma_*\langle l\rangle. \end{aligned}$$

Consequently,

(6.7) $$|k|(|\omega_*|+1) > \frac{1}{2}\varepsilon_*^{-2}\langle l\rangle - \cdot(|B|\rho_*^b + o(\varepsilon_*) + \gamma_*)\langle l\rangle \geq \frac{1}{3}\varepsilon_*^{-2}\langle l\rangle.$$

This completes the proof. ■

Lemma 6.2. *If* $0 < |k| \leq K_\nu$, *then* $R_{kl}^{\nu+1} = \emptyset$.

Proof. Due to the definitions of $\varepsilon_\nu, K_\nu, \gamma_\nu$, for all $\nu \geq 0$, we have $\varepsilon_\nu K_\nu^{\tau+1} < \varepsilon_\nu K_\nu^{2\tau+1} < \varepsilon_\nu^{\frac{9}{10}} \ll \frac{\varepsilon_*^{\frac{1}{20}}}{2^{\nu+2}} = \gamma_\nu - \gamma_{\nu+1}$, since $|l| \leq 2\langle l\rangle, \forall |l| \leq 2$, we have

$$\begin{aligned} |\langle k, \omega_{\nu+1}\rangle + \langle l, \Omega^{\nu+1}\rangle| &\geq |\langle k, \omega_\nu\rangle + \langle l, \Omega^\nu\rangle| - |k||\omega_{\nu+1} - \omega_\nu| - |l||\Omega^{\nu+1} - \Omega^\nu| \\ &\geq |\langle k, \omega_\nu\rangle + \langle l, \Omega^\nu\rangle| - \cdot|k|\langle l\rangle|X_{P^\nu}|_{D(r_\nu, s_\nu), \mathcal{O}_\nu} \\ &\geq \frac{\gamma_\nu\langle l\rangle}{|k|^\tau} - \cdot\varepsilon_\nu K_\nu\langle l\rangle \geq \frac{\gamma_{\nu+1}\langle l\rangle}{|k|^\tau}. \end{aligned}$$

The proof is finished. ■

Lemma 6.3. *If* $R_{kl}^{\nu+1} \neq \emptyset$, *we have*

$$meas(R_{kl}^{\nu+1}) < \cdot\frac{\rho_*^{b-1}\gamma_*}{|k|^\tau}.$$

Proof. Lemma 6.2 implies that $|k| > K_\nu$, let

$$\begin{aligned}\widehat{R}_{kl}^{\nu+1} &= \omega_{\nu+1}(R_{kl}^{\nu+1})\\ &= \{\zeta \in \widehat{\mathcal{O}} = \omega_{\nu+1}([\rho_*, 2\rho_*]^b) : \langle k, \zeta\rangle + \langle l, \Omega^{\nu+1}(\omega_{\nu+1}^{-1}(\zeta))\rangle| < \frac{\gamma_{\nu+1}\langle l\rangle}{|k|^\tau}\},\end{aligned}$$

where $\zeta = sv + w, w \perp v, s \in \mathbb{R}, v \in \{-1, 1\}^b, \langle k, v\rangle = |k|$. Consider the set

$$\widehat{R}_{kl}^{\nu+1} = \{\zeta = sv + w \in \widehat{\mathcal{O}} : |f_{kl}(s)| < \frac{\gamma_{\nu+1}\langle l\rangle}{|k|^\tau}\},$$

where

$$f_{kl}(s) = s|k| + \langle l, \Omega^{\nu+1}(\omega_{\nu+1}^{-1}(sv + w))\rangle.$$

For $s_2 > s_1$, since $|k| > K_\nu$, we have

$$\begin{aligned}f_{kl}(s_2) - f_{kl}(s_1) &\geq |k|(s_2 - s_1) - |l||D\Omega^{\nu+1}| \cdot |(D\omega_{\nu+1})^{-1}|(s_2 - s_1)\\ &\geq |k|(s_2 - s_1) - \cdot|B|M_2^0(s_2 - s_1) \geq \frac{1}{2}|k|(s_2 - s_1).\end{aligned}$$

By Fubini's theorem, we get

$$\widehat{R}_{kl}^{\nu+1} \leq \frac{2}{|k|}\frac{\gamma_{\nu+1}\langle l\rangle}{|k|^\tau}(diam\widehat{\mathcal{O}})^{b-1}. \tag{6.8}$$

Notice that $diam(\widehat{\mathcal{O}}) \leq |D\omega_{\nu+1}|diam([\rho_*, 2\rho_*]^b) < \cdot diam([\rho_*, 2\rho_*]^b)$, and that $\frac{1}{2}\gamma_* < \gamma_\nu < \gamma_*, \gamma_\nu \searrow \frac{1}{2}\gamma_*$ (as $\nu \to \infty$), by Lemma 6.1, we immediately have

$$meas(R_{kl}^{\nu+1}) < \cdot\frac{\rho_*^{b-1}\gamma_*}{|k|^\tau}.$$

Therefore, Lemma 6.3 follows. ■

Lemma 6.4.

$$card\Big\{l : \langle l\rangle \leq c_0|k|\Big\} < \cdot|k|^2.$$

Proof. The result follows by a simple calculation. ■

With the help of Lemmata 6.2, 6.3, 6.4 and $\gamma_* = \varepsilon_*^{\frac{1}{20}}, \tau > 3b + 4$, we have

$$\begin{aligned}meas([\rho_*, 2\rho_*]^b \setminus \mathcal{O}_\infty) &= \sum_{K_\nu<|k|\leq K_{\nu+1}, |l|\leq 2, \langle l\rangle\leq c_0|k|, \nu\geq 0} meas(R_{kl}^{\nu+1})\\ &< \cdot\rho_*^{b-1}\sum_{|k|>K_\nu, \nu\geq 0}\frac{|k|^2\gamma_*}{|k|^\tau}\\ &< \cdot\rho_*^{b-1}\sum_{\nu\geq 0}\frac{\varepsilon_*^{\frac{\tau-b-1}{12(2\tau+1)}}\gamma_*}{2^{\nu(\tau-b-1)}}\\ &< \cdot\rho_*^{b-1}\varepsilon_*^{\frac{1}{40}}\gamma_* = \cdot\rho_*^{b-1}\gamma_*^{\frac{3}{2}}.\end{aligned} \tag{6.9}$$

Since $\rho_* = \gamma_*^{\frac{4}{5}}$, we have

$$\frac{meas([\rho_*, 2\rho_*]^b \setminus \mathcal{O}_\infty)}{meas([\rho_*, 2\rho_*]^b)} < \cdot \frac{\rho_*^{b-1}\gamma_*^{\frac{3}{2}}}{\rho_*^b} < \gamma_*^{\frac{1}{2}}.$$

This finishes Proposition 6.1.

Lemma 6.5. *There exists a constant* $\delta_* > 0$, *such that*

$$|\langle l, \beta - BA^{-1}\alpha\rangle| > \delta_* > 0, \ \forall 1 \le |l| \le 2. \tag{6.10}$$

Proof. Due to the proof of Du-Yuan [8, Lemma 6.1], $\forall (k, l) \in \mathbb{Z}^b \times \mathbb{Z}^\infty, 1 \le |l| \le 2$, we have

$$\langle \alpha, k\rangle + \langle \beta, l\rangle \neq 0 \ \text{ or } \ Ak + B^T l \neq 0. \tag{6.11}$$

Suppose $Ak + B^T l = 0$, then (6.11) implies that

$$\begin{aligned}\langle \alpha, k\rangle + \langle \beta, l\rangle &= \langle \alpha, -A^{-1}B^T l\rangle + \langle \beta, l\rangle = -\alpha^T A^{-1} B^T l + \beta^T l \\ &= -\alpha^T (A^{-1})^T B^T l + \beta^T l = -(BA^{-1}\alpha)^T l + \beta^T l \\ &= (\beta - BA^{-1}\alpha)^T l = \langle \beta - BA^{-1}\alpha, l\rangle \neq 0.\end{aligned}$$

This finishes the proof. ■

Lemma 6.6. $\widetilde{\mathcal{R}}_{k0} = \emptyset$. *For* γ_* *small enough,* $k \in \mathbb{Z}^b \setminus 0, |l| \le 2$, *we have*

$$meas(\widetilde{\mathcal{R}}_{kl}) < \cdot \frac{\varepsilon_*^2 \gamma_*}{|k|^\tau}.$$

Proof. It suffices to consider the two cases (a) and (b) as follows.

(a) $l = 0$. Since $\omega_* \in \mathcal{D}_{\iota\gamma_*, \tau}$, $\iota \ge 4$, $\langle l\rangle = 1$, we have

$$|h_{k0}(\lambda)| = |\lambda\langle k, \omega_*\rangle| \ge \frac{\frac{1}{2}\iota\gamma_*}{|k|^\tau} \ge \frac{2\gamma_*\langle l\rangle}{|k|^\tau}.$$

(b) $l \neq 0$. If $|\langle k + A^{-1}B^T l, \omega_*\rangle| < \frac{1}{3}\varepsilon_*^{-2}\delta_*$, we have

$$|h_{kl}(\lambda)| \ge \varepsilon_*^{-2}\delta_* - \frac{1}{2}\varepsilon_*^{-2}\delta_* - O(\varepsilon_*) \gg \frac{2\gamma_*\langle l\rangle}{|k|^\tau}.$$

If $|\langle k + A^{-1}B^T l, \omega_*\rangle| \ge \frac{1}{3}\varepsilon_*^{-2}\delta_*$, for $\lambda_1 < \lambda_2$, we have

$$|h_{kl}(\lambda_2) - h_{kl}(\lambda_1)| \ge \Big[|\langle k + A^{-1}B^T l, \omega_*\rangle| + O(\varepsilon_*)\Big]|\lambda_2 - \lambda_1| \ge \frac{1}{4}\varepsilon_*^{-2}\delta_*|\lambda_2 - \lambda_1|.$$

This completes the proof. ∎

In view of Lemma 6.4 and Lemma 6.6, we have

$$
\begin{aligned}
meas(\omega_\infty([\rho_*,2\rho_*]^b\setminus\mathcal{O}_\infty)\bigcap\omega_*\mathbb{R}^+) &\le meas(\bigcup_{k\in\mathbb{Z}^b\setminus 0,|l|\le 2,\langle l\rangle\le c_0|k|}\widetilde{\mathcal{R}}_{kl})\\
&< \cdot\sum_{|k|\ge\frac{\langle l\rangle}{c_0}}\frac{\varepsilon_*^2|k|^2\gamma_*}{|k|^\tau}\\
&< \varepsilon_*^{2+\frac{1}{24}}meas([\frac{1}{2},\frac{3}{2}]), \qquad (6.12)
\end{aligned}
$$

where we have used that $\tau > 3b+4$. This finishes Proposition 6.2.

Remark 6.1. Proposition 6.1 implies that $(A4)$ in Section 3 can still be fulfilled through the KAM process. Moreover, there exist positive-measure Cantor-like parameter subsets $\mathcal{O}_\infty\subset[\rho_*,2\rho_*]^b$ and $\Lambda_\star\subset[\frac{1}{2},\frac{3}{2}]$, such that $\Psi_\infty(\mathbb{T}^b\times\{(\xi,\lambda)\})$ is an embedded invariant torus of the original perturbed Hamiltonian system at $(\xi,\lambda)\in\mathcal{O}_\infty\times\Lambda_\star$. Denote

$$\widehat{\xi}=\varepsilon_*^2\xi\in\widetilde{\mathcal{O}}:=\varepsilon_*^2\mathcal{O}_\infty\subset[\varepsilon_*^2\rho_*,2\varepsilon_*^2\rho_*]^b,\ \widehat{\omega}^*=\varepsilon_*^2\omega_*=\alpha+A\widehat{\xi}:=\widehat{\omega}(\widehat{\xi}),$$

then the tangential frequencies mapping $\widehat{\xi}\to\widehat{\omega}(\widehat{\xi})$ satisfies conditions $(C1)$ and $(C2)$ with respect to $(\widehat{M_1^*},\widehat{M_2^*}):=(\varepsilon_*^2M_1^*,M_2^*)$. Let

$$\omega^\star=\omega^\star([\varepsilon_*^2\rho_*,2\varepsilon_*^2\rho_*]^b)=\lambda\widehat{\omega}^*=\varepsilon_*^2\lambda\omega_*. \qquad (6.13)$$

For any fixed $\iota\ge 4$, since $\rho_*=\gamma_*^{\frac{4}{5}}>\iota\gamma_*$, we can choose $\omega_*\in\mathcal{D}_{\rho_*,\tau}$, thus $\widehat{\omega}^*\in\mathcal{D}_{\varepsilon_*^2\rho_*,\tau}$, such that $0\notin\omega_*([\rho_*,2\rho_*]^b)$, from (6.13), we also have $0\notin\omega^\star([\varepsilon_*^2\rho_*,2\varepsilon_*^2\rho_*]^b)$. Since (6.12) is equivalent to

$$meas([-\frac{1}{2},\frac{1}{2}]\setminus\{\Lambda_\star-\{1\}\})<\varepsilon_*^{2+\frac{1}{24}}meas([-\frac{1}{2},\frac{1}{2}]),$$

where $\Lambda_\star-\{1\}=\{\lambda-1:\lambda\in\Lambda_\star\}$, we can guarantee that the segment $[1-\varepsilon_*^{2+\frac{1}{24}},1+\varepsilon_*^{2+\frac{1}{24}}]\widehat{\omega}^*$ belong to $\omega^\star([\varepsilon_*^2\rho_*,2\varepsilon_*^2\rho_*]^b)$. Finally, for $\widehat{\omega}^*$, we can apply Proposition 6.1 and Proposition 6.2 to obtain Theorem in Section 1.

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Xiufang Ren
Department of Mathematics
Nanjing Agricultural University
Nanjing 210095
P. R. China
E-mail: xiufangren@gmail.com