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CHAOS EXPANSION AND REGULARITY OF THE LOCAL TIME OF THE SOLUTION TO THE STOCHASTIC HEAT EQUATION WITH ADDITIVE FRACTIONAL-COLORED NOISE

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Abstract. Using multiple stochastic integrals, we study the chaotic expansion and the regularity in the Sobolev-Watanabe spaces of the local time of the solution to the stochastic heat equation with fractional-colored noise.

1. Introduction

The local time of a stochastic process measures the time spent by the process in a given Borel set. It is an important characteristic of a stochastic process. Especially for Gaussian processes, the local time has been widely studied. We refer, among others, to the recent monograph [15] for a complete presentation.

A special interest has been focused on the local time of the fractional Brownian motion and related processes. Firstly, an important literature has been developed in the sixties-seventies due to the introduction of the concept of *local nondeterminism* and then, in the last decade, new results has been obtained simultaneously with the development of the stochastic calculus for fractional processes. We refer, among others, to [3, 4, 21, 7, 9] and [10] for various properties of the local times of the fractional Brownian motion and fields and to [20, 19] for a study of the local times and additive functionals of the so-called bifractional Brownian motion, that constitutes an extension of the fractional Brownian motion (see below the definition of this Gaussian process). A related result on the local time of the fractional Ornstein-Uhlenbeck process has been proven in [14].

The purpose of this work is to analyze the local time of a particular class of Gaussian process that are solution to the heat equation driven by an additive Gaussian

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process with non-trivial covariance both in time and in space. Consider the stochastic heat equation with additive noise

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \dot{W}^H, \quad t \in [0,T],\ x \in \mathbb{R}^d \tag{1}$$
$$u(0,x) = 0, \quad x \in \mathbb{R}^d,$$

where Δ denotes the Laplacian on $\mathbb{R}^d$ and the noise W is defined by (2). The driving noise W^H is defined as a centered Gaussian process $W^H = \{W^H(t,A); t \in [0,T], A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with covariance: for every $s,t \geq 0$

$$\begin{aligned} &\mathbf{E}(W^H(t,A)W^H(s,B)) \\ &= R_H(t,s)\int_A\int_B f(y-y')dydy' =: \langle 1_{[0,t]\times A}, 1_{[0,s]\times B}\rangle_{\mathcal{HP}} \end{aligned} \tag{2}$$

where f is spatial covariance kernel and R_H denotes the covariance of the fractional Brownian motion

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{3}$$

for every $s,t \in [0,T]$. Such a noise W is usually referred as a *fractional-colored noise* because it behaves as a fractional Brownian motion with respect to its time variable and has colored spatial covariance given by the kernel f. We will give in the sequel more details on this kernel.

Using multiple Wiener-Itô stochastic integrals, we will give the chaotic expansion of the local time of this stochastic process and we will discuss its regularity in the Sobolev-Watanabe spaces that we introduce in Section 2.

Before discussing the case of the fractional-colored noise, let us recall few facts on the heat equation driven by a Wiener process. First, consider a centered Gaussian field $\big(W(t,A), t \in [0,T], A \in \mathcal{B}(\mathbb{R}^d)\big)$ with covariance given by

$$\mathbf{E}\left(W(t,A)W(s,B)\right) = (s \wedge t)\lambda(A \cap B) \tag{4}$$

where λ denotes the Lebesque measure and $A, B \in \mathcal{B}(\mathbb{R}^d)$. The noise W is usually referred to as a *space-time white noise*. It is well-known (see for example the now classical paper by Dalang [8]) that the heat equation (1) admits an unique mild solution if and only if $d = 1$. This mild solution is defined as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)W(ds,dy), \quad t \in [0,T], x \in \mathbb{R}^d \tag{5}$$

where the above integral is a Wiener integral with respect to the Gaussian process W and G is the Green kernel of the heat equation given by

$$G(t,x) = \begin{cases} (2\pi t)^{-d/2}\exp\left(-\frac{|x|^2}{2t}\right) & \text{if } t > 0, x \in \mathbb{R}^d \\ 0 & \text{if } t \leq 0, x \in \mathbb{R}^d. \end{cases} \tag{6}$$

Consequently the process $(u(t,x), t \in [0,T], x \in \mathbb{R})$ is a centered Gaussian process. It has been proved that the solution (5) exists if and only if $d = 1$ and in this case the covariance of the solution (5) satisfies the following : for every $x \in \mathbb{R}$ we have

$$\mathbf{E}\left(u(t,x)u(s,x)\right) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{t+s} - \sqrt{|t-s|}\right), \text{ for every } s,t \in [0,T]. \tag{7}$$

This fact establishes an interesting connection between the law of the solution (5) and the so-called *bifractional Brownian motion.* Recall that the bifractional Brownian motion $(B_t^{H,K})_{t\in[0,T]}$ is a centered Gaussian process, starting from zero, with covariance

$$R^{H,K}(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), \quad s,t \in [0,T] \tag{8}$$

with $H \in (0,1)$ and $K \in (0,1]$. We refer to [11] and [19] for the definition and the basic properties of this process. Note that, if $K = 1$ then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$ (compare (8) with (3). When $K = 1$ and $H = \frac{1}{2}$ then it reduces to the standard Brownian motion.

Relation (7) implies that the solution $(u(t,x), t \in [0,T], x \in \mathbb{R})$ to equation (1) (with time-space white noise) is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ multiplied by the constant $2^{-K}\frac{1}{\sqrt{2\pi}}$. Therefore, the chaos expansion and the regularity of the local time of the solution to the heat equation with space-time white noise can be obtained from the [20].

The restriction $d = 1$ for the existence of the solution with space -time white noise lead the researchers in the last decades to consider other types of noise that would allow to consider higher dimension. An approach is to consider the so -called *white-colored noise*, meaning a Gaussian process with zero mean and covariance

$$\mathbf{E}W(t,A)W(s,B) = (t \wedge s)\int_A\int_B f(z-z')dzdz'. \tag{9}$$

Here the kernel f should be the Fourier transform of a tempered non-negative measure μ on $\mathbb{R}^d$, i.e. a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^l \mu(d\xi) < \infty, \quad \text{for some } l > 0. \tag{10}$$

Under this assumption the right hand side of (9) is a covariance function.

There are many examples of kernels f that satisfy the above condition (10) (see e.g. [1, 2]). We will use in this work the Riesz kernel. The Riesz kernel of order α is defined by

$$f(x) = R_\alpha(x) := \gamma_{\alpha,d}|x|^{-d+\alpha}, \quad 0 < \alpha < d, \tag{11}$$

where $\gamma_{\alpha,d} = 2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

It has been proved by Dalang (see [8]) that (even in the non linear case) the stochastic heat equation with white -colore noise admits a unique solution if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right) \mu(d\xi) < \infty.$$

Obviously, this condition allows to consider higher dimension. For example in the case of the Riesz kernel, the stochastic heat equation with white-colore noise admits an unique solution if and only if

$$d < 2 + \alpha$$

and this implies that one can consider every dimension $d \geq 1$.

It is possible to compute the covariance of the solution with respect to the time variable. Actually for fixed $x \in \mathbb{R}^d$, $d \geq 1$ and for every $s \leq t$ we have (see e.g. [1])

$$\mathbf{E}u(t,x)u(s,x) = (2\pi)^{-d} \int_0^s du \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{1}{2}(t-u)|\xi|^2} e^{-\frac{1}{2}(s-u)|\xi|^2}$$

and in the case of the Riesz kernel f (see (11)) we get

$$\begin{aligned} \mathbf{E}u(t,x)u(s,x) &= (2\pi)^{-d} \int_0^s du (t+s-2u)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{1}{2}|\xi|^2} \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{1}{2}|\xi|^2} \frac{1}{-\frac{d}{2}+1} \left((t+s)^{-\frac{d}{2}+1} - (t-s)^{-\frac{d}{2}+1} \right). \qquad (12) \end{aligned}$$

As a consequence, in the case of the spatial covariance given by the Riesz kernel, the solution of the heat equation with white noise in time coincides distribution to, modulo a constant, a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d}{2}$. The sample path properties of this process can be therefore again deduced from [20].

In the case $\alpha = 0$ and $d = 1$ (that corresponds to the space-time white noise case) we retrieve the formula (7) because $\int_{\mathbb{R}} \mu(d\xi) e^{-\frac{1}{2}|\xi|^2} = \sqrt{2\pi}$.

Our purpose is to study the case of the linear heat equation driven by a fractional noise in time. That is, we will consider a Gaussian field W^H with covariance given by (2) and we will assume throughout the paper that the Hurst parameter H is contained in the interval $(\frac{1}{2}, 1)$. We also consider the linear stochastic heat equation (1) where W^H is a centered Gaussian noise with covariance (2). The existence of the mild solution to the above heat equation has been studied in [1]. The mild solution of (1) can be written as a Wiener integral with respect to the Gaussian process W^H (see the next section of our paper). Let us point out that the law of the solution to (1) has a more complicated structure. Actually it is not a bifractional Brownian motion anymore. An analysis of this solution has been done in [6] where it has been show that in the covariance structure of u appear, besides the bifractional Brownian motion, two other Gaussian processes whose covariance is not trivial. Therefore, the study of the local times of the solution to (1) cannot be deduced from the literature (e.g. [19], [20]) and new techniques need to be developed.

We start by recalling in Section 2 some preliminaries on the stochastic heat equation with fractional-colored noise and on the multiple stochastic integrals. In Section 3 we give the chaos expansion of the local time of this solution and we analyze its regularity as a Watanabe functional.

2. Preliminaries

We recall here the basic facts on the existence and properties of the solution to (1) and we introduce the elements of the stochastic calculus on Wiener space that we will need in the paper.

2.1. The solution to the stochastic heat equation with fractional-colored noise

Let us consider the equation (1) with the covariance of the noise W^H given by (2) and recall that the solution can be written in the mild form

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-u, x-y) W^H(ds, dy), \qquad t \in [0,T], x \in \mathbb{R}^d \tag{13}$$

where the above integral is a standard Wiener integral with respect to the Gaussian noise W^H (see [1]). Recall the result from [1, 2].

Theorem 1. *The process $(u(t,x))_{t\in[0,T], x\in\mathbb{R}^d}$ exists and satisfies*

$$\sup_{t\in[0,T], x\in\mathbb{R}^d} \mathbf{E}\left(u(t,x)^2\right) < +\infty$$

if and only if

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{2H} \mu(d\xi) < \infty.$$

Remark 1. *For the Riesz kernel (11) the condition translates into*

$$d < 4H + \alpha. \tag{14}$$

The covariance $\mathbf{E}u(t,x)u(s,x)$ (here $x \in \mathbb{R}^d$ is fixed) is given by (12).

Remark 2. *When the spatial covariance is given by the Riesz kernel (11), the process $u(t,x)$ is self-similar with parameter*

$$H - \frac{d-\alpha}{2}$$

with respect to t. This follows easily from relation (12). The self-similarity index is strictly positive under (14).

The increments of the solution to (1) satisfy the following (see [18]): there exists two strictly positive constants C_1, C_2 such that for any $t, s \in [0,T]$ and for any $x \in \mathbb{R}^d$

$$C_1|t-s|^{2H-\frac{d-\alpha}{2}} \leq \mathbf{E}\left|u(t,x)-u(s,x)\right|^2 \leq C_2|t-s|^{2H-\frac{d-\alpha}{2}}. \tag{15}$$

2.2. Multiple stochastic integrals and Watanabe spaces

Here we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, that is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi\rangle_{\mathcal{H}}$. Denote by I_n the multiple stochastic integral with respect to B (see [16]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$

$$H_n(x) = \frac{(-1)^n}{n!}\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as: for m, n positive integers,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle f, g\rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n,\\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{16}$$

It also holds that

$$I_n(f) = I_n\big(\tilde{f}\big)$$

where $\tilde{f}$ denotes the symmetrization of f defined by $\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n\geq 0} I_n(f_n) \tag{17}$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}\left[F\right]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n\geq 0} nI_n(f_n)$$

if F is given by (17).

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(I-L)^{\frac{\alpha}{2}}\|_{L^p(\Omega)}$$

where I represents the identity. In this way, a random variable F as in (17) belongs $\mathbb{D}^{\alpha,2}$ if and only if

$$\sum_{n\geq 0}(1+n)^{\alpha}\|I_n(f_n)\|^2_{L^2(\Omega)} = \sum_{n\geq 0}(1+n)^{\alpha} n!\|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty. \tag{18}$$

Throughout this paper we will denote by $p_s(x)$ the Gaussian kernel of variance $s > 0$ given by $p_s(x) = \frac{1}{\sqrt{2\pi s}}e^{-\frac{x^2}{2s}}, x \in \mathbb{R}$ and for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by $p^d_s(x) = \prod_{i=1}^d p_s(x_i)$.

3. Existence and Regularity of the Local Times

Let us first define the local time of a stochastic process $(X_t)_{t\in T}$. For any Borel set $I \subset T$ the occupation measure of X on I is defined as

$$\mu_I(A) = \lambda\left(t \in I, X_t \in I\right), \quad A \in \mathcal{B}(\mathbb{R})$$

where λ denotes the Lebesque measure. If μ_I is absolutely continuous with respect to the Lebesque measure, we say that X has local time on I. The local time is defined as the Radon-Nykodim derivative of μ_I

$$L(I,x) = \frac{d\mu_I}{d\lambda}(x), \quad x \in \mathbb{R}.$$

We will use the notation

$$L(t,x) := L([0,t],x), \quad t \in \mathbb{R}_+, x \in \mathbb{R}.$$

The local time satisfies the occupation time formula

$$\int_I f(X_t)dt = \int_{\mathbb{R}} f(x)L(I,x)dx \tag{19}$$

for any Borel set I in T and for any measurable function $f : \mathbb{R} \to \mathbb{R}$.

Let now X be isonormal Gaussian process with variance $R(s,t)$ as introduced in Section 2. The local time can be formally written as

$$L(t,x) = \int_0^t \delta(x - X_s)ds \tag{20}$$

where δ denotes the Dirac function and the quantity $\delta(x - X_s)$ can be understood as a distribution in the Watanabe spaces. We will use the following decomposition of the delta Dirac function (see Nualart and Vives [17], Imkeller et al. [12], Eddahbi et al. [9]; see also [13] for a general theory) into orthogonal multiple Wiener-Itô integrals

$$\delta(x - X_s) = \sum_{n\geq 0} R(s)^{-\frac{n}{2}} p_{R(s)}(x) H_n \left(\frac{x}{R(s)^{\frac{1}{2}}} \right) I_n \left(1_{[0,s]}^{\otimes n} \right) \tag{21}$$

where $R(s) := R(s,s)$, $p_{R(s)}$ is the Gaussian kernel of variance $R(s)$, H_n is the Hermite polynomial of degree n and I_n represents the multiple Wiener-Itô integral of degree n with respect to the Gaussian process X as defined in Section 2.

Since the covariance function of the mild solution (13) does not depend on $x \in \mathbb{R}^d$, we will consider in the sequel a Gaussian process $(U_t)_{t\in[0,T]}$ with covariance function

$$R(t,s) = d(\alpha, H) \int_0^t \int_0^s |u-v|^{2H-2} (t+s-u-v)^{-\frac{d-\alpha}{2}} dv du \tag{22}$$

with $d(\alpha, H)$ a strictly positive constant and as mentioned before $H \in (\frac{1}{2}, 1)$ and $0 < \alpha < d < 4H + \alpha$.

Before proving our main result, let us state an auxiliary lemma.

Lemma 1. *Let R be the covariance function (22) of the process U.*

(1) *For every $a, u \in [0,T]$ such that $au \in [0,T]$ we have*

$$R(a, au) = a^{2H - \frac{d-\alpha}{2}} R(1,u).$$

(2) *Let $F(z) = \frac{R(1,z)}{\sqrt{R(1)R(z)}}$. Then for every $z \in [0,T]$, it holds*

$$0 \leq F(z) \leq 1.$$

(3) *There exists a strictly positive constant $c(H,\alpha,d)$ not depending on n such that for n large enough*

$$\int_0^1 dz\, (F(z))^n\, R(z)^{-\frac{1}{2}} \leq c(H,\alpha,d) n^{-\frac{1}{2H - \frac{d-\alpha}{2}}}.$$

Proof. Point 1. is a consequence of Remark 2. It can be proven also directly by a change of variables. The point 2. follows from a well-known for the covariance function. Let us focus on the proof of the third point. Let γ be an arbitrary point between 0 and 1. We can write

$$\begin{aligned}
&\int_0^1 dz \left(\frac{R(1,z)}{\sqrt{R(1)R(z)}}\right)^n R(z)^{-\frac{1}{2}} \\
&= \int_0^{1-\gamma} dz \left(\frac{R(1,z)}{\sqrt{R(1)R(z)}}\right)^n R(z)^{-\frac{1}{2}} + \int_{1-\gamma}^1 dz \left(\frac{R(1,z)}{\sqrt{R(1)R(z)}}\right)^n R(z)^{-\frac{1}{2}} \\
&:= J_1 + J_2.
\end{aligned}$$

The function $F(z) = \frac{R(1,z)}{\sqrt{R(1)R(z)}}$ with $z \in [0,1]$ satisfies $F(0) = 0, F(1) = 1$, is continuous and $F(z) = 1$ if and only if $z = 1$ (obviously U_1 and U_z are proportional if and only if $z = 1$). Consequently, there exists a a constant $c_0(\gamma) \in (0,1)$ such that $F(z) \leq c_0(\gamma)$ for every $z \in [0, 1-\gamma]$. Thus

$$J_1 \leq c(H,d,\alpha,\gamma) c_o(\gamma)^n. \tag{23}$$

Concerning the term J_2, we can write, by the inequality (3.15) in [5] and the inequality (15)

$$\begin{aligned}
J_2 &\leq \int_{1-\gamma}^1 dz \left(1 - c\frac{(1-z)^{2H-\frac{d-\alpha}{2}}}{\sqrt{R(1)R(z)}}\right)^{\frac{n}{2}} R(z)^{-\frac{1}{2}} \\
&\leq c(H,d,\alpha,\gamma) \int_{1-\gamma}^1 dz \left(1 - c(1-z)^{2H-\frac{d-\alpha}{2}}\right)^{\frac{n}{2}} \\
&= c(H,d,\alpha,\gamma) \int_{1-\gamma}^1 dz e^{\frac{n}{2}\log\left(1-c(1-z)^{2H-\frac{d-\alpha}{2}}\right)}
\end{aligned}$$

where the constant c in the exponential function also depends on H, γ, d and we noticed the trivial fact that $R(z)$ is bounded below by a strictly positive constant for z outside the origin. Using the inequality $-\log z \geq 1 - z$ for every $z \in (0,1]$ and hence for every $z \in (1-\gamma, 1)$, we get

$$\begin{aligned}
J_2 &\leq c(H,d,\alpha,\gamma) \int_{1-\gamma}^1 dz e^{-\frac{n}{2}c(1-z)^{2H-\frac{d-\alpha}{2}}} \\
&= c(H,d,\alpha,\gamma) \int_0^{\gamma} dz e^{-\frac{n}{2}cz^{2H-\frac{d-\alpha}{2}}}
\end{aligned}$$

and by $\frac{n}{2}z^{2H-\frac{d-\alpha}{2}} = y$ we find that

$$J_2 \leq c(H,\alpha,d,\gamma) n^{-\frac{1}{2H-\frac{d-\alpha}{2}}}.$$

By using the inequalities (23) and (), we found the conclusion. ■

Our main result is stated below. It gives the chaos expansion of the local time if the process U and its regularity in the Watanabe spaces.

Theorem 1. *For every $t \in [0,T]$ and $x \in \mathbb{R}$ the local time $L(t,x)$ of the process U admits the following chaos expansion into multiple Wiener-Itô integrals*

$$
\begin{aligned}
L(t,x) &= \int_0^t \delta(x-U_s)ds \\
&= \int_0^t \sum_{n\geq 0} R(s)^{-\frac{n}{2}} p_{R(s)}(x) H_n\left(\frac{x}{R(s)^{\frac{1}{2}}}\right) I_n(1_{[0,s]^{\otimes n}})ds.
\end{aligned}
\tag{24}
$$

Moreover $L(t,y)$ belongs to the Sobolev-Watanabe space $\mathbb{D}^{\gamma,2}$ for every

$$
\gamma < \frac{1}{2H - \frac{d-\alpha}{2}} - \frac{1}{2}. \tag{25}
$$

Proof. The chaos expansion (24) is a consequence of (21) and (20) and of the techniques used in [7] or [9]. Let us compute the $\mathbb{D}^{\gamma,2}$ norm of the random variable $L(t,x)$ with $t \in [0,T]$ and $x \in \mathbb{R}$. Using the isometry of multiple Wiener-Itô integrals (16) and the expression of the Sobolev-Watanabe norm (18), we can write

$$
\begin{aligned}
&\|L(t,x)\|^2_{\gamma,2} \\
&= \sum_{n\geq 0}(1+n)^\gamma \int_0^t\int_0^t dudv R(u)^{-\frac{n}{2}} R(v)^{-\frac{n}{2}} p_{R(u)}(x) H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right) \\
&\quad p_{R(v)}(x) H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right) \\
&\quad \mathbf{E} I_n(1_{[0,u]^{\otimes n}}) I_n(1_{[0,v]^{\otimes n}}) \\
&= \sum_{n\geq 0}(1+n)^\gamma n! \int_0^t\int_0^t dudv R(u)^{-\frac{n}{2}} R(v)^{-\frac{n}{2}} \\
&\quad p_{R(u)}(x) H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right) p_{R(v)}(x) H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right) \langle 1^{\otimes n}_{[0,u]}, 1^{\otimes n}_{[0,v]}\rangle \\
&= \sum_{n\geq 0}(1+n)^\gamma n! \int_0^t\int_0^t dudv R(u)^{-\frac{n}{2}} R(v)^{-\frac{n}{2}} \\
&\quad p_{R(u)}(x) H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right) p_{R(v)}(x) H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right) R(u,v)^n \\
&= 2\sum_{n\geq 0}(1+n)^\gamma n! \int_0^t dv \int_0^u dv R(u)^{-\frac{n}{2}} R(v)^{-\frac{n}{2}} \\
&\quad p_{R(u)}(x) H_n\left(\frac{x}{R(u)^{\frac{1}{2}}}\right) p_{R(v)}(x) H_n\left(\frac{x}{R(v)^{\frac{1}{2}}}\right) R(u,v)^n.
\end{aligned}
$$

Above $\langle \cdot, \cdot \rangle$ denotes the scalar product in the Hilbert space associated with the Gaussian process U. We use the identity (see e.g. [7])

$$H_n(y)e^{-\frac{y^2}{2}} = (-1)^{[n/2]}2^{\frac{n}{2}}\frac{2}{n!\pi}\int_0^\infty u^n e^{-u^2}g(uy\sqrt{2})du \tag{26}$$

where $g(r) = \cos(r)$ if n is even and $g(r) = \sin(r)$ if n is odd. Since $|g(r)| \le 1$, we have the bound

$$\left|H_n(y)e^{-\frac{y^2}{2}}\right| \le 2^{\frac{n}{2}}\frac{2}{n!\pi}\Gamma(\frac{n+1}{2}) := c_n \tag{27}$$

Thus (with C a generic strictly positive constant)

$$\begin{aligned}\|L(t,x)\|_{\gamma,2}^2 &\le C\sum_{n\ge 0}(1+n)^\gamma n!c_n^2\int_0^t\int_0^t dudvR(u)^{-\frac{n+1}{2}}R(v)^{-\frac{n+1}{2}}R(u,v)^n\\ &= C\sum_{n\ge 0}(1+n)^\gamma n!c_n^2\int_0^t duR(u)^{-\frac{n+1}{2}}u\int_0^1 dzR(uz)^{-\frac{n+1}{2}}R(uz,u)^n\end{aligned}$$

where we made the change of variables $\frac{v}{u} = z$ in the integral dv.

Lemma 1 point 1. implies that for every u, z

$$R(u,uz) = u^{2H-\frac{d}{2}}R(1,z)$$

and thus

$$\begin{aligned}&\|L(t,x)\|_{\gamma,2}^2\\ &\le C\sum_{n\ge 0}n!c_n^2\int_0^t u^{1-2H+\frac{d-\alpha}{2}}du\int_0^1 dzR(1,z)^nR(z)^{-\frac{n+1}{2}}R(1)^{-\frac{n+1}{2}}\\ &= C\sum_{n\ge 0}n!c_n^2\int_0^1 dz\left(\frac{R(1,z)}{\sqrt{R(1)R(z)}}\right)^n R(z)^{-\frac{1}{2}}\end{aligned} \tag{28}$$

where we noticed that the integral du is finite since $2H < 2 + \frac{d-\alpha}{2}$ and the constant C may change from one line to another. Notice also that, by Stirling's formula, the sequence $n!c_n^2$ behaves as $n^{\frac{1}{2}}$ when n goes to infinity. We conclude, by (28), that

$$\|L(t,x)\|_{\gamma,2}^2 \le c(H,\alpha,d)\sum_{n\ge 0}(1+n)^\gamma n!c_n^2 n^{-\frac{1}{2H-\frac{d-\alpha}{2}}}$$

and this is convergent if $\gamma < \frac{1}{2H-\frac{d-\alpha}{2}} - \frac{1}{2}$. ∎

Corollary 1. *The solution to the fractional-colored heat equation (1) admits a local time in $L^2(\Omega)$ if*

$$2H - \frac{d-\alpha}{2} \le \frac{3}{2}. \tag{29}$$

Proof. The space $L^2(\Omega)$ coincides with the Sobolev-Watanabe space $\mathbb{D}^{0,2}$. By applying Theorem 1, we obtain that (29). ■

Remark 3. • *In the case of the fractional-white noise* ($\alpha = 0$), (29) *is always true since*

$$2H - \frac{d}{2} \leq 2 - \frac{1}{2} = \frac{3}{2}.$$

• *Recall that the local time of the fractional Brownian motion belongs to the Watanabe space* $\mathbb{D}^{\gamma,2}$ *with* $\gamma < \frac{1}{2H} - \frac{1}{2}$ (*see [9]*). *This is consistent with the condition* (25) *since* $H - \frac{d-\alpha}{4}$ *is the self-similarity index of the solution* U (*see Remark* 2).

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