TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 5, pp. 1727-1750, October 2013
DOI: 10.11650/tjm.17.2013.3044
This paper is available online at http://journal.taiwanmathsoc.org.tw

STABILITY ANALYSIS FOR GENERALIZED f-PROJECTION OPERATORS WITH AN APPLICATION

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Abstract. In this paper, stability results for generalized f-projection operators with parametric perturbations are given in reflexive, smooth, and strictly convex Banach spaces. By using the stability of generalized f-projection operators and the degree theory for the generalized set-valued variational inequality introduced by Wang and Huang [30], the lower continuity of the solution mapping for the parametric generalized set-valued variational inequality is established under some suitable conditions.

1. Introduction

Throughout this paper, unless otherwise stated, assume that B is a real Banach space with the dual space B^*, the norm and the dual pair between B and B^* are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed convex subset of B.

It is well known that the projection operators in finite dimensional spaces and infinite dimensional spaces are widely used in different areas of mathematics such as functional and numerical analysis, theory of optimization and approximation, and also for the problems of optimal control and operations research, nonlinear and stochastic programming and game theory, variational inequalities, complementarity problems, etc. (for example, see [2, 3, 4, 21] and the references therein). Alber [3] introduced the generalized projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$ in uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. Alber [4] studied

Received March 10, 2013, accepted April 4, 2013.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 49J40, 47J20.
Key words and phrases: Generalized f-projection operator, Degree theory, Generalized set-valued variational inequality, Lower semicontinuity, Mosco-convergence.
This work was supported by the Key Program of NSFC (Grant No. 70831005), the National Natural Science Foundation of China (11171237, 11101069) and the Key Program of the Fundamental Research Funds for the Central Universities (No. SWJTU11CX156).
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the stability of Π_K and π_K with respect to a perturbation of the set and presented some new properties of the generalized projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$. Li [21] extended the generalized projections from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces. Recently, Wu and Huang [32] introduced and studied generalized f-projection operators $\pi_K^f : B^* \to 2^K$ in Banach spaces, which extended the definition of the generalized projections $\pi_K$introduced and studied by Alber [2, 4] and Li [21]. Some properties of the generalized f-projection operators π_K^f are given in [13, 32] and further study for variational inequalities with the generalized f-projection operators π_K^f can be found in [23, 24, 33, 34]. Let B be a reflexive, smooth, and strictly convex Banach space, C be a nonempty closed convex subset of B, and $\{K_n\} \subset C$ be a sequence of nonempty closed convex subsets. Ibaraki et al. [14] showed that for any $x \in C$, $\Pi_{K_n}x$ converges strongly to $\Pi_K x$, as K_n Mosco-converging to K. However, it remains unknown whether there is the similar results for generalized f-projection operators π_K^f.

On the other hand, it is well known that the variational inequalities theory has wide applications in finance, economics, transportation, optimization, operations research, and engineering sciences. Among many desirable properties of the solution sets for variational inequalities, stability analysis of solution set is of considerable interest (see, for example, [19, 12, 15, 16, 17, 18, 20, 27, 28] and the references therein). Pang [27] used degree theory to obtain interesting results on sensitivity of a parametric nonsmooth equation with multivalued perturbed solution sets. This paper has been very influential for the optimization community. By using the metric projection method of Dafermos [8], Yen [35] obtained a theorem on Höder continuity of the solution to a parametric variational inequalities in Hilbert spaces for strongly monotone operators. Based on the normal map and the degree-theoretic method, Robinson [28] established a result on the solution stability of the variational conditions in finite dimensional spaces. By using homeomorhisms between the solution set of variational inequalities and the solution set of generalized normal maps, Kien and Yao [18] proved that the solution map of parametric variational inequalities is lower semicontinuous. By using the degree theory and the natural map, Kien and Wong [16] showed that under certain conditions, the solution mapping of parametric single-valued variational inequalities is lower semicontinuous with respect to parameters in finite dimensional spaces.

Note that although there have been many papers which study solution stability of parametric variational inequalities, very few paper which focus on a such study for parametric generalized variational inequalities and parametric generalized set-valued variational inequalities by using degree theory. Recently, Kien et al. [17] built a degree theory for the generalized variational inequality in finite dimensional spaces and employed the degree to prove some results on existence and stability of the solutions for the generalized variational inequality. Very recently, Wang and Huang [30] introduced and studied the degree theory for a generalized set-valued variational inequality in

Banach spaces and gave an existence result of solutions for the generalized set-valued variational inequality under some suitable conditions.

We remark that the result about stability of the solutions for the generalized variational inequality in Kien et al. [17] deal with the case of finite dimensional spaces, where the compactness of sets are easy to obtain and the duality mapping and metric projection have many nice properties. Such problem will be more difficult if the space is infinite-dimensional because some properties of duality mappings and metric projections are no longer valid.

Motivated and inspired by the work mentioned above, in this paper, we study the stability of generalized f-projection operators π_K^f with parametric perturbations in a reflexive, smooth, and strictly convex Banach space. By using the stability of generalized f-projection operators π_K^f and the degree theory for a generalized set-valued variational inequality introduced by [30], we show that the solution mapping of the parametric generalized set-valued variational inequality is lower semicontinuous with respect to the parameters. The results presented in this paper extend and improve some corresponding results in [14, 16, 17, 18]. The rest of the paper is organized as follows. In Section 2, we recall some concepts and preliminary results. In Section 3, we establish the stability of generalized f-projection operators. In Section 4, by using the stability of generalized f-projection operators and the degree theory for a generalized set-valued variational inequality introduced by [30],we give sufficient conditions for the lower semicontinuity of the solution mapping for the parametric generalized set-valued variational inequality.

2. Preliminaries

We firstly recall some definitions and results of the degree theory (see, for example, [25]). Let $R = (-\infty, +\infty)$ and Ω_1 be an open bounded set in R^n. We denote by $\partial\Omega_1$ the boundary of Ω_1 and $\overline{\Omega_1}$ the closure of Ω_1. Let $C^1(\overline{\Omega_1}) = C^1(\Omega_1) \cap C(\overline{\Omega_1})$, where $C^1(\Omega_1)$ is the set of all continuously differentiable functions $h : \Omega_1 \mapsto R^n$ and $C(\overline{\Omega_1})$ is the set of all continuous functions on $\overline{\Omega_1}$. We will denote by $\rho(x, A_1)$ the distance from a point $x \in R^n$ to a set $A_1 \subset R^n$, i.e.,

$$\rho(x, A_1) := \inf\{|x - y| : y \in A_1\}.$$

If $h \in C^1(\overline{\Omega_1})$, let $J_h(x) = \det(\text{grad } h(x))$ and

$$Z_h = \{x \in \overline{\Omega_1} : J_h(x) = 0\}.$$

Here $\text{grad } h(x)$ denotes the gradient of function h with respect to vector x. It is well known that if $h \in C^1(\overline{\Omega_1})$ and $p \notin h(Z_h)$, then the set h^{-1}(p) is finite (see, for example, Theorem 1.1.2 in [25]).

Definition 2.1. (a_1) Let $h \in C^1(\overline{\Omega_1})$ and $p \notin h(Z_h) \cup h(\partial\Omega_1)$. The degree of h at p with respect to Ω_1 is defined by

$$d(h, \Omega_1, p) := \sum_{x \in h^{-1}(p)} \operatorname{sgn}(J_h(x)).$$

(a_2) Let $h \in C^1(\overline{\Omega_1})$ and $p \notin h(\partial\Omega_1)$ such that $p \in h(Z_h)$. We define the degree of h at p with respect to Ω_1 to be the number $d(h, \Omega_1, q)$ such that $|p - q| < \rho(p, h(\partial\Omega_1))$ for all $q \notin h(Z_h) \cup h(\partial\Omega_1)$.

(a_3) Let $h \in C(\overline{\Omega_1})$ and $p \notin h(\partial\Omega_1)$. We define $d(h, \Omega_1, p)$, the degree of h at p with respect to Ω_1, to be $d(h_1, \Omega, p)$ such that $|h(x) - h_1(x)| < \rho(p, h(\partial\Omega_1))$ for all $h_1 \in C^1(\overline{\Omega_1})$ and $x \in \overline{\Omega_1}$.

Let D be an open, bounded set in a Banach space X with the boundary ∂D and the closure $\overline{D}$. We say that a single-valued mapping $\hat{F_1} : \overline{D} \to X$ is compact if $\hat{F_1}$ is continuous and for every bounded subset A of $\overline{D}$, $\hat{F_1}(A) = \cup_{x\in A}\hat{F_1}(x)$ is a relatively compact set, i.e., $\overline{\hat{F_1}(A)}$ is a compact set. We denote by $\hat{K}(\overline{D})$ the set of all mappings $\phi : \overline{D} \to X$ such that $\phi = I - \tilde{T}$, where $\tilde{T} : \overline{D} \to X$ is compact.

Definition 2.2. Suppose that $\phi = I - \tilde{T}$, where $\tilde{T} : \overline{D} \to X$ is a compact mapping, I is the identity mapping on $\overline{D}$ and $p \in X \setminus \phi(\partial D)$. Let $\hat{\phi} = I - \hat{T}$, where $\hat{T}$ is a continuous mapping defined in $\overline{D}$ with finite dimensional range such that $\|\tilde{T}(x) - \hat{T}(x)\| < \rho(p, \phi(\partial D))$ for all $x \in \overline{D}$. Choose a finite dimensional linear space V to contain $\hat{T}(\overline{D})$ and let $D_V = D \cap V$. Then define $d(\phi, D, p) = d(\hat{\phi}, D_V, p)$.

The following Theorem lists some basic properties of the degree.

Theorem 2.1. *The Leray-Schauder degree has the following properties:*

- (a_1) *(Existence). If $\phi \in \hat{K}(\overline{D})$ and $d(\phi, D, p) \neq 0$ then there is $x \in D$ such that $\phi(x) = p$;*
- (a_2) *(Homotopy invariance). $H : [0,1] \times \overline{D} \to X$ is a compact mapping. Put $\phi_t(x) = x - H(t, x)$. If $p \notin \phi_t(\partial D)$ for all $t \in [0,1]$, then $d(\phi_t, D, p)$ is independent of $t \in [0,1]$;*
- (a_3) *(Excision). If $\phi \in \hat{K}(\overline{D})$ $p \notin \phi(\partial D)$, $D_0 \subset \overline{D}$ is closed and $p \notin \phi(D_0)$, then $d(\phi, D, p) = d(\phi, D \setminus D_0, p)$.*

Recall that a point $\bar{x}_0$ is said to be an isolated solution of the equation $\phi(x) = p$ if there exists a bounded open set $U \subset X$ such that $\bar{x}_0$ is the unique solution in $\overline{U} \subset D$. We denote by $\mathcal{U}$ the collection of such bounded open sets. By the excision property, it follows that $d(\phi, U_1, p) = d(\phi, U_2, p)$ for all U_1, $U_2 \in \mathcal{U}$. The index of $\bar{x}_0$, denoted by $\operatorname{index}(\phi, \bar{x}_0, p)$, is the common value $d(\phi, U, p)$ for all $U \in \mathcal{U}$.

Throughout this paper, unless otherwise stated, we use $\to$ for convergence in strong sense and $\rightharpoonup$ for convergence in weak sense. Let $T : K \to 2^{B^*}$ be a set-valued mapping, $f : K \to R\cup\{+\infty\}$ be a proper, convex and lower semi-continuous function and $\Omega \subset B$ be an open bounded set with the boundary $\partial\Omega$ and the closure $\overline{\Omega}$ such that $K \cap \Omega \neq \emptyset$.

Next we recall the concept of the normalized duality mapping. The normalized duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \{j(x) \in B^*\colon\ \langle j(x), x\rangle = \|j(x)\|\|x\| = \|x\|^2 = \|j(x)\|^2\}\ , \quad \forall\ x \in B.$$

Without confusion, one understands that $\|j(x)\|$ is the B^*-norm and $\|x\|$ is the B-norm. In this paper, we consider the following generalized set-valued variational inequality (in short, GSVI): find $x \in K$ and $x^* \in F(x)$ such that

$$\langle x^*, y - x\rangle + f(y) - f(x) \geq 0, \quad \text{for all } y \in K. \tag{2.1}$$

where $F(x) = J(x) - T(x)$ for every $x \in K$. It is well known that GSVI (2.1) is encountered in many applications, in particular, in mechanical problems and equilibrium problems (see, for example, [7, 12, 19]).

For any $x \in B$, the generalized f-normal cone of K at x, denoted by $N_K^f(x)$, is defined as follows (see [30], Definition 1.2):

$$N_K^f(x) = \begin{cases} \{\varphi \in B^* : \langle \varphi, y - x\rangle + f(x) - f(y) \leq 0, \forall y \in K\}, & \text{if } x \in K,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Remark 2.1. (a_1) If $f = 0$, then the generalized f-normal cone of K at x reduces to the generalized normal cone of K at x considered in [22], that is,

$$\mathbb{N}_K(x) = \begin{cases} \{\varphi \in B^* : \langle \varphi, y - x\rangle \leq 0, \forall y \in K\}, & \text{if } x \in K,\\ \emptyset, & \text{otherwise.}\end{cases}$$

(a_2) If B is a Hilbert space and $f = 0$, then the generalized f-normal cone reduces to the normal cone defined as follows:

$$N_K(x) = \begin{cases} \{\varphi \in H : \langle \varphi, y - x\rangle \leq 0, \forall y \in K\}, & \text{if } x \in K,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Definition 2.3. Let $F : K \to 2^{B^*}$ be a set-valued mapping. (i) F is said to be upper semicontinuous at $x \in K$ if, for any open set $V \subset B^*$ with $F(x) \subset V$, there exists an open neighborhood U of x such that $F(y) \subset V$ for all $y \in U \cap K$. If F is upper semicontinuous at every $x \in K$, we say that F is upper semicontinuous on K.

(ii) F is said to be compact if F is upper semicontinuous and $F(A)$ is relatively compact for any bounded subset A of K, that is, $\overline{F(A)}$ is a compact set.

For any fixed $\rho > 0$, let $G : B^* \times K \to (-\infty, +\infty]$ be a function defined as follows:

$$G(\varphi, x) = \|x\|^2 - 2\langle \varphi, x\rangle + \|\varphi\|^2 + 2\rho f(x),\ \forall\ \varphi \in B^*,\ \forall\ x \in K, \tag{2.2}$$

where $\varphi \in B^*$, $x \in K$ and $f : K \subset B \to R\bigcup\{+\infty\}$ is proper, convex and lower semi-continuous. It is easy to see that $G(\varphi, x) \geq (\|x\| - \|\varphi\|)^2 + 2\rho f(x)$ for all $\varphi \in B^*$ and $x \in K$.

Definition 2.4. We say that B has the property (h) if $x_n \rightharpoonup x$ weakly and $\|x_n\| \to \|x\|$ implies $x_n \to x$.

Remark 2.2. It is well known that any locally uniformly convex Banach space has the property (h) and B^* has a Fré́chet differentiable norm if and only if B is reflexive, strictly convex, and has the property (h) (see, for example, [29]).

Definition 2.5. [32]. We say that $\pi_K^f : B^* \to 2^K$ is a generalized f-projection operator if

$$\pi_K^f \varphi = \{u \in K : G(\varphi, u) = \inf_{y \in K} G(\varphi, y)\}, \quad \forall\, \varphi \in B^*.$$

Remark 2.3. (i) If $f(x) = 0$ for all $x \in K$, then the generalized f-projection operator π_K^f reduces to the generalized projection operator π_k defined by Alber [3] and Li [21], that is,

$$\pi_K \varphi = \{u \in K : G_1(\varphi, u) = \inf_{y \in K} G_1(\varphi, y)\}, \quad \forall\, \varphi \in B^*,$$

where $G_1(\varphi, x) = \|x\|^2 - 2\langle \varphi, x\rangle + \|\varphi\|^2$ for all $\varphi \in B^*$ and $x \in K$.

(ii) If $f(x) = 0$ for all $x \in K$ and $B = \mathcal{H}$ is a Hilbert space, then the generalized f-projection operator π_K^f is equivalent to the following metric projection operator

$$P_K(\varphi) = \{u \in K : \|u - \varphi\| = \inf_{y \in K} \|y - \varphi\|\}, \quad \forall\, \varphi \in \mathcal{H}.$$

Theorem 2.2. [13, 32]. *If B is a reflexive Banach space with dual space B^* and K is a nonempty closed convex subset of B, then the following conclusions hold:*

(i) *For any given $\varphi \in B^*$, $\pi_K^f \varphi$ is a nonempty, closed and convex subset of K;*

(ii) *If B is smooth, then for any given $\varphi \in B^*$, $x \in \pi_K^f \varphi$ if and only if*

$$\langle \varphi - J(x), x - y\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall\, y \in K;$$

(iii) *If B is strictly convex, then the operator $\pi_K^f : B^* \to K$ is single-valued.*

Theorem 2.3. [13]. *Let B be a reflexive and strictly convex Banach space with dual space B^* and K be a nonempty closed convex subset of B. Suppose that $f : K \subset B \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Then*

(i) *$\pi_K^f : B^* \to K$ is norm-weak continuous;*

(ii) *If B has the property (h), then $\pi_K^f : B^* \to K$ is continuous.*

Lemma 2.1. [26]. *Let $\mathcal{B}$ be a reflexive Banach space, $C \subset \mathcal{B}$ be a closed convex set. Let $b : C \to (-\infty, +\infty]$ be a lower semicontinuous and convex function with bounded from below such that* $\lim_{\|x\|\to\infty} b(x) = +\infty$. *Then there exists $x_0 \in C$ such that* $\inf_{x\in C} b(x) = b(x_0)$.

Lemma 2.2. [30]. *Let $\mathbb{T} : K \to 2^{B^*}$ be an upper semicontinuous mapping with closed convex values. For any $\epsilon > 0$, there is a continuous single-valued mapping $T_\epsilon : B \to B^*$ such that for every $x \in K$, there exist $y \in K$ and $z \in \mathbb{T}(y)$ such that*

$$\|y - x\| < \epsilon \quad \text{and} \quad \|z - T_\epsilon(x)\| < \epsilon. \tag{2.3}$$

Moreover, $R(T_\epsilon|_K) \subset \overline{\text{co}}(R(\mathbb{T}))$, where $T_\epsilon|_K$ denotes the restriction of T_ϵ on K, $R(\mathbb{T})$ is the range of $\mathbb{T}$ and $R(T_\epsilon|_K)$ is the range of $T_\epsilon|_K$. In particular, $T_\epsilon|_K$ is compact if $\mathbb{T}$ is compact.

We consider $F(x) = J(x) - T(x)$ for all $x \in K$, where $J : B \to 2^{B^*}$ is the normalized duality mapping and $T : K \to B^*$ is a compact mapping with closed convex values. For any $\epsilon > 0$, we define a mapping $\Phi_\epsilon : B \to B$ as follows:

$$\Phi_\epsilon(x) = x - \pi_K^f(J(x) - F_\epsilon(x)) = x - \pi_K^f(T_\epsilon(x)), \quad \forall\, x \in K,$$

where $F_\epsilon = J - T_\epsilon$ and T_ϵ is an approximate continuous selection of T which satisfies the conclusions of Lemma 2.2.

Through the rest of this paper, unless otherwise stated, we take $\rho = 1$ in (2.2). By Theorem 2.2, it is easy to know the following theorem holds.

Theorem 2.4. [30]. *Assume that B is a reflexive and smooth Banach space with dual space B^* and K is a nonempty closed convex subset of B. Then the following statements are mutually equivalent:*

(a_1) *$x \in K$ and $x^* \in F(x)$ is a solution of GSVI (1);*

(a_2) *$x \in K$ satisfies $x \in \pi_K^f(J(x) - F(x))$;*

(a_3) *$x \in K$ satisfies $0 \in F(x) + N_K^f(x)$.*

Lemma 2.3. [30]. *Assume that B is a reflexive, smooth and strictly convex Banach space with the dual space B^*, B has the property (h) and K is a nonempty closed convex subset of B. Let $f : K \to (-\infty, +\infty]$ be proper, convex and lower semi-continuous and $F(x) = J(x) - T(x)$ for all $x \in K$, where $J : B \to 2^{B^*}$ is the normalized duality mapping and $T : K \to 2^{B^*}$ is a compact mapping with compact convex values. Moreover, $0 \notin (F + N_K^f)(\partial\Omega)$. Then the following assertions hold:*

(i) *There exists $\epsilon_1 > 0$ such that $0 \notin \Phi_\epsilon(\partial\Omega)$ for all $\epsilon \in (0, \epsilon_1]$;*

(ii) *There exists $\epsilon_2 > 0$ such that*

$$d(\Phi_\epsilon, \Omega, 0) = d(\Phi_{\epsilon'}, \Omega, 0),\ \forall\ \epsilon,\ \epsilon' \in (0, \epsilon_2].$$

It follows from Lemma 2.3 that there exists $\bar{\epsilon} > 0$ such that $0 \notin \Phi_\epsilon(\partial\Omega)$ and $d(\Phi_\epsilon, \Omega, 0) = d(\Phi_{\epsilon'}, \Omega, 0)$ for all $\epsilon,\ \epsilon' \in (0,\ \bar{\epsilon}]$. Based on Lemma 2.3, we can introduce the following definition.

Definition 2.6. [30]. Assume that B is a reflexive, smooth and strictly convex Banach space with the dual space B^*, B has the property (h) and K is a nonempty closed convex subset of B. Let $f : K \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Let $F(x) = J(x) - T(x)$ for all $x \in K$, where $J : B \to 2^{B^*}$ is the normalized duality mapping and $T : K \to 2^{B^*}$ is a compact mapping with compact convex values. Moreover, $0 \notin (F + N_K^f)(\partial\Omega)$. The degree of the generalized set-valued variational inequality defined by F and K respect to Ω at 0 is the common value $d(\Phi_\epsilon, \Omega, 0)$ for $\epsilon > 0$ sufficiently small and denoted by $d(F + N_K^f, \Omega, 0)$.

Remark 2.4. If $B = R^n$ and $f = 0$ is a finite dimensional space, then Definition 2.6 is equivalent to the degree of the generalized variational inequality defined by Definition 2.1 in [17]. For details, see [30].

The following theorem contains some properties of the degree of the generalized set-valued variational inequality in Banach spaces.

Theorem 2.5. [30]. *Assume that B is a reflexive, smooth and strictly convex Banach space with the dual space B^*, B has the property (h) and K is a nonempty closed convex subset of B. Suppose that $f : K \to (-\infty, +\infty]$ is proper, convex and lower semi-continuous. Let $F(x) = J(x) - T(x)$ for all $x \in K$, where $J : B \to 2^{B^*}$ is the normalized duality mapping and $T : K \to 2^{B^*}$ is a compact mapping with compact convex values. Then the following assertions hold:*

(a_1) (*Normalization*). *Suppose that there exists $\hat{x} \in \Omega \cap K$ such that $f(\hat{x}) = \inf_{x\in K} f(x)$. If $0 \notin (F + N_K^f)(\partial\Omega)$ with $F = J - J(\hat{x})$, then $d(F + N_k^f, \Omega, 0) = 1$.*

(a_2) (*Existence*). *If $0 \notin (F + N_K^f)(\partial\Omega)$ and $d(F + N_K^f, \Omega, 0) \neq 0$, then there exists $x \in K \cap \Omega$ such that*

$$0 \in F(x) + N_K^f(x).$$

(a_3) (*Homotopy invariance*). *For $\tilde{i} \in \{1, 2\}$, $F_{\tilde{i}}(x) = J(x) - T^{\tilde{i}}(x)$ for all $x \in K$, where $J : B \to 2^{B^*}$ is the normalized duality mapping and $T^{\tilde{i}} : K \to 2^{B^*}$ is a compact mapping with compact convex values. Moreover, $0 \notin (tF_1 + (1 - t)F_2 + N_K^f)(\partial\Omega)$ for all $t \in [0, 1]$. Then*

$$d(F_1 + N_K^f, \Omega, 0) = d(F_2 + N_K^f, \Omega, 0).$$

(a_4) (*Excision*). *If $0 \notin (F + N_K^f)(\partial\Omega)$ and $D \subset \overline{\Omega}$ is a closed set such that $0 \notin (F + N_k^f)(D)$ then*

$$d(F + N_K^f, \Omega, 0) = d(F + N_K^f, \Omega \setminus D, 0).$$

(a_5) *If* $0 \notin (F + N_K^f)(\partial\Omega)$ *and* $T_1 : B \to B^*$ *is a single-valued continuous mapping such that* $T_1(x) \in T(x)$ *for all* $x \in K$, *then* $d(F + N_K^f, \Omega, 0) = d(\Phi, \Omega, 0)$, *where* $\Phi(x) = x - \pi_K^f(T_1(x))$.

Remark 2.5. If f is bounded from below and $\lim_{\|x\|\to\infty} f(x) = +\infty$, then Lemma 2.1 shows that $f(\hat{x}) = \inf_{x\in K} f(x)$ is true. Thus, the assumption of (a_1) holds under some suitable conditions.

Definition 2.7 A vector $x_0 \in K$ is called an isolated solution of GSVI (2.1) if there exists a neighborhood $V_1 \subset K$ of x_0 such that x_0 is the unique solution of GSVI (2.1) in $\overline{V_1}$.

Theorem 2.6. *Assume that* B *is a reflexive, smooth and strictly convex Banach space with the dual space* B^*, B *has the property (h) and* K *is a nonempty closed convex subset of* B. *Suppose that* $f : K \subset B \to R \cup \{+\infty\}$ *is proper, convex and lower semi-continuous. Let* $F(x) = J(x) - T(x)$ *for all* $x \in K$, *where* $J : B \to 2^{B^*}$ *is the normalized duality mapping and* $T : K \to 2^{B^*}$ *is a compact mapping with compact convex values. Suppose that* x_0 *is an isolated solution of GSVI (2.1) and* $\mathfrak{U}$ *is the collection of all open bounded neighborhoods* V_1 *of* x_0 *such that* $\overline{V_1}$ *does not contain another solution of GSVI (2.1). Then*

$$d(F + N_K^f, V_2, 0) = d(F + N_K^f, V_3, 0)$$

for all V_2, $V_3 \in \mathfrak{U}$. *The common value* $d(F + N_K^f, V_1, 0)$ *for* $V_1 \in \mathfrak{U}$ *is called the index of* $F + N_K^f$ *and denoted by* $i(F + N_K^f, x_0, 0)$.

Proof. Taking any $\tilde{V} \in \mathfrak{U}$ we have $0 \notin (F+N_K^f)(\partial\tilde{V})$. Therefore $d(F+N_K^f, \tilde{V}, 0)$ is well defined. We now assume that V_2, $V_3 \in \mathfrak{U}$. Put $V_1 = V_2 \cup V_3 \in \mathfrak{U}$ and $D_1 = \overline{V}_2 \cap V_3^c$, where $V_3^c = B \setminus V_3$. We have that D_1 is a bound and closed set in $\overline{V}$ and $0 \notin (F + N_K^f)(D_1)$. By (a_4) in Theorem 2.5, we get

$$d(F + N_K^f, V_1, 0) = d(F + N_K^f, V_1 \setminus D_1, 0) = d(F + N_K^f, V_3, 0).$$

Using a similar argument for $D_1 = \overline{V}_3 \cap V_2^c$, we get

$$d(F + N_K^f, V_1, 0) = d(F + N_K^f, V_1 \setminus D_1, 0) = d(F + N_K^f, V_2, 0)$$

and so

$$d(F + N_K^f, V_2, 0) = d(F + N_K^f, V_3, 0), \quad \forall\, V_2, V_3 \in \mathfrak{U}.$$

This completes the proof. ∎

Remark 2.6. Theorem 2.6 and Definition 2.7 generalize, extend and improve Theorem 2.3 and Definition 2.2 in [17] from finite dimensional spaces to Banach spaces, respectively.

Remark 2.7 It is well known that Hilbert spaces and R^n are reflexive, uniformly convex and uniformly smooth. Therefore, Lemmas 2.2 and 2.3, Definitions 2.6 and 2.7, Theorems 2.4, 2.5 and 2.6 still hold in Hilbert spaces and R^n.

3. Stability Analysis for Generalized f-Projection Operators

Lemma 3.1. [9]. *Let $f : B \to (-\infty, +\infty]$ be a convex and lower semicontinuous function. Then there exist $x^* \in B^*$ and $\beta \in R$ such that*

$$f(x) \geq \langle x^*, x\rangle + \beta, \ \forall\, x \in B.$$

Definition 3.1. [23]. Let B be a reflexive, smooth and strictly convex Banach space with the dual space B^*, and K be a nonempty closed convex subset of B. We say that $\Pi_K^f : B \to 2^K$ is a generalized f-projection operator if

$$\Pi_K^f x = \{u \in K : G_3(Jx, u) = \inf_{y\in K} G_3(Jx, y)\}, \quad \forall\, x \in B,$$

where

$$G_3(Jx, \xi) = \|x\|^2 - 2\langle Jx, \xi\rangle + \|\xi\|^2 + 2\rho f(\xi), \ \forall\, x \in B, \ \forall\, \xi \in K.$$

Remark 3.1.

(i) If $f(x) = 0$ for all $x \in K$, then the generalized f-projection operator Π_K^f reduces to the generalized projection operator Π_k defined by Alber [3] and Li [21], that is,

$$\Pi_K x = \{u \in K : G_2(Jx, u) = \inf_{y\in K} G_2(Jx, y)\}, \quad \forall\, x \in B,$$

where $G_2(Jx, \xi) = \|x\|^2 - 2\langle Jx, \xi\rangle + \|\xi\|^2$ for all $x \in B$ and $\xi \in K$.

(ii) If $f(x) = 0$ for all $x \in K$ and $B = \mathcal{H}$ is a Hilbert space, then the generalized f-projection operator Π_K^f is equivalent to the following metric projection operator

$$P_K(\varphi) = \{u \in K : \|u - \varphi\| = \inf_{y\in K} \|y - \varphi\|\}, \quad \forall\, \varphi \in \mathcal{H}.$$

(iii) Let B be a reflexive, smooth and strictly convex Banach space with the dual space B^*, and K be a nonempty closed convex subset of B. According to Definitions 2.5 and 3.1, it is easy to see that $\Pi_K^f = \pi_K^f J$ and $\pi_K^f = \Pi_K^f J^*$, where $J^* : B^* \to B$ is a normalized duality mapping in B^*.

Definition 3.2. Let O be a nonempty subset of B and $K' : O \to 2^B$ be a set-valued mapping. For any $\{w_n\} \subset O$ with $w_n \to w_0 \in O$, we say that the sequence of sets $K(w_n)$ Mosco-converges to $K(w)$ if the following two assumptions are satisfied:

(i) for every sequence $u_n \in K'(w_n)$ such that u_n weakly converges to u_0, then $u_0 \in K'(w_0)$;

(ii) for every $u_0 \in K'(w_0)$, there exists $u_n \in K'(w_n)$ (for n large enough) such that u_n strongly converges to u_0.

Let A and B be nonempty subsets in B. The Hausdorff metric between A and B is defined as follows:

$$H(A, B) = \max\{\sup_{a \in A} d(a, B); \sup_{b \in B} d(A, b)\},$$

where $d(a, B) = \inf_{b \in B} \parallel a - b \parallel$ and $d(A, b) = \inf_{a \in A} \parallel a - b \parallel$.

Proposition 3.1. *Assume that* (Λ, d) *is a metric space and* $K : \Lambda \to 2^B$ *is a set-valued mapping with nonempty closed convex values. For any sequence* $\{\lambda_n\}$ *with* $\lambda_n \to \lambda_0 \in \Lambda$ *as* $n \to +\infty$*,* $H(K(\lambda_n), K(\lambda_0)) \to 0$*. Then* $K(\lambda_n)$ *Mosco-converges to* $K(\lambda_0)$.

Proof. Since $H(K(\lambda_n), K(\lambda_0)) \to 0$ as $\lambda_n \to \lambda_0$, for any $\hat{\epsilon} > 0$, there exist $N > 0$, for any $n \geq N$, such that

$$K(\lambda_0) \subset K(\lambda_n) + \hat{\epsilon}B(0, 1) \text{ and } K(\lambda_n) \subset K(\lambda_0) + \hat{\epsilon}B(0, 1), \tag{3.1}$$

where $B(0, 1)$ is the open unit ball of B. For any $u_0 \in K(\lambda_0)$, (3.1) implies that there exists $u_n \in K(\lambda_n)$ (for n large enough) such that $u_n \to u_0$. On the other hand, for every sequence $u_n \in K(w_n)$ such that u_n converges weakly to $\hat{u}_0$, by (3.1), we know there exists $y_n \in K(\lambda_0)$ such that $\|u_n - y_n\| \leq \hat{\epsilon}$. Thus $y_n \rightharpoonup \hat{u}_0$. Since $K(\lambda_0)$ is a nonempty, closed and convex set and $y_n \in K(\lambda_0)$, we get $\hat{u}_0 \in K(\lambda_0)$. Therefore $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$. This completes the proof. ■

Proposition 3.1 shows that, for any sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, $K(\lambda_n)$ converges to $K(\lambda_0)$ in Hausdorff metric implies that $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$. However, the following example shows that the inverse is not true, in general.

Example 3.1. Let $K : R \to R^2$ be defined by $K(\lambda) = \{(x, y)|y = \lambda x, x \in R\}$. We claim that for any $\lambda_n \to \lambda_0$, $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$. In fact, for any $(\bar{x}, \bar{y}) \in K(\lambda_0)$, taking $x_n = \bar{x}$ and $y_n = \lambda_n\bar{x}$, we get that $(x_n, y_n) \in K(\lambda_n)$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$. On the other hand, for any $(\bar{x}_n, \bar{y}_n) \in K(\lambda_n)$ with $(\bar{x}_n, \bar{y}_n) \to (\bar{x}_0, \bar{y}_0)$, it is easy to see that $(\bar{x}_0, \bar{y}_0) \in K(\lambda_0)$ and so the claim is true. In addition, we know that K is not Hausdorff lower semicontinuous at 0 and so K is not Hausdorff continuous at 0. Indeed, fix $\epsilon_0 > 0$. For all $\delta_1 > 0$, we can choose $\bar{\lambda} \in B(0, \delta_1)$ and $(\bar{x}, 0) = (\frac{2\epsilon_0\sqrt{1+\bar{\lambda}^2}}{|\bar{\lambda}|}, 0) \in K(0)$. It is easy to get that $d((\bar{x}, 0), K(\bar{\lambda})) = \frac{|\bar{\lambda}||\bar{x}|}{\sqrt{1+\bar{\lambda}^2}} = 2\epsilon_0 > \epsilon_0$. Thus, for any sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ as $n \to +\infty$, $H(K(\lambda_n), K(0)) \not\to 0$.

Remark 3.2. If $O = X$ and $w_n \rightharpoonup w_0$, then Definition 3.1 becomes Definition 2 in [1].

Theorem 3.1. *Assume that (Λ, d) is a metric space, B is a reflexive and strictly convex Banach space with the dual space B^* and $K : \Lambda \to 2^B$ is a set-valued mapping with nonempty closed convex values. Suppose that $f : B \to (-\infty, +\infty)$ is convex and continuous, and for any sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$. Then for each sequence $\{(u_n^*, \lambda_n)\} \subset B^* \times \Lambda$ such that $u_n^* \to u_0^*$ and $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, $\pi^f_{K(\lambda_n)} u_n^*$ converges weakly to $\pi^f_{K(\lambda_0)} u_0^*$.*

Proof. For any fixing sequence $\{(u_n^*, \lambda_n)\} \subset B^* \times \Lambda$ with $u_n^* \to u_0^*$ and $\lambda_n \to \lambda_0$ as $n \to +\infty$, put $x_n = \pi^f_{K(\lambda_n)} u_n^*$. Since $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$, for any fixing $\bar{y} \in K(\lambda_0)$, there exists $y_n \in K(\lambda_n)$ such that $y_n \to \bar{y}$ as $n \to +\infty$. Since $f : B \to (-\infty, +\infty)$ is convex and continuous, applying Lemma 3.1, we know there exist $h \in B^*$ and $\alpha \in R$ such that

$$f(y) \geq \langle h, y \rangle + \alpha, \ \forall \ y \in B.$$

It follows that

$$\begin{aligned} G(u_n^*, x_n) &= \|u_n^*\|^2 - 2\langle u_n^*, x_n \rangle + \|x_n\|^2 + 2\rho f(x_n) \\ &\geq \|u_n^*\|^2 - 2\langle u_n^*, x_n \rangle + \|x_n\|^2 + 2\rho \langle h, x_n \rangle + 2\rho\alpha \\ &= \|u_n^*\|^2 - 2\langle u_n^* - \rho h, x_n \rangle + \|x_n\|^2 + 2\rho\alpha \\ &\geq \|u_n^*\|^2 - 2\|u_n^* - \rho h\| \|x_n\| + \|x_n\|^2 + 2\rho\alpha \\ &= (\|x_n\| - \|u_n^* - \rho h\|)^2 + \|u_n^*\|^2 - \|u_n^* - \rho h\|^2 + 2\rho\alpha. \end{aligned} \tag{3.2}$$

Since f is continuous, it is easy to see that $G(\cdot, \cdot)$ is continuous and so $G(u_n^*, y_n) \to G(u_0^*, \bar{y})$ as $n \to +\infty$. Hence the sequence $\{G(u_n^*, y_n)\}$ is bounded. Now $x_n = \pi^f_{K(\lambda_n)} u_n^*$ and (3.2) imply that

$$G(u_n^*, y_n) \geq G(u_n^*, x_n) \geq (\|x_n\| - \|u_n^* - \rho h\|)^2 + \|u_n^*\|^2 - \|u_n^* - \rho h\|^2 + 2\rho\alpha$$

and so the sequence $\{x_n\}$ is bounded. Since B is reflexive, there exists a subsequence, again denoted by $\{x_n\}$, such that it converges weakly to $x_0 \in B$. $x_n \in K(\lambda_n)$ and $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$ imply that $x_0 \in K(\lambda_0)$.

Next we prove that $x_0 = \pi^f_{K(\lambda_0)} u_0^*$. Since f is convex and continuous, the norm is weakly lower semicontinuous and $u_n^* \to u_0^*$ as $n \to +\infty$, we get

$$\begin{aligned} \liminf_{n \to +\infty} G(u_n^*, x_n) &= \liminf_{n \to +\infty} (\|u_n^*\|^2 - 2\langle u_n^*, x_n \rangle + \|x_n\|^2 + 2\rho f(x_n)) \\ &\geq \|u_0^*\|^2 - 2\langle u_0^*, x_0 \rangle + \|x_0\|^2 + 2\rho f(x_0) = G(u_0^*, x_0). \end{aligned} \tag{3.3}$$

On the other hand, since $x_n = \pi^f_{K(\lambda_n)} u^*_n$ and $G(\cdot, \cdot)$ is continuous, we get

$$\liminf_{n\to+\infty} G(u^*_n, x_n) \le \liminf_{n\to+\infty} G(u^*_n, y_n) = G(u^*_0, \bar{y}). \tag{3.4}$$

By $x_0 \in K(\lambda_0)$, (3.3) and (3.4), we know $G(u^*_0, x_0) = \min_{\bar{y}\in K(\lambda_0)} G(u^*_0, \bar{y})$. Definition 2.5 and Theorem 2.2 imply that $x_0 = \pi^f_{K(\lambda_0)} u^*_0$. According to our consideration above, each sequence $\{x_n\}$ has, in turn, a subsequence which converges weakly to the unique point $\pi^f_{K(\lambda_0)} u^*_0$. Therefore, the sequence $\{x_n\}$ converges weakly to $\pi^f_{K(\lambda_0)} u^*_0$. This completes the proof. ■

Corollary 3.1. *Let B be a smooth Banach space such that B^* has a Fréchet differentiable norm, and C be a nonempty closed convex subset of B. Let $C_1, C_2, C_3, \cdots$ be nonempty closed convex subsets of C. If C_n Mosco-converges to C_0, as $n \to \infty$ and C_0 is nonempty, then C_0 is a closed convex subset of C and, for each $x \in C$, $\Pi_{C_n} x$ converges weakly to $\Pi_{C_0} x$.*

Proof. It is easy to prove that C_0 is closed and convex if C_n is a closed convex subset of C. By taking $f = 0$, $\Lambda = R$, $K(0) = C_0$ and $K(\frac{1}{n}) = C_n$ $(n = 1, 2, \cdots)$ in Theorem 3.1, we get $\pi_{C_n} J(x)$ converges strongly to $\pi_{C_0} J(x)$. It follows from [4] that $\pi_{C_n} J(x) = \Pi_{C_n} x$ and $\pi_{C_0} J(x) = \Pi_{C_0} x$. Thus, $\Pi_{C_n} x$ converges weakly to $\Pi_{C_0} x$. The proof is complete. ■

Remark 3.3. We would like to mention that Corollary 3.1 is Theorem 3.1 of [14].

Theorem 3.2. *Assume that (Λ, d) is a metric space, B is a reflexive and strictly convex Banach space with the dual space B^*, B has the property (h) and $K : \Lambda \to 2^B$ is a set-valued mapping with nonempty closed convex values. Suppose that $f : B \to (-\infty, +\infty)$ is convex and continuous, and for any sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$. Then for each sequence $\{(u^*_n, \lambda_n)\} \subset B^* \times \Lambda$ such that $u^*_n \to u^*_0$ and $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, $\pi^f_{K(\lambda_n)} u^*_n$ converges strongly to $\pi^f_{K(\lambda_0)} u^*_0$.*

Proof. Fix $\{(u^*_n, \lambda_n)\} \subset B^* \times \Lambda$ with $u^*_n \to u^*_0$ and $\lambda_n \to \lambda_0$ as $n \to +\infty$, arbitrarily. We write $x_n = \pi^f_{K(\lambda_n)} u^*_n$ and $x_0 = \pi^f_{K(\lambda_0)} u^*_0$. By Theorem 3.1, we obtain $\{x_n\}$ converges weakly to x_0. Since B has the property (h), it is sufficient to prove that $\|x_n\| \to \|x_0\|$ as $n \to +\infty$. Since $x_0 \in K(\lambda_0)$ and $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$, there exists a sequence $\{y_n\}$ such that $y_n \in K(\lambda_n)$ and $y_n \to x_0$ as $n \to +\infty$. Since $x_n = \pi^f_{K(\lambda_n)} u^*_n$ and $G(\cdot, \cdot)$ is continuous, by (3.3) we get

$$\begin{aligned} G(u^*_0, x_0) &\le \liminf_{n\to+\infty} G(u^*_n, x_n) \\ &\le \limsup_{n\to+\infty} G(u^*_n, x_n) \le \limsup_{n\to+\infty} G(u^*_n, y_n) = G(u^*_0, x_0), \end{aligned} \tag{3.5}$$

which implies that $\lim_{n\to+\infty} G(u_n^*, x_n) = G(u_0^*, x_0)$. Since $f : B \to (-\infty, +\infty)$ is convex and continuous and f is subdifferentiable, there exists an element $\bar{x}^* \in B^*$ such that

$$f(x) - f(x_0) \geq \langle \bar{x}^*, x - x_0\rangle, \ \forall\, x \in \ B.$$

It follows that

$$f(x_n) \geq f(x_0) + \langle \bar{x}^*, x_n - x_0\rangle. \tag{3.6}$$

Now (3.6) implies that

$$\begin{aligned} &\|u_n^*\|^2 - 2\langle u_n^*, x_n\rangle + \|x_n\|^2 + 2\rho f(x_0) + 2\rho\langle \bar{x}^*, x_n - x_0\rangle \\ &\quad \leq \|u_n^*\|^2 - 2\langle u_n^*, x_n\rangle + \|x_n\|^2 + 2\rho f(x_n) = G(u_n^*, x_n). \end{aligned} \tag{3.7}$$

Since $x_n \rightharpoonup x_0$ and $u_n^* \to u_0^*$, we get

$$\begin{aligned} &\lim_{n\to+\infty}(-2\langle u_n^*, x_n\rangle + \|u_n^*\|^2 + 2\rho f(x_0) + 2\rho\langle \bar{x}^*, x_n - x_0\rangle) \\ &= -2\langle u_0^*, x_0\rangle + \|u_0^*\|^2 + 2\rho f(x_0). \end{aligned} \tag{3.8}$$

From (3.7), (3.8) and $\lim_{n\to+\infty} G(u_n^*, x_n) = G(u_0^*, x_0)$, we have

$$\begin{aligned} &\limsup_{n\to+\infty} \|x_n\|^2 - 2\langle u_0^*, x_0\rangle + \|u_0^*\|^2 + 2\rho f(x_0) \\ &= \limsup_{n\to+\infty}(\|x_n\|^2 - 2\langle u_n^*, x_n\rangle + \|u_n^*\|^2 + 2\rho f(x_0) + 2\rho\langle \bar{x}^*, x_n - x_0\rangle) \\ &\leq \limsup_{n\to+\infty}(\|x_n\|^2 - 2\langle u_n^*, x_n\rangle + \|u_n^*\|^2 + 2\rho f(x_n)) \\ &= \limsup_{n\to+\infty} G(u_n^*, x_n) \\ &= G(u_0^*, x_0) = \|x_0\|^2 - 2\langle u_0^*, x_0\rangle + \|u_0^*\|^2 + 2\rho f(x_0), \end{aligned} \tag{3.9}$$

which implies $\limsup_{n\to+\infty} \|x_n\|^2 \leq \|x_0\|^2$. Thus,

$$\limsup_{n\to+\infty} \|x_n\| \leq \|x_0\|. \tag{3.10}$$

On the other hand, the weakly lower semi-continuity of the norm implies that

$$\liminf_{n\to+\infty} \|x_n\| \geq \|x_0\|. \tag{3.11}$$

Now (3.10) and (3.11) show that $\lim_{n\to+\infty} \|x_n\| = \|x_0\|$. Using the property (h) of B, we obtain that $\{x_n\}$ converges strongly to x_0. This completes the proof. ■

Remark 3.4. By using similar arguments in Corollary 3.1, from Theorem 3.2, it is easy to get Theorem 4.1 of [14].

Corollary 3.2. *Assume that* (Λ, d) *is a metric space,* B *is a reflexive and locally uniformly convex Banach space with the dual space* B^**,* B *has a Fréchet differentiable norm and* $K : \Lambda \to 2^B$ *is a set-valued mapping with nonempty closed convex values. Suppose that* $f : B \to (-\infty, +\infty)$ *is convex and continuous, and for any sequence* $\{\lambda_n\}$ *with* $\lambda_n \to \lambda_0 \in \Lambda$ *as* $n \to +\infty$*,* $K(\lambda_n)$ *Mosco-converges to* $K(\lambda_0)$*. Then for each sequence* $\{(u_n, \lambda_n)\} \subset B \times \Lambda$ *such that* $u_n \to u_0$ *and* $\lambda_n \to \lambda_0 \in \Lambda$ *as* $n \to +\infty$*,* $\Pi^f_{K(\lambda_n)} u_n$ *converges strongly to* $\Pi^f_{K(\lambda_0)} u_0$.

Proof. Since $u_n \to u_0$ and J is continuous, we know that $J(u_n) \to J(u_0)$. It follows from Theorem 3.2 that $\pi^f_{K(\lambda_n)} J(u_n)$ converges strongly to $\pi^f_{K(\lambda_0)} J(u_0)$. Since $\pi^f_{K(\lambda_n)} J(u_n) = \Pi^f_{K(\lambda_n)}(u_n)$ and $\pi^f_{K(\lambda_0)} J(u_0) = \Pi^f_{K(\lambda_0)}(u_0)$, $\Pi^f_{K(\lambda_n)} u_n$ converges strongly to $\Pi^f_{K(\lambda_0)} u_0$. The proof is complete. ∎

The following example shows that the assumptions of Theorem 3.2 can be satisfied.

Example 3.2. Let U be a bounded domain of R^N with Lipschitz boundary and $1 < p < N$. Let $B = W_0^{1,p}(U)$, $W_1 = W_0^{1,p}(U) \cap W^{2,p}(U)$ and $\Lambda = W_1 \times W_1$. Set

$$\mathcal{U} = \{w = (\varphi, \psi) \in W_1 \times W_1 = \Lambda : \varphi \leq \psi \ a.e. \ \Omega\}.$$

For any $\lambda = (\varphi, \psi) \in \Lambda$, we define

$$K(\lambda) = \{v \in W_0^{1,p}(U), \ \varphi \leq v \leq \psi \ a.e. \ \Omega\}.$$

By Lemma 3.1 in [31] , we know for any sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, $K(\lambda_n)$ Mosco-converges to $K(\lambda_0)$. Let $f(x) = 3x$, for all $x \in B$. Therefore, all the conditions of Theorems 3.2 are satisfied. By Theorem 3.2, for each sequence $\{(u_n^*, \lambda_n)\} \subset B^* \times \Lambda$ with $u_n^* \to u_0^*$ and $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to +\infty$, we know that $\pi^f_{K(\lambda_n)} u_n^*$ converges strongly to $\pi^f_{K(\lambda_0)} u_0^*$.

4. An Application

As an application of Theorem 3.2, in this section, we will discuss the stability of solutions for a class of parametric generalized set-valued variational inequalities under some suitable conditions.

Through the rest of this paper, unless otherwise stated, we assume that B is a reflexive, strictly convex and smooth Banach space with the dual space B^* and B has the property (h). Let (M, d) and (Λ, d) be complete metric spaces, $K : \Lambda \to 2^B$ be a set-valued mapping and for each $\lambda \in \Lambda$, $f : K(\lambda) \subset B \to R$ be single-valued mapping. Let $F(\mu, x) = J(x) - T(\mu, x)$ for all $(\mu, x) \in M \times B$, where $J : B \to 2^{B^*}$ is the normalized duality mapping and $T : M \times B \to 2^{B^*}$ is a set-valued mapping.

In this section, we consider the following parametric generalized set-valued variational inequality: for each given $(\lambda, \mu) \in \Lambda \times M$, find $x \in K(\lambda)$ and $x^* \in F(\mu, x)$ such that

$$\langle x^*, y - x\rangle + f(y) - f(x) \geq 0, \quad \text{for all } y \in K(\lambda). \tag{4.1}$$

By Theorem 2.4, we know $x \in K(\lambda)$ and $x^* \in F(\mu, x)$ is a solution of the problem (4.1) if and only if $x \in K(\lambda)$ satisfies

$$0 \in F(\mu, x) + N^f_{K(\lambda)}(x), \tag{4.2}$$

where $N^f_{K(\lambda)}(x)$ is the value at x of the generalized f-normal cone operator associated with the set $K(\lambda)$ and (μ, λ) are parameters.

If $f = 0$, $M = R^k$, $\Lambda = R^m$, $B = R^n$ and $F = F_1$ is a set-valued mapping, then the problem (4.2) reduces to the following problem: find $x \in K(\lambda)$ satisfies

$$0 \in F_1(\mu, x) + N_{K(\lambda)}(x), \tag{4.3}$$

The problem (4.3) is called the parametric generalized variational inequalities, introduced and studied by Kien et al. [17].

If $f = 0$ and $F = F_2$ is a single valued mapping, then the problem (4.2) reduces to the following problem: find $x \in K(\lambda)$ satisfies

$$0 \in F_2(\mu, x) + \mathbb{N}_{K(\lambda)}(x), \tag{4.4}$$

The problem (4.4) is introduced and studied by Kien and Yao [18].

We denote by $S(\mu, \lambda)$, $S_1(\mu, \lambda)$ and $S_2(\mu, \lambda)$ the solution set of the problems (4.1), (4.3) and (4.4) corresponding to (μ, λ), respectively. Our main concern is now to investigate the behavior of $S(\mu, \lambda)$ when (μ, λ) varies around (μ_0, λ_0). This problem and special cases has been addressed by many authors in the last two decades (see, for example, [8, 15, 16, 17, 18, 20, 27, 28, 35, 36] and the references therein).

A set-valued mapping $K : \Lambda \to 2^B$ is said to have the Aubin property of order $\alpha > 0$ at a point (λ_0, x_0) if there exist positive constants k, ϵ_0 and β_0 such that

$$K(\lambda') \cap \overline{B(x_0, \epsilon_0)} \subset K(\lambda) + kd(\lambda', \lambda)^\alpha \overline{B(0,1)}, \quad \forall\ \lambda,\ \lambda' \in B(\lambda_0, \beta_0), \tag{4.5}$$

where $\overline{B(0,1)}$ is the closed unit ball of B,

$$\overline{B(x_0, \epsilon_0)} = x_0 + \epsilon_0 \overline{B(0,1)} = \{y : \|y - x_0\| \leq \epsilon_0\}$$

and $B(\lambda_0, \beta)$ is an open ball centered at λ_0 with radius β in the metric space Λ. If K satisfies property (4.5) for $\alpha = 1$ then K is said to be pseudo-Lipschitz around (λ_0, x_0), for example, see [5]. For each $\epsilon > 0$, we put

$$K_\epsilon = K(\lambda) \cap \overline{B(x_0, \epsilon)}. \tag{4.6}$$

Lemma 4.1. [15]. *Suppose that $K : \Lambda \to 2^B$ is set-valued mapping with nonempty closed and convex values and satisfies (4.5). Then for any ϵ in $(0, \epsilon_0]$ and any β with $0 < \beta < \min\{\beta_0, (\frac{\epsilon}{4k})^{\frac{1}{\alpha}}\}$, the multifunction K_ϵ defined by (4.6), is Hölder continuous with constant $5k$ on the ball $B(\lambda_0, \beta)$, that is,*

$$K_\epsilon(\lambda') \subset K_\epsilon(\lambda) + 5kd(\lambda', \lambda)^\alpha \overline{B(0,1)} \quad \forall\ \lambda,\ \lambda' \in B(\lambda_0, \beta). \tag{4.7}$$

Lemma 4.2. [10]. *Let X be an arbitrary metric space, A be a closed subset of X, L be a locally convex linear space and $\mathcal{G} : A \to L$ be a continuous mapping. Then there exists an extension $\mathcal{F} : X \to L$ of $\mathcal{G}$, i.e, $\mathcal{F}(a) = \mathcal{G}(a)$ for all $a \in A$. Furthermore, $\mathcal{F}(X) \subset \mathrm{con}(\mathcal{G}(\mathrm{A}))$, where $\mathrm{con}(\mathcal{G}(\mathrm{A}))$ is convex hull of $\mathcal{G}(A)$.*

Proposition 4.1. *Let $\Lambda_0 \subset \Lambda$ be a neighborhood of λ_0, $X_0 \subset B$ be a neighborhood of x_0 and $K : \Lambda_0 \to 2^B$ be a set-valued mapping with nonempty closed convex values and satisfies (4.5). Then there exist positive constants β_1, r_1 such that, for any*

$$\{\lambda_n\} \subset B(\lambda_0, \beta_1) = \{\bar{\lambda} \in \Lambda : \|\bar{\lambda} - \lambda_0\| < \beta_1\} \subset \Lambda_0$$

with $\lambda_n \to \lambda_0' \in B(\lambda_0, \beta_1)$ as $n \to +\infty$, $K(\lambda_n) \cap \overline{B(x_0, r_1)}$ Mosco-converges to $K(\lambda_0') \cap \overline{B(x_0, r_1)}$, where

$$\overline{B(x_0, r_1)} = \{y \in B : \|y - x_0\| \le r_1\} \subset X_0.$$

Proof. Choose $r_1 = \epsilon_0$ and $\beta_1 = \beta$ for some β with $0 < \beta < \min\{\beta_0, (\frac{\epsilon_0}{4k})^{\frac{1}{\alpha}}\}$ in Lemma 4.1. Putting $\lambda' = \lambda_0'$ and $\lambda = \lambda_n$ in (4.7), we know for every $u_0 \in K(\lambda_0') \cap \overline{B(x_0, r_1)}$, there exists $u_n \in K(\lambda_n) \cap \overline{B(x_0, r_1)}$ such that

$$\|u_n - u_0\| \le 5kd(\lambda_0', \lambda_n)^\alpha.$$

Since $\lambda_n \to \lambda_0'$, we know that $u_n \to u_0$ as $n \to +\infty$.

On the other hand, fix any sequence $\{\bar{u}_n\}$ with $\bar{u}_n \in K(\lambda_n) \cap \overline{B(x_0, r_1)}$ and $\bar{u}_n \rightharpoonup \bar{u}_0$ as $n \to +\infty$. Letting $\lambda' = \lambda_n$ and $\lambda = \lambda_0'$ in (4.7), it is easy to see that for each $\bar{u}_n \in K(\lambda_n) \cap \overline{B(x_0, r_1)}$, there exists $y_n \in K(\lambda_0') \cap \overline{B(x_0, r_1)}$ such that

$$\|\bar{u}_n - y_n\| \le 5kd(\lambda_0', \lambda_n)^\alpha.$$

Since $\lambda_n \to \lambda_0'$, $\|\bar{u}_n - y_n\| \le 5kd(\lambda_0', \lambda_n)^\alpha$ and $\bar{u}_n \rightharpoonup \bar{u}_0$, we know $y_n \rightharpoonup \bar{u}_0$ as $n \to +\infty$. Since $K(\lambda_0')$ and $\overline{B(x_0, r_1)}$ are closed convex set and $y_n \in K(\lambda_0') \cap \overline{B(x_0, r_1)}$, it follows that $\bar{u}_0 \in K(\lambda_0') \cap \overline{B(x_0, r_1)}$ and so $K(\lambda_n) \cap \overline{B(x_0, r_1)}$ Mosco-converges to $K(\lambda_0') \cap \overline{B(x_0, r_1)}$. This completes the proof. ■

Theorem 4.1. *Suppose $x_0 \in S(\mu_0, \lambda_0)$ is an isolated solution. Let X_0, Λ_0 and M_0 be neighborhoods of x_0, λ_0 and μ_0, respectively. Let $K : \Lambda_0 \to 2^B$ be a set-valued mapping with nonempty closed convex values and $f : B \to R$ be convex and continuous. Assume that the following conditions are satisfied:*

(i) *$T(\cdot,\cdot)$ is a lower semicontinuous mapping with nonempty closed convex values on $M_0 \times X_0$ and $T(\mu_0,\cdot)$ is a compact mapping with compact convex values on X_0;*

(ii) *there exist positive constants β_1 and r_1 such that, for any*

$$\{\lambda_n\} \subset B(\lambda_0, \beta_1) = \{\bar{\lambda} \in \Lambda : \|\bar{\lambda} - \lambda_0\| < \beta_1\} \subset \Lambda_0$$

with $\lambda_n \to \lambda_0' \in B(\lambda_0, \beta_1)$ as $n \to +\infty$, $K(\lambda_n) \cap \overline{B(x_0, r_1)}$ Mosco-converges to $K(\lambda_0') \cap \overline{B(x_0, r_1)}$;

(iii) *the mapping $\pi^f_{K(\cdot)}(T(\cdot,\cdot))$ has finite dimensional range and*

$$i(F(\mu_0,\cdot) + N^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}, x_0, 0) \neq 0,$$

where

$$\overline{B(x_0, r_1)} = \{y \in B : \|y - x_0\| \leq r_1\} \subset X_0.$$

Then there exist a neighborhood M_1 of μ_0, a neighborhood Λ_1 of λ_0 and an open bounded neighborhood Q_0 of x_0 such that the following assertions are fulfilled:

(a) *The solution map $\hat{S} : M_1 \times \Lambda_1 \to 2^B$ of the problem (4.1) defined by $\hat{S}(\mu, \lambda) = S(\mu, \lambda) \cap Q_0$ is nonempty for all $(\mu, \lambda) \in M_1 \times \Lambda_1$ and $\hat{S}(\mu_0, \lambda_0) = \{x_0\}$;*

(b) *$\hat{S}$ is lower semicontinuous at (μ_0, λ_0).*

Proof. By (i) and the continuous selection theorem due to Michael (see [11]), there exists a continuous mapping $T_1 : M_0 \times X_0 \to B^*$ such that $T_1(\mu, x) \in T(\mu, x)$, $\forall\ (\mu, x) \in M_0 \times X_0$. By Lemma 4.2, we can assume that T_1 is continuous on $B \times M$. We know $T_1(\mu_0,\cdot)$ is compact on X_0 when $T(\mu_0,\cdot)$ is compact on X_0. Put $g(\mu, x) = J(x) - T_1(\mu, x)$, then $g(\mu, x) \in J(x) - T(\mu, x) \subset F(\mu, x)$. For any $(\mu, \lambda, x) \in M_0 \times \Lambda_0 \times X_0$, consider the mapping

$$\Phi(\mu, \lambda, x) = x - \pi^f_{K(\lambda)\cap\overline{B(x_0,r_1)}}(J(x) - g(\mu, x)) = x - \pi^f_{K(\lambda)\cap\overline{B(x_0,r_1)}}(T_1(\mu, x)).$$

We claim that Φ is continuous on $M_0 \times B(\lambda_0, \beta) \times \overline{B(x_0, r_1)}$. Indeed, for any fixed point $(\tilde{\mu}_0, \tilde{\lambda}_0, \tilde{x}_0) \in M_0 \times B(\lambda_0, \beta) \times \overline{B(x_0, r_1)}$ and for any sequence $(\tilde{\mu}_n, \tilde{\lambda}_n, \tilde{x}_n)$ in $M_0 \times B(\lambda_0, \beta) \times \overline{B(x_0, r_1)}$ with $(\tilde{\mu}_n, \tilde{\lambda}_n, \tilde{x}_n) \to (\tilde{\mu}_0, \tilde{\lambda}_0, \tilde{x}_0)$ as $n \to +\infty$, from the condition (ii), it follows that $K(\tilde{\lambda}_n)\cap\overline{B(x_0, r_1)}$ Mosco-converges to $K(\tilde{\lambda}_0)\cap\overline{B(x_0, r_1)}$. The continuity of T_1 implies that $T_1(\tilde{\mu}_n, \tilde{x}_n) \to T_1(\tilde{\mu}_0, \tilde{x}_0)$ as $n \to +\infty$. By Theorem 3.2, we get

$$\pi^f_{K(\tilde{\lambda}_n)\cap\overline{B(x_0,r_1)}}(T_1(\tilde{\mu}_n, \tilde{x}_n)) \to \pi^f_{K(\tilde{\lambda}_0)\cap\overline{B(x_0,r_1)}}(T_1(\tilde{\mu}_0, \tilde{x}_0)), \quad \text{as } n \to +\infty.$$

Thus, Φ is continuous on $M_0 \times B(\lambda_0, \beta) \times \overline{B(x_0, r_1)}$.

Moreover, since $x_0 \in S(\mu_0, \lambda_0)$ is an isolated solution, there exists an open bounded neighborhood $Q_0 \subset X_0$ of x_0 such that x_0 is the unique solution in $\overline{Q_0}$ of the generalized equation

$$0 \in F(\mu_0, x) + N^f_{K(\lambda_0)}(x).$$

Since x_0 belongs to the interior of X_0, it is also the unique solution in $\overline{Q}_0$ of the generalized equation

$$0 \in F(\mu_0, x) + N^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}(x).$$

By (iii) and Theorem 2.6, we have

$$d(F(\mu_0,\cdot) + N^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}, Q_0, 0) = i(F(\mu_0,\cdot) + N^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}, x_0, 0) \neq 0.$$

(a_5) in Theorem 2.5 implies

$$d(\Phi(\mu_0,\lambda_0,\cdot), Q_0, 0) = d(F(\mu_0,\cdot) + N^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}, Q_0, 0) \neq 0.$$

Note that any solution of the equation $\Phi(\mu_0,\lambda_0,x) = 0$ is also a solution of

$$0 \in F(\mu_0, x) + N^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}(x).$$

Hence, x_0 is a unique solution of the equation $\Phi(\mu_0,\lambda_0,x) = 0$ in $\overline{Q_0}$. Since the mapping $\pi^f_{K(\cdot)}(T(\cdot,\cdot))$ has finite dimensional range, we can choose a finite dimensional subspace Y containing the range of $\pi^f_{K(\cdot)}(T(\cdot,\cdot))$ on $M_0 \times \Lambda_0 \times X_0$. Thus,

$$d(\Phi(\mu_0,\lambda_0,\cdot), Q_0, 0) = d(\Phi(\mu_0,\lambda_0,\cdot), Q_0 \cap Y, 0) \neq 0.$$

In the rest of the proof we will use some techniques from Robinson [28].

(a) Taking any $w \in \partial(Q_0 \cap Y)$, we have $\Phi(\mu_0,\lambda_0,w) \neq 0$. This implies that there exists a $\delta_w > 0$ such that

$$0 \notin B(\Phi(\mu_0,\lambda_0,w),\delta_w) := \{\tilde{y} \in B : \|\tilde{y} - \Phi(\mu_0,\lambda_0,w)\| < \delta_w\} = B_w.$$

By the continuity of Φ, there exists a neighborhood $U_w \subset M_0$ of μ_0, a neighborhood $\Lambda_w \subset \Lambda_0$ of λ_0 and a neighborhood Q_w of w such that $\Phi(\mu,\lambda,z) \in B_w$ for all $(\mu,\lambda,z) \in U_w \times \Lambda_w \times Q_w$. Since $\partial(Q_0 \cap Y)$ is a compact set, there exist some $w_1,\ w_2,\cdots,w_n$ such that $\partial(Q_0 \cap Y) \subset \cup_{i=1}^n Q_{w_i}$. Put $M_1 = \cap_{i=1}^n U_{w_i}$ and $\Lambda_1 = \cap_{i=1}^n \Lambda_{w_i}$. We shall show that M_1, Λ_1 and Q_0 satisfy the conclusion of the theorem. In fact, we fix any $(\mu,\lambda) \in M_1 \times \Lambda_1$. For any $x \in \overline{Q}_0$ and any $t \in [0,1]$, let

$$H(t,x) = x - (1-t)\pi^f_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}(T_1(\mu_0,x)) - t\pi^f_{K(\lambda)\cap\overline{B(x_0,r_1)}}(T_1(\mu,x)).$$

Then we have $H(0,x) = \Phi(\mu_0,\lambda_0,x)$ and $H(1,x) = \Phi(\mu,\lambda,x)$. Choose any $\bar{x} \in \partial(Q_0 \cap Y)$, then $\bar{x} \in Q_{w_i}$ for some $i \in \{1,\cdots,n\}$ and hence $(\mu,\lambda) \in U_{w_i} \times \Lambda_{w_i}$. By the convexity of B_{w_i}, $H(t,\bar{x}) \in B_{w_i}$. Hence $H(t,\bar{x}) \neq 0$. This means that $0 \notin H(t,\partial(Q_0 \cap Y))$. By (a_2) of Theorem 2.1, we have

$$d(\Phi(\mu_0,\lambda_0,\cdot),Q_0\cap Y,0)=d(\Phi(\mu,\lambda,\cdot),Q_0\cap Y,0)\neq 0.$$

By (a_1) of Theorem 2.1, there exists $\hat{x}=\hat{x}(\mu,\lambda)\in Q_0$ such that $\Phi(\mu,\lambda,\hat{x})=0$. Hence $S(\mu,\lambda)\cap Q_0\neq\emptyset$ for all $(\mu,\lambda)\in M_1\times\Lambda_1$. Moreover, by the assumptions of Q_0 we get $\hat{S}(\mu_0,\lambda_0)=S(\mu_0,\lambda_0)\cap Q_0=\{x_0\}$.

(b) Suppose that $V\subset B$ is an open set such that $\hat{S}(\mu_0,\lambda_0)\cap V\neq\emptyset$. Since $\hat{S}(\mu_0,\lambda_0)=\{x_0\}$, $x_0\in V$. By the boundedness of Q_0, the set $\tilde{G}:=V\cap Q_0$ is bounded and open. By excising $\overline{Q_0}\setminus\tilde{G}$, we we obtain from (a_3) of Theorem 2.1 that

$$d(\Phi(\mu_0,\lambda_0,\cdot),Q_0\cap Y,0)=d(\Phi(\mu_0,\lambda_0,\cdot),\tilde{G}\cap Y,0)\neq 0. \tag{4.8}$$

For any $w\in\partial(\tilde{G}\cap Y)$ we have $\Phi(\mu_0,\lambda_0,w)\neq 0$. Hence there exists a $\theta_w>0$ such that

$$0\notin B(\Phi(\mu_0,\lambda_0,w),\theta_w):=\{y'\in B:\|y'-\Phi(\mu_0,\lambda_0,w)\|<\theta_w\}=B'_w.$$

By the continuity of Φ, there exist a neighborhood $U'_w\subset M_0$ of μ_0, a neighborhood $\Lambda'_w\subset\Lambda_0$ of λ_0 and a neighborhood Q'_w of w such that $\Phi(\mu,\lambda,z)\in B'_w$ for all $(\mu,\lambda,z)\in U'_w\times\Lambda'_w\times Q'_w$. Since $\partial(\tilde{G}\cap Y)$ is a compact set, there are some $w_1,\ w_2,\cdots,w_n$ such that $\partial(\tilde{G}\cap Y)\subset\cup_{i=1}^n Q'_{w_i}$. Put $M_2=\cap_{i=1}^n U'_{w_i}$ and $\Lambda_2=\cap_{i=1}^n\Lambda'_{w_i}$. By the similar argument as the proof of the part (a) and using (4.8), for any fixed $(\tilde{\mu},\tilde{\lambda})\in M_2\times\Lambda_2$, we can show that

$$d(\Phi(\mu_0,\lambda_0,\cdot),\tilde{G}\cap Y,0)=d(\Phi(\tilde{\mu},\tilde{\lambda},\cdot),\tilde{G}\cap Y,0)\neq 0.$$

According to (a_1) of Theorem 2.1, there exists $\hat{x'}=\hat{x'}(\tilde{\mu},\tilde{\lambda})\in\tilde{G}$ such that $\Phi(\tilde{\mu},\tilde{\lambda},\hat{x'})=0$. This means that $S(\tilde{\mu},\tilde{\lambda})\cap\tilde{G}=\hat{S}(\tilde{\mu},\tilde{\lambda})\cap V\neq\emptyset$ for all $(\tilde{\mu},\tilde{\lambda})\in M_2\times\Lambda_2$. Hence $\hat{S}$ is lower semicontinuous at (μ_0,λ_0). The proof is complete. ■

Remark 4.1.

(a) If we replace the condition (ii) with the following condition $(ii)'$:

$(ii)'$ $K:\Lambda_0\to 2^B$ is a set-valued mapping with nonempty closed convex values and $K:\Lambda_0\to 2^B$ is pseudo-Lipschitz continuous at a point (λ_0,x_0), i.e., there exist positive constants k, ϵ_0 and β_0 such that

$$K(\lambda')\cap\overline{B(x_0,\epsilon_0)}\subset K(\lambda)+kd(\lambda',\lambda)\overline{B(0,1)}\quad\forall\ \lambda,\ \lambda'\in B(\lambda_0,\beta_0),$$

then by Proposition 4.1, we know the conclusions of Theorem 4.1 still hold. The condition $(ii)'$ has been applied to establish the continuity of the projection operators (see, for example, [16, 17, 18]). (b) The conditions and proof method of Theorem 4.1 are quite different from ones in Theorem 1.1 of [15].

Corollary 4.1. *Suppose $x_0\in S_1(\mu_0,\lambda_0)$ is an isolated solution. Let $X_0\subset R^n$, $\Lambda_0\subset R^k$ and $M_0\subset R^m$ be neighborhoods of x_0, λ_0 and μ_0, respectively. Let $K:\Lambda_0\to 2^{R^n}$ be a set-valued mapping with nonempty closed convex values. Assume that the following conditions are satisfied:*

(i) *$F_1(\cdot,\cdot)$ is a lower semicontinuous mapping with nonempty closed convex values on $M_0 \times X_0$ and $F_1(\mu_0,\cdot)$ is an upper semicontinuous mapping with compact convex values on X_0;*

(ii) *there exist positive constants β_1, r_1 such that for any*

$$\{\lambda_n\} \subset B(\lambda_0,\beta_1) = \{\bar{\lambda} \in R^k : \|\bar{\lambda} - \lambda_0\| < \beta_1\} \subset \Lambda_0$$

with $\lambda_n \to \lambda_0' \in B(\lambda_0,\beta_1)$ as $n \to +\infty$, $K(\lambda_n) \cap \overline{B(x_0,r_1)}$ Mosco-converges to $K(\lambda_0') \cap \overline{B(x_0,r_1)}$;

(iii) *$i(F_1(\mu_0,\cdot) + N_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}, x_0, 0) \neq 0$, where $\overline{B(x_0,r_1)} = \{y \in R^n : \|y - x_0\| \leq r_1\} \subset X_0$.*

Then there exist a neighborhood M_1 of μ_0, a neighborhood Λ_1 of λ_0 and an open bounded neighborhood Q_0 of x_0 such that the following assertions are fulfilled:

(a) *The solution map $\hat{S}_1 : M_1 \times \Lambda_1 \to 2^{R^n}$ of the problem (4.3) defined by $\hat{S}_1(\mu,\lambda) = S_1(\mu,\lambda) \cap Q_0$ is nonempty for all $(\mu,\lambda) \in M_1 \times \Lambda_1$ and $\hat{S}_1(\mu_0,\lambda_0) = \{x_0\}$;*

(b) *$\hat{S}_1$ is lower semicontinuous at (μ_0,λ_0).*

Proof. *For any bounded set $\tilde{A} \subset X_0$, it is easy to see that $\overline{F_1(\mu_0,\tilde{A})} \subset \overline{F_1(\mu_0,\overline{\tilde{A}})}$. Since $F_1(\mu_0,\cdot) : X_0 \to 2^{R^n}$ is an upper semicontinuous mapping with compact convex values, $\overline{F_1(\mu_0,\overline{\tilde{A}})}$ is a compact set and so $\overline{F_1(\mu_0,\tilde{A})}$ is compact. Hence $F_1(\mu_0,\cdot)$ is a compact mapping with compact convex values on X_0. Letting $f = 0$, $M = R^k$, $\Lambda = R^m$, $B = R^n$ and $T(\mu,x) = x - F_1(\mu,x)$ for any $(\mu,x) \in R^m \times R^n$ in Theorem 4.1, we know that Corollary 4.1 holds.* ■

Remark 4.2. Proposition 4.1 and Example 4.1 imply that the condition (ii) in Corollary 4.1 is weaker than the one (ii) in Theorem 3.2 of [17]. Therefore, Corollary 4.1 extends and improves Theorem 2.4 in [16] and Theorem 3.2 in [17].

Corollary 4.2. *Suppose that B and B^* are locally uniformly convex and $x_0 \in S_2(\mu_0,\lambda_0)$ is an isolated solution. Let X_0, Λ_0 and M_0 be neighborhoods of x_0, λ_0 and μ_0, respectively. Let $K : \Lambda_0 \to 2^B$ be a set-valued mapping with nonempty closed convex values. Assume that the following conditions are satisfied:*

(i) *$F_2 : M_0 \times X_0 \to B^*$ is a continuous mapping;*

(ii) *there exist positive constants β_1, r_1 such that for any*

$$\{\lambda_n\} \subset B(\lambda_0,\beta_1) = \{\bar{\lambda} \in \Lambda : \|\bar{\lambda} - \lambda_0\| < \beta_1\} \subset \Lambda_0$$

with $\lambda_n \to \lambda_0' \in B(\lambda_0,\beta_1)$ as $n \to +\infty$, $K(\lambda_n) \cap \overline{B(x_0,r_1)}$ Mosco-converges to $K(\lambda_0') \cap \overline{B(x_0,r_1)}$;

(iii) *the mapping* $\pi_{K(\cdot)}(J(\cdot) - F_2(\cdot,\cdot))$ *has finite dimensional range and*

$$\text{idex}(\Phi(\mu_0, \lambda_0, \cdot), x_0, 0) \neq 0,$$

where

$$\Phi(\mu_0, \lambda_0, x) = x - \pi_{K(\lambda_0)\cap\overline{B(x_0,r_1)}}(J(x) - F_2(\mu_0, x))$$

for all $x \in X_0$ *and* $\overline{B(x_0, r_1)} = \{y \in X_0 : \|y - x_0\| \leq r_1\}$.

Then there exist a neighborhood M_1 *of* μ_0*, a neighborhood* Λ_1 *of* λ_0 *and an open bounded neighborhood* Q_0 *of* x_0 *such that the following assertions are fulfilled:*

(a) *The solution map* $\hat{S}_2 : M_1 \times \Lambda_1 \to 2^B$ *of the problem (4.4) defined by* $\hat{S}_2(\mu, \lambda) = S_2(\mu, \lambda) \cap Q_0$ *is nonempty for all* $(\mu, \lambda) \in M_1 \times \Lambda_1$ *and* $\hat{S}_2(\mu_0, \lambda_0) = \{x_0\}$*;*

(b) $\hat{S}_2$ *is lower semicontinuous at* (μ_0, λ_0).

Proof. Since J and F_2 are continuous, by using the similar arguments to the proof of Theorem 4.1, we obtain that Φ is continuous on $M_0 \times B(\lambda_0, \beta) \times \overline{B(x_0, r_1)}$. Since $x_0 \in S_2(\mu_0, \lambda_0)$ is an isolated solution and $\text{index}(\Phi(\mu_0, \lambda_0, \cdot), x_0, 0) \neq 0$, there exists an open bounded neighborhood $Q_0 \subset X_0$ of x_0 such that x_0 is the unique solution of the equation $\Phi(\mu_0, \lambda_0, x) = 0$ in $\overline{Q_0}$. Moreover, since the mapping $\pi_{K(\cdot)}(J(\cdot) - F_2(\cdot,\cdot))$ has finite dimensional range, we can choose a finite dimensional subspace Y containing the range of $\pi_{K(\cdot)}(J(\cdot) - F_2(\cdot,\cdot))$ on $M_0 \times \Lambda_0 \times X_0$. Thus,

$$d(\Phi(\mu_0, \lambda_0, \cdot), Q_0, 0) = d(\Phi(\mu_0, \lambda_0, \cdot), Q_0 \cap Y, 0) \neq 0.$$

By using similar arguments to the proof of Theorem 4.1, we know that Corollary 4.2 holds. ∎

Remark 4.3. Corollary 4.2 is proved directly without the homeomorphic result between the solution sets, which is a different version of Theorem 3.1 in [18].

References

1. S. Adly, M. Bergounioux and M. A. Mansour, Optimal control of a quasi-variational obstacle problem, *J. Glob. Optim.*, **47** (2010), 421-435.

2. Y. Alber, The regularization method for variational inequalities with nonsmooth unbounded operator in Banach space, *Appl. Math. Lett.*, **6** (1993), 63-68.

3. Y. Alber, Generalized projection operators in Banach spaces: properties and applications. in: *Proceedings of the Israel Seminar Ariel, Israel, Funct. Differ. Equ.*, **1** (1994), 1-21.

4. Y. Alber, Metric and generalized projection operators in Banach spaces: properties and applications. in: *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. Kartsatos, ed., Dekker, New York, 1996, pp. 15-50.

5. J. P. Aubin, Lipschitz behavior of solutions to convex minimization problem, *Math. Oper. Res.*, **10** (1984), 87-111.

6. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, Applica-tions to Free-Boundary Problems, John Wiley and Sons, 1984.

7. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, **63** (1994), 123-145.

8. S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res.*, **13** (1988), 421-434.

9. K. Deimling, *Nonlinear Functional Analysis*, Berlin, Heidelberg: Springer-Verlag, 1985.

10. J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.*, **1** (1951), 353-367.

11. Dušan Repovš and Pavel V. Semenov, Ernest Michael and theory of continuous selections, *Topology Appl.*, **155** (2008), 755-763.

12. F. Facchinei and J. S. Pang, *Finite dimensional variational inequalities and complementarity problems*, Springer-Verlag, New York, 2003.

13. J. H. Fan, X. Liu and J. L. Li, Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces, *Nonlinear Anal. TMA*, **70** (2009), 3997-4007.

14. T. Ibaraki, Y. Kimura and W. Takahashi, Convergence theorem for generalized prejections and maximal monotone operators in Banach spaces, *Abstract Appl. Anal.*, **10** (2003), 621-629.

15. B. T. Kien, Lower semicontinuity of the solution map to a parametric generalized variational inequality in reflexive Banach spaces, *Set-Valued Anal.*, **16** (2008), 1089-1105.

16. B. T. Kien and M.-M. Wong, On the solution stability of variational inequalities, *J. Glob. Optim.*, **39** (2007), 101-111.

17. B. T. Kien, M.-M. Wong, N. C. Wong and J. C. Yao, Degree theory for generalized variational inequalities and applications, *European J. Oper. Res.*, **192** (2009), 730-736.

18. B. T. Kien and J. C. Yao, Localization of generalized normal maps and stability of variational inequalities in reflexive Banach spaces, *Set-Valued Anal.*, **16** (2008), 399-412.

19. I. V. Konnov and E. O. Volotskaya, Mixed variational inequalities and economic equilibrium problems, *J. Appl. Math.*, **2** (2002), 289-314.

20. A. B. Levy, Sensitivity of solutions to variational inequalities on Banach spaces, *SIAM J. Control Optim.*, **38** (1999), 50-60.

21. J. L. Li, The generalized projection operator on reflexive Banach spaces and its applications, *J. Math. Anal. Appl.*, **306** (2005), 55-71.

22. J. L. Li and J. Whitaker, Exceptional family of elements and solvability of variational inequalities for mappings defined only on closed convex cones in Banach spaces, *J. Math. Anal. Appl.*, **310** (2005), 254-261.

23. X. Li, N. J. Huang and D. O'Regan, Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications, *Comput. Math. Appl.*, **60** (2010), 1322-1331.

24. X. Li, N. J. Huang and Y. Z. Zou, On the stability of generalized f-projection operators with an application, *Acta Math. Sinica, Chinese Series*, **54** (2011), 811-822.

25. N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, England, 1978.

26. S. Mazur and S. J. Schauder, Über ein prinzip in der variationscechnung, *Proc. Int. Congress Math. Oslo.*, **65** (1936).

27. J. S. Pang, A degree-theoretic approach to parametric nonsmooth equations with multivalued perturbed solution sets, *Math. Program.*, **62** (1993), 359-383.

28. S. M. Robinson, Localized normal maps and the stability of variational conditions, *Set-Valued Anal*, **12** (2004), 259-274.

29. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

30. Z. B. Wang and N. J. Huang, Degree theory for a generalized set-valued variational inequality with an application in Banach spaces, *J. Glob. Optim.*, **49** (2011), 343-357.

31. Z. B. Wang, N. J. Huang and C. F. Wen, The existence results for optimal control problems governed by quasi-variational inequalities in reflexive Banach spaces, *Taiwanese J. Math.*, **16** (2012), 1221-1243.

32. K. Q. Wu and N. J. Huang, The generalized f-projection operator with an application, *Bull. Austral. Math. Soc.*, **73** (2006), 307-317.

33. K. Q. Wu and N. J. Huang, Properties of the generalized f-projection operator and its applications in Banach spaces, *Comput. Math. Appl.*, **54** (2007), 399-406.

34. K. Q. Wu and N. J. Huang, The generalized f-projection operator and set-valued variational inequalities in Banach spaces, *Nonlinear Anal.*, **71** (2009), 2481-2490.

35. N. D. Yen, Hölder continuity of solution to a parametric variational inequality, *Appl. Math. Optim.*, **31** (1995), 245-55.

36. N. D. Yen, Lipshitz continuity of solutions of variational inequalities with a parametric polyhedral constraint, *Math. Oper. Res.*, **20** (1995), 695-707.

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