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FIBONACCI NUMBERS MODULO CUBES OF PRIMES

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Abstract. Let p be an odd prime. It is well known that $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, where $\{F_n\}_{n\geqslant 0}$ is the Fibonacci sequence and $(-)$ is the Jacobi symbol. In this paper we show that if $p \neq 5$ then we may determine $F_{p-(\frac{p}{5})} \bmod p^3$ in the following way:

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{p}{5}\right)\left(1+\frac{F_{p-(\frac{p}{5})}}{2}\right) \pmod{p^3}.$$

We also use Lucas quotients to determine $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/m^k$ modulo p^2 for any integer $m \not\equiv 0 \pmod p$; in particular, we obtain

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p}\right) \pmod{p^2}.$$

In addition, we pose three conjectures for further research.

1. Introduction

The well known Fibonacci sequence $\{F_n\}_{n\geqslant 0}$, defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots),$$

plays an important role in many fields of mathematics. This sequence has nice number-theoretic properties; for example, E. Lucas showed that $(F_m, F_n) = F_{(m,n)}$ for any $m, n \in \mathbb{N} = \{0, 1, \ldots\}$, where (m, n) denotes the greatest common divisor of m and n.

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Let $p \neq 2, 5$ be a prime. It is known that $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, where $(-)$ denotes the Jacobi symbol. In 1992 Z. H. Sun and Z. W. Sun [15] proved that if $p^2 \nmid F_{p-(\frac{p}{5})}$ then the Fermat equation $x^p + y^p = z^p$ has no integral solutions with $p \nmid xyz$. When $F_{p-(\frac{p}{5})} \equiv 0 \pmod{p^2}$, p is called a Wall-Sun-Sun prime (cf. [2] and [3, p. 32]). It is conjectured that there should be infinitely many (but rare) Wall-Sun-Sun primes though none of them has been found. There are some congruences for the Fibonacci quotient $F_{p-(\frac{p}{5})}/p$ modulo p (cf. [22, 15] and [21]); for example, in 1982 H. C. Williams [22] proved that

$$\frac{F_{p-(\frac{p}{5})}}{p} \equiv \frac{2}{5}\sum_{k=1}^{\lfloor \frac{4}{5}p \rfloor} \frac{(-1)^k}{k} \pmod p.$$

Quite recently H. Pan and Z. W. Sun [10] proved that for any $a \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ we have

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right)\left(1 - 2F_{p^a-(\frac{p^a}{5})}\right) \pmod{p^3},$$

which was a conjecture in [21].

Now we give the first theorem of this paper.

Theorem 1.1. *Let p be an odd prime and let a be a positive integer. If $p \neq 5$, then*

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{p^a}{5}\right)\left(1 + \frac{F_{p^a-(\frac{p^a}{5})}}{2}\right) \pmod{p^3}. \tag{1.1}$$

If $p \neq 3$, then

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-32)^k} \equiv \left(\frac{2}{p^a}\right)\left(1 + \frac{2^{p^a-1}-1}{6} - \frac{(2^{p^a-1}-1)^2}{8}\right) \pmod{p^3}. \tag{1.2}$$

Let p be an odd prime and let $a \in \mathbb{Z}^+$. For $k = 0, 1, \ldots, (p^a-1)/2$, clearly

$$\frac{\binom{(p^a-1)/2}{k}}{\binom{-1/2}{k}} = \prod_{0 \leqslant j < k}\left(1 - \frac{p^a}{2j+1}\right) \equiv 1 \pmod p$$

and hence

$$\binom{(p^a-1)/2}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod p. \tag{1.3}$$

Thus, for any integer $m \not\equiv 0$ (mod p) we have

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p^a}\right) \pmod p, \tag{1.4}$$

since

$$\sum_{k=0}^{(p^a-1)/2}\binom{(p^a-1)/2}{k}\left(-\frac{4}{m}\right)^k=\left(1-\frac{4}{m}\right)^{(p^a-1)/2}$$

and

$$\binom{2k}{k}=\binom{p^a+(2k-p^a)}{0p^a+k}\equiv\binom{2k-p^a}{k}=0\pmod p$$

for each $k=(p^a+1)/2,\ldots,p^a-1$ by Lucas' theorem (cf. [11, p. 44]). Recently the author [17] determined $\sum_{k=0}^{p^a-1}\binom{2k}{k}/m^k \bmod p^2$ in terms of Lucas sequences. See also [12, 7] and [18] for related results on p-adic valuations.

Let $A,B\in\mathbb{Z}$. The Lucas sequences $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ are defined by

$$u_0=0,\ u_1=1,\ \text{and } u_{n+1}=Au_n-Bu_{n-1}\ (n=1,2,3,\ldots),$$

and

$$v_0=2,\ v_1=A,\ \text{and } v_{n+1}=Av_n-Bv_{n-1}\ (n=1,2,3,\ldots).$$

The sequence $\{v_n\}_{n\geqslant 0}$ is called the companion of $\{u_n\}_{n\geqslant 0}$. (Note that $F_n=u_n(1,-1)$, and those $L_n=v_n(1,-1)$ are called Lucas numbers.) It is known that for any prime p not dividing $2B$ we have

$$u_p\equiv\left(\frac{\Delta}{p}\right)\pmod p\quad\text{and}\quad u_{p-(\frac{\Delta}{p})}\equiv 0\pmod p$$

where $\Delta=A^2-4B$ (see, e.g., [17, Lemma 2.3]); the integer $u_{p-(\frac{\Delta}{p})}/p$ is called a *Lucas quotient*. The reader may consult [16] for connections between Lucas quotients and quadratic fields.

Our second theorem is as follows.

Theorem 1.2. *Let p be an odd prime and let $a\in\mathbb{Z}^+$. Let m be any integer not divisible by p. Then*

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m(m-4)}{p^a}\right)+\left(\frac{-m}{p}\right)\left(\frac{m(m-4)}{p^{a-1}}\right)\bar{m}u_{p-(\frac{4-m}{p})}(4,m)\pmod{p^2},\tag{1.5}$$

where

$$\bar{m}=\begin{cases}1 & \text{if } m\equiv 4\pmod p,\\ 2 & \text{if } (\frac{4-m}{p})=1,\\ 2/m & \text{if } (\frac{4-m}{p})=-1.\end{cases}$$

We also have

$$\sum_{k=0}^{(p^a-1)/2}\frac{C_k}{m^k}\equiv\frac{4-m}{2}\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{m^k}+\frac{m}{2}-2p\,\delta_{a,1}\left(\frac{-m}{p}\right)\pmod{p^2},\tag{1.6}$$

where C_k denotes the Catalan number $\frac{1}{k+1}\binom{2k}{k}=\binom{2k}{k}-\binom{2k}{k+1}$, and the Kronecker symbol $\delta_{s,t}$ takes 1 or 0 according as $s=t$ or not.

Remark 1.1. For any $m\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{N}$, the sum $\sum_{k=0}^{n}k\binom{2k}{k}/m^k$ is closely related to $\sum_{k=0}^{n}\binom{2k}{k}/m^k$ via the identity

$$\sum_{k=0}^{n}\left(1-\frac{m-4}{2}k\right)\frac{\binom{2k}{k}}{m^k}=(2n+1)\frac{\binom{2n}{n}}{m^n}$$

which can be easily proved by induction.

Now we present two consequences of Theorem 1.2.

Corollary 1.1. *Let p be an odd prime and let $a\in\mathbb{Z}^+$. Then*

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p^a}\right)\pmod{p^2}\tag{1.7}$$

and

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p^a}\right)\pmod{p^2}.\tag{1.8}$$

Corollary 1.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(2k-1)^2 16^k}\equiv\left(\frac{-1}{p}\right)\frac{3(\frac{p}{3})+1}{4}\pmod{p^2},\tag{1.9}$$

that is,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(2k-1)^2 16^k}\equiv\begin{cases}1\pmod{p^2} & \text{if } p\equiv 1\pmod{12},\\ -1/2\pmod{p^2} & \text{if } p\equiv 5\pmod{12},\\ -1\pmod{p^2} & \text{if } p\equiv 7\pmod{12},\\ 1/2\pmod{p^2} & \text{if } p\equiv 11\pmod{12}.\end{cases}\tag{1.10}$$

We will show Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Section 4 is devoted to our proofs of Corollaries 1.1–1.2.

To conclude this section we pose three conjectures.

Conjecture 1.1. For any $n \in \mathbb{N}$ we have

$$\frac{1}{(2n+1)^2\binom{2n}{n}}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{16^k} \equiv \begin{cases} 1 \pmod 9 & \text{if } 3 \mid n, \\ 4 \pmod 9 & \text{if } 3 \nmid n. \end{cases}$$

Also,

$$\frac{1}{3^{2a}}\sum_{k=0}^{(3^a-1)/2}\frac{\binom{2k}{k}}{16^k} \equiv (-1)^a 10 \pmod{27}$$

for every $a = 1, 2, 3, \ldots$.

Let $p > 3$ be a prime. In 2007 A. Adamchuk [1] conjectured that if $p \equiv 1 \pmod 3$ then

$$\sum_{k=1}^{\lfloor \frac{2}{3}p \rfloor}\binom{2k}{k} \equiv 0 \pmod{p^2}.$$

Motivated by this and Theorems 1.1 and 1.2, we pose the following conjecture based on the author's computation via the software `Mathematica`.

Conjecture 1.2. Let p be an odd prime and let $a \in \mathbb{Z}^+$.
(i) If $p \equiv 1 \pmod 3$ or $a > 1$, then

$$\sum_{k=0}^{\lfloor \frac{5}{6}p^a \rfloor}\frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

(ii) Suppose $p \neq 5$. If $p^a \equiv 1, 2 \pmod 5$ or $p \equiv 2 \pmod 5$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor \frac{4}{5}p^a \rfloor}(-1)^k\binom{2k}{k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod 5$ or $p \equiv 3 \pmod 5$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor \frac{3}{5}p^a \rfloor}(-1)^k\binom{2k}{k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

(iii) If $p^a \equiv 1, 2 \pmod 5$ or $p \equiv 2 \pmod 5$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor \frac{7}{10}p^a \rfloor}\frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

If $p^a \equiv 1, 3 \pmod 5$ or $p \equiv 3 \pmod 5$ or $a > 2$, then

$$\sum_{k=0}^{\lfloor \frac{9}{10}p^a \rfloor} \frac{\binom{2k}{k}}{(-16)^k} \equiv \left(\frac{5}{p^a}\right) \pmod{p^2}.$$

Conjecture 1.3. Let $p \neq 2, 5$ be a prime and set $q := F_{p-(\frac{p}{5})}/p$. Then

$$p\sum_{k=1}^{p-1} \frac{F_{2k}}{k^2\binom{2k}{k}} \equiv -\left(\frac{p}{5}\right)\left(\frac{3}{2}q + \frac{5}{4}p\,q^2\right) \pmod{p^2}$$

and

$$p\sum_{k=1}^{p-1} \frac{L_{2k}}{k^2\binom{2k}{k}} \equiv -\frac{5}{2}q - \frac{15}{4}p\,q^2 \pmod{p^2}.$$

Remark 1.2. It is interesting to compare Conjecture 1.3 with the two identities

$$\sum_{k=1}^{\infty} \frac{F_{2k}}{k^2\binom{2k}{k}} = \frac{4\pi^2}{25\sqrt{5}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{L_{2k}}{k^2\binom{2k}{k}} = \frac{\pi^2}{5}$$

obtained by putting $x = (\sqrt{5} \pm 1)/2$ in the known formula

$$2\arcsin^2\frac{x}{2} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k^2\binom{2k}{k}} \quad (|x| < 2).$$

2. Proof of Theorem 1.1

Lemma 2.1. *Let p be an odd prime and let $k \in \{0, \ldots, (p^a - 1)/2\}$ with $a \in \mathbb{Z}^+$. Then*

$$\begin{aligned} &\binom{(p^a-1)/2+k}{2k} - \frac{\binom{2k}{k}}{(-16)^k} \\ \equiv&(-1)^{k-1}\left(\frac{-1}{p^a}\right)\binom{p^a-1-2k}{(p^a-1)/2-k}\sum_{0<j\leqslant k}\frac{p^{2a}}{(2j-1)^2} \pmod{p^3}. \end{aligned} \tag{2.1}$$

Proof. Clearly (2.1) holds for $k = 0$. Below we assume $1 \leqslant k \leqslant (p^a - 1)/2$. Note that

$$\begin{aligned} \binom{(p^a-1)/2+k}{2k} &= \frac{\prod_{j=1}^{k}(p^{2a} - (2j-1)^2)}{4^k(2k)!} \\ &= \frac{\prod_{j=1}^{k}(-(2j-1)^2)}{4^k(2k)!}\prod_{j=1}^{k}\left(1 - \frac{p^{2a}}{(2j-1)^2}\right) \\ &\equiv \frac{\binom{2k}{k}}{(-16)^k}\left(1 - \sum_{j=1}^{k}\frac{p^{2a}}{(2j-1)^2}\right) \pmod{p^4} \end{aligned}$$

(which was observed by Z. H. Sun [14, Lemma 2.2] in the case $a = 1$). Thus, in view of (1.3) we have

$$\begin{aligned}\frac{\binom{2k}{k}}{(-16)^k} &\equiv \binom{(p^a-1)/2}{k}4^{-k} = \binom{(p^a-1)/2}{(p^a-1)/2-k}4^{-k} \\ &\equiv \binom{p^a-1-2k}{(p^a-1)/2-k}(-4)^{-((p^a-1)/2-k)}4^{-k} \\ &\equiv \left(\frac{-1}{p^a}\right)(-1)^k\binom{p^a-1-2k}{(p^a-1)/2-k} \pmod p.\end{aligned}$$

So (2.1) holds.

Lemma 2.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \neq 3$, then*

$$1 + \frac{2^{p^a-1}-1}{6} + \frac{(2^{p^a-1}-1)^2}{24} \equiv \left(\frac{2}{p^a}\right)\frac{2^{p^a}+1}{3 \times 2^{(p^a-1)/2}} \pmod{p^3}. \tag{2.2}$$

When $p \neq 5$, we have

$$\frac{L_{p^a}-1}{5} - \left(\frac{p^a}{5}\right)F_{p^a} + 1 \equiv -\frac{1}{2}F^2_{p^a-(\frac{p^a}{5})} \pmod{p^4}. \tag{2.3}$$

Proof. Note that

$$2^{(p^a-1)/2} = \left(2^{\frac{p-1}{2}}\right)^{\sum_{k=0}^{a-1}p^k} \equiv \left(\frac{2}{p}\right)^a = \left(\frac{2}{p^a}\right) \pmod p$$

and

$$\begin{aligned}\frac{2^{p^a-1}-1}{p} &= \frac{2^{(p^a-1)/2}-(\frac{2}{p^a})}{p}\left(2^{(p^a-1)/2}+\left(\frac{2}{p^a}\right)\right) \\ &\equiv 2\left(\frac{2}{p^a}\right)\frac{2^{(p^a-1)/2}-(\frac{2}{p^a})}{p} \pmod p.\end{aligned}$$

Thus

$$\begin{aligned}(2^{p^a-1}-1)^2 &\equiv 4\left(2^{(p^a-1)/2}-\left(\frac{2}{p^a}\right)\right)^2 \\ &= 4(2^{p^a-1}-1)+8-8\left(\frac{2}{p^a}\right)2^{(p^a-1)/2} \pmod{p^3}.\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{2^{(p^a-1)/2}}{2}\left(2^{p^a-1}-1\right)+\frac{1}{8}\left(\frac{2}{p^a}\right)\left(2^{p^a-1}-1\right)^2\\
&\equiv \frac{2^{(p^a-1)/2}}{8}\left(\left(2^{p^a-1}-1\right)^2-8+8\left(\frac{2}{p^a}\right)2^{(p^a-1)/2}\right)\\
&\quad+\frac{1}{8}\left(\frac{2}{p^a}\right)\left(2^{p^a-1}-1\right)^2\\
&\equiv \frac{1}{4}\left(\frac{2}{p^a}\right)\left(2^{p^a-1}-1\right)^2-2^{(p^a-1)/2}+\left(\frac{2}{p^a}\right)2^{p^a-1}\\
&\equiv \left(\frac{2}{p^a}\right)\left(2^{p^a}+1\right)-3\times 2^{(p^a-1)/2} \pmod{p^3},
\end{aligned}$$

which is equivalent to (2.2) times $3\times 2^{(p^a-1)/2}$. So (2.2) is valid if $p>3$.

Now assume that $p\neq 5$. As $L_{2n}=5F_n^2+2(-1)^n=L_n^2-2(-1)^n$ for all $n\in\mathbb{N}$, by [15, Corollary 1] we have $L_{p-(\frac{5}{p})}\equiv 2(\frac{p}{5}) \pmod{p^2}$. Thus, in view of [17, Lemma 2.3],

$$L_{p^a-(\frac{p^a}{5})}\equiv(-1)^{((\frac{5}{p})-(\frac{5}{p^a}))/2}L_{p-(\frac{5}{p})}\equiv 2\left(\frac{p^a}{5}\right) \pmod{p^2}.$$

Write $L_{p^a-(\frac{p^a}{5})}=2(\frac{p^a}{5})+p^2Q$ with $Q\in\mathbb{Z}$. Then

$$\begin{aligned}
5F^2_{p^a-(\frac{p^a}{5})}&=L^2_{p^a-(\frac{p^a}{5})}-4(-1)^{p^a-(\frac{p^a}{5})}\\
&=-4+\left(2\left(\frac{p^a}{5}\right)+p^2Q\right)^2\equiv 4p^2\left(\frac{p^a}{5}\right)Q \pmod{p^4}.
\end{aligned}$$

Note that

$$L_{p^a}=F_{p^a}+2F_{p^a-1}=2F_{p^a+1}-F_{p^a}=2F_{p^a-(\frac{p^a}{5})}+\left(\frac{p^a}{5}\right)F_{p^a}$$

and

$$2L_{p^a}=5F_{p^a-1}+L_{p^a-1}=5F_{p^a+1}-L_{p^a+1}=5F_{p^a-(\frac{p^a}{5})}+\left(\frac{p^a}{5}\right)L_{p^a-(\frac{p^a}{5})}.$$

Therefore

$$\begin{aligned}
&\frac{L_{p^a}-1}{5}-\left(\frac{p^a}{5}\right)F_{p^a}+1\\
&=2F_{p^a-(\frac{p^a}{5})}-\frac{4}{5}(L_{p^a}-1)\\
&=2F_{p^a-(\frac{p^a}{5})}-\frac{4}{5}\left(\frac{5}{2}F_{p^a-(\frac{p^a}{5})}+\frac{1}{2}\left(\frac{p^a}{5}\right)L_{p^a-(\frac{p^a}{5})}-1\right)\\
&=\frac{4}{5}-\frac{2}{5}\left(\frac{p^a}{5}\right)\left(2\left(\frac{p^a}{5}\right)+p^2Q\right)\equiv-\frac{1}{2}F^2_{p^a-(\frac{p^a}{5})} \pmod{p^4}.
\end{aligned}$$

This proves (2.3). ■

The following Lemma was posed as [20, Conjecture 1.1].

Lemma 2.3. *Let p be an odd prime and let $H_k^{(2)} = \sum_{0<j\leqslant k} 1/j^2$ for $k = 0, 1, 2, \ldots$. If $p \neq 5$, then*

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)} \equiv \left(\frac{p}{5}\right)\frac{5}{2}q^2 - \delta_{p,3} \pmod p, \tag{2.4}$$

where q denotes the Fibonacci quotient $F_{p-(\frac{p}{5})}/p$. If $p > 3$, then

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv \frac{2}{3}q_p(2)^2 \pmod p, \tag{2.5}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Proof. It is easy to verify (2.4) for $p = 3$. Below we assume $p > 3$.

The desired congruences essentially follow from [8, (37)]. Here we provide the details. Putting $t = -1, -1/2$ in [8, (37)] we get

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)} \equiv -2\sum_{k=1}^{p-1}\frac{u_k(3,1)}{k^2} \pmod p \tag{2.6}$$

and

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv -2\sum_{k=1}^{p-1}\frac{u_k(5/2,1)}{k^2} \pmod p. \tag{2.7}$$

Note that $u_k(3,1) = F_{2k}$ and $u_k(5/2,1) = 2(2^k - 2^{-k})/3$ for all $k = 0, 1, 2, \ldots$.

Since

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -q_p(2)^2 \pmod p \quad \text{and} \quad \sum_{k=1}^{p-1}\frac{1}{k^2 2^k} \equiv -\frac{q_p(2)^2}{2} \pmod p$$

by [5] and [13, Theorem 4.1(iv)] respectively, (2.7) implies that

$$\sum_{k=0}^{p-1}(-2)^k\binom{2k}{k}H_k^{(2)} \equiv -\frac{4}{3}\sum_{k=1}^{p-1}\left(\frac{2^k}{k^2} - \frac{1}{k^2 2^k}\right) \equiv \frac{2}{3}q_p(2)^2 \pmod p.$$

Now we work with $p > 5$. Recall that for any $n \in \mathbb{N}$ we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where α and β are the two roots of the equation $x^2-x-1=0$. By [10, (3.2) and (3.7)],

$$2\beta^{2p}\sum_{k=1}^{p-1}\frac{\alpha^{2k}}{k^2}\equiv -2\sum_{k=1}^{p-1}\frac{\alpha^k}{k^2}-\left(\frac{L_p-1}{p}\right)^2 \equiv \left(\frac{2\alpha^p-1}{5}-1\right)\left(\frac{L_p-1}{p}\right)^2 \pmod p.$$

Since $\alpha\beta=-1$ and $\alpha^{2p}=(\alpha+1)^p\equiv\alpha^p+1 \pmod p$, we have

$$\sum_{k=1}^{p-1}\frac{\alpha^{2k}}{k^2}\equiv\frac{(\alpha^p+1)(\alpha^p-3)}{5}\left(\frac{L_p-1}{p}\right)^2\equiv-\frac{\alpha^p+2}{5}\left(\frac{L_p-1}{p}\right)^2 \pmod p.$$

Hence

$$\begin{aligned}5\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2}&=(\alpha-\beta)^2\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2}=(\alpha-\beta)\sum_{k=1}^{p-1}\frac{\alpha^{2k}-\beta^{2k}}{k^2}\\&\equiv(\alpha-\beta)\frac{\beta^p-\alpha^p}{5}\left(\frac{L_p-1}{p}\right)^2\equiv-\frac{(\alpha-\beta)^{p+1}}{5}\left(\frac{L_p-1}{p}\right)^2\\&=-5^{(p-1)/2}\left(\frac{L_p-1}{p}\right)^2\equiv-\left(\frac{5}{p}\right)\frac{25}{4}q^2 \pmod p\end{aligned}$$

since $2(L_p-1)\equiv 5F_{p-(\frac{p}{5})} \pmod{p^2}$ by [21, p. 139]. Combining this with (2.6) we obtain

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}H_k^{(2)}\equiv 2\times\left(\frac{5}{p}\right)\frac{5}{4}q^2=\left(\frac{p}{5}\right)\frac{5}{2}q^2 \pmod p.$$

The proof of Lemma 2.3 is now complete. ■

Proof of Theorem 1.1. Let us first recall the following two identities:

$$F_{n+1}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\quad\text{and}\quad\sum_{k=0}^{n}\frac{\binom{n+k}{2k}}{2^k}=\frac{2^{2n+1}+1}{3\times 2^n}.$$

Thus we have

$$F_{p^a}=\sum_{k=0}^{(p^a-1)/2}\binom{p^a-1-k}{p^a-1-2k}=\sum_{j=0}^{(p^a-1)/2}\binom{(p^a-1)/2+j}{2j}$$

and

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{(p^a-1)/2+k}{2k}}{2^k}=\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}.$$

Therefore, with the help of (2.1),

$$\begin{aligned}
&\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\\
\equiv&\sum_{k=0}^{(p^a-1)/2}\binom{p^a-1-2k}{(p^a-1)/2-k}(-1)^{(p^a-1)/2-k}\sum_{0<j\leqslant k}\frac{p^{2a}}{(2j-1)^2}\\
=&\sum_{k=0}^{(p^a-1)/2}\binom{2k}{k}(-1)^k\sum_{0<j\leqslant (p^a-1)/2-k}\frac{p^{2a}}{(2j-1)^2}\pmod{p^3}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\\
\equiv&\sum_{k=0}^{(p^a-1)/2}\binom{2k}{k}\frac{(-1)^k}{2^{(p^a-1)/2-k}}\sum_{0<j\leqslant (p^a-1)/2-k}\frac{p^{2a}}{(2j-1)^2}\pmod{p^3}.
\end{aligned}$$

For $k=0,\ldots,(p^a-1)/2$, clearly

$$\begin{aligned}
\sum_{j=1}^{(p^a-1)/2-k}\frac{p^{2a}}{(2j-1)^2}&=\sum_{j=k+1}^{(p^a-1)/2}\frac{p^{2a}}{(2((p^a-1)/2-j+1)-1)^2}\\
&\equiv\sum_{j=1}^{(p^a-1)/2}\frac{p^{2a}}{4j^2}-\sum_{0<j\leqslant k}\frac{p^{2a}}{4j^2}\\
&\equiv\sum_{i=1}^{(p-1)/2}\frac{p^{2a}}{4(p^{a-1}i)^2}-\sum_{0<i\leqslant\lfloor k/p^{a-1}\rfloor}\frac{p^{2a}}{4(p^{a-1}i)^2}\pmod{p^3}.
\end{aligned}$$

Since

$$2\sum_{i=1}^{(p-1)/2}\frac{1}{i^2}\equiv\sum_{i=1}^{(p-1)/2}\left(\frac{1}{i^2}+\frac{1}{(p-i)^2}\right)=\sum_{i=1}^{p-1}\frac{1}{i^2}\equiv 2\delta_{p,3}\pmod p$$

with the help of Wolstenholme's congruence (cf. [23] and [24]), by the above we have

$$\begin{aligned}
&\frac{-4}{p^2}\left(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\right)\\
&\equiv\sum_{k=0}^{p^a-1}\binom{2k}{k}(-1)^k\left(H^{(2)}_{\lfloor k/p^{a-1}\rfloor}-\delta_{p,3}\right)\pmod p
\end{aligned}\tag{2.8}$$

and

$$\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\bigg)\equiv\Big(\frac{2}{p^a}\Big)\sum_{k=0}^{p^a-1}\binom{2k}{k}(-2)^kH^{(2)}_{\lfloor k/p^{a-1}\rfloor}\pmod p. \tag{2.9}$$

Our next task is to simplify the right-hand sides of congruences (2.8) and (2.9). Let $u\in\{1,2\}$. Then

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\binom{2k}{k}(-u)^kH^{(2)}_{\lfloor k/p^{a-1}\rfloor}\\&=\sum_{k=0}^{p-1}\sum_{r=0}^{p^{a-1}-1}\binom{2p^{a-1}k+2r}{p^{a-1}k+r}(-u)^{p^{a-1}k+r}H_k^{(2)}\\&\equiv\sum_{k=0}^{p-1}(-u)^kH_k^{(2)}\sum_{r=0}^{p^{a-1}-1}\binom{2p^{a-1}k+2r}{p^{a-1}k+r}(-u)^r\pmod p.\end{aligned}$$

For $k\in\{0,\ldots,p-1\}$ and $r\in\{0,\ldots,p^{a-1}-1\}$, by the Chu-Vandermonde identity (cf. [6, p. 169]) we have

$$\binom{2p^{a-1}k+2r}{p^{a-1}k+r}=\sum_{j=0}^{p^{a-1}k+r}\binom{2p^{a-1}k}{j}\binom{2r}{p^{a-1}k+r-j}.$$

If $p^{a-1}\nmid j$, then

$$\binom{2p^{a-1}k}{j}=\frac{2p^{a-1}k}{j}\binom{2p^{a-1}k-1}{j-1}\equiv 0\pmod p.$$

Thus

$$\begin{aligned}\binom{2p^{a-1}k+2r}{p^{a-1}k+r}&\equiv\sum_{j=0}^{k}\binom{2p^{a-1}k}{p^{a-1}j}\binom{2r}{p^{a-1}(k-j)+r}=\binom{2p^{a-1}k}{p^{a-1}k}\binom{2r}{r}\\&\equiv\binom{2k}{k}\binom{2r}{r}\pmod p\quad\text{(by Lucas' theorem)}.\end{aligned}$$

Therefore

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}(-u)^kH^{(2)}_{\lfloor k/p^{a-1}\rfloor}\equiv\sum_{k=0}^{p-1}(-u)^kH_k^{(2)}\binom{2k}{k}\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-u)^r\pmod p. \tag{2.10}$$

In view of (1.4),

$$\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-1)^r \equiv \left(\frac{5}{p^{a-1}}\right) \pmod p,$$

and also

$$\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-2)^r \equiv \left(\frac{(p-1)/2\times((p-1)/2-4)}{p^{a-1}}\right) = 1 \pmod p$$

provided $p \neq 3$. Combining this with (2.8) and (2.10), we obtain

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\bigg)\\ &\equiv \sum_{k=0}^{p-1}\binom{2k}{k}(-1)^k H_k^{(2)}\sum_{r=0}^{p^{a-1}-1}\binom{2r}{r}(-1)^r-\delta_{p,3}\sum_{k=0}^{p^a-1}\binom{2k}{k}(-1)^k\\ &\equiv \sum_{k=0}^{p-1}\binom{2k}{k}(-1)^k H_k^{(2)}\left(\frac{5}{p^{a-1}}\right)-\delta_{p,3}\left(\frac{5}{p^a}\right) \pmod p,\end{aligned}$$

and hence

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-16)^k}-F_{p^a}\bigg)\\ &\equiv\left(\frac{5}{p^{a-1}}\right)\left(\sum_{k=0}^{p-1}\binom{2k}{k}(-1)^k H_k^{(2)}+\delta_{p,3}\right) \pmod p.\end{aligned}\tag{2.11}$$

Similarly, when $p \neq 3$ we have

$$\begin{aligned}&\frac{-4}{p^2}\bigg(\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}}\bigg)\\ &\equiv\left(\frac{2}{p^a}\right)\sum_{k=0}^{p-1}\binom{2k}{k}(-2)^k H_k^{(2)} \pmod p.\end{aligned}\tag{2.12}$$

Now assume that $p \neq 3$. By (2.5) and (2.12),

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{(-32)^k}-\frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}} \equiv -\left(\frac{2}{p^a}\right)\frac{p^2}{6}q_p(2)^2 \pmod{p^3}.$$

Since $p^a \equiv p \pmod{\varphi(p^2)}$, we have $2^{p^a} \equiv 2^p \pmod{p^2}$ and hence

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-32)^k} - \frac{2^{p^a}+1}{3\times 2^{(p^a-1)/2}} \equiv -\left(\frac{2}{p^a}\right)\frac{(2^{p^a-1}-1)^2}{6} \pmod{p^3}.$$

Combining this with (2.2) we immediately obtain (1.2).

Below we suppose that $p \neq 5$. By (2.4) and (2.11),

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} - F_{p^a} \equiv -\frac{5}{8}\left(\frac{p^a}{5}\right) F^2_{p-(\frac{p}{5})} \pmod{p^3}.$$

In view of [17, Lemma 2.3],

$$\frac{F_{p^a-(\frac{p^a}{5})}}{p} \equiv (-1)^{((\frac{5}{p})-(\frac{5}{p^a}))/2}\left(\frac{5}{p^{a-1}}\right)\frac{F_{p-(\frac{p}{5})}}{p} = \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}$$

and thus

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} - F_{p^a} \equiv -\frac{5}{8}\left(\frac{p^a}{5}\right) F^2_{p^a-(\frac{p^a}{5})} \pmod{p^3}.$$

Combining this with (2.3) we get

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} - F_{p^a} \equiv \frac{L_{p^a}-1}{4}\left(\frac{p^a}{5}\right) - \frac{5}{4}F_{p^a} + \frac{5}{4}\left(\frac{p^a}{5}\right) \pmod{p^3}.$$

Therefore

$$\begin{aligned}\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{(-16)^k} &\equiv \left(\frac{p^a}{5}\right)\frac{L_{p^a}}{4} - \frac{F_{p^a}}{4} + \left(\frac{p^a}{5}\right) \\ &= \left(\frac{p^a}{5}\right)\left(1+\frac{1}{4}\left(L_{p^a} - \left(\frac{p^a}{5}\right)F_{p^a}\right)\right) \\ &= \left(\frac{p^a}{5}\right)\left(1+\frac{1}{2}F_{p^a-(\frac{p^a}{5})}\right) \pmod{p^3}.\end{aligned}$$

This proves (1.1).

So far we have completed the proof of Theorem 1.1. ∎

3. Proof of Theorem 1.2

We need some preliminary results about Lucas sequences.

Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. The equation $x^2 - Ax + B = 0$ has two roots

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2}$$

which are algebraic integers. It is well known that for any $n \in \mathbb{N}$ we have

$$u_n(A, B) = \sum_{0 \leqslant k < n} \alpha^k \beta^{n-1-k} \quad \text{and} \quad v_n(A, B) = \alpha^n + \beta^n.$$

If p is a prime then

$$v_p(A, B) = \alpha^p + \beta^p \equiv (\alpha + \beta)^p = A^p \equiv A \pmod{p}.$$

Lemma 3.1. *Let $A, B \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then*

$$u_{n+1}(A, B) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} A^{n-2k} (-B)^k. \tag{3.1}$$

Remark 3.1. (3.1) is a well-known formula due to Lagrange, see, e.g., H. Gould [G, (1.60)].

Lemma 3.2. *Let $A, B \in \mathbb{Z}$ and let p be an odd prime not dividing $B\Delta$ where $\Delta = A^2 - 4B$. Then*

$$u_p(A, B) \equiv \frac{A}{2} B^{((\frac{\Delta}{p})-1)/2} u_{p-(\frac{\Delta}{p})}(A, B) + \left(\frac{\Delta}{p}\right) \frac{B^{p-1}+1}{2} \pmod{p^2}. \tag{3.2}$$

Proof. For convenience we let $u_n = u_n(A, B)$ and $v_n = v_n(A, B)$ for all $n \in \mathbb{N}$. Let α and β be the two roots of the equation $x^2 - Ax + B = 0$. Then

$$v_n^2 - \Delta u_n^2 = (\alpha^n + \beta^n)^2 - (\alpha^n - \beta^n)^2 = 4(\alpha\beta)^n = 4B^n$$

for any $n \in \mathbb{N}$. As $p \mid u_{p-(\frac{\Delta}{p})}$ (see, e.g., [17, Lemma 2.3]), p^2 divides

$$\begin{aligned} &v^2_{p-(\frac{\Delta}{p})} - 4B^{p-(\frac{\Delta}{p})} \\ &= \left(v_{p-(\frac{\Delta}{p})} - 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2}\right)\left(v_{p-(\frac{\Delta}{p})} + 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2}\right). \end{aligned}$$

On the other hand, by [17, Lemma 2.3] we have

$$v_{p-(\frac{\Delta}{p})} \equiv 2B^{(1-(\frac{\Delta}{p}))/2} \equiv 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2} \pmod{p}.$$

Therefore

$$v_{p-(\frac{\Delta}{p})} \equiv 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2} \pmod{p^2}.$$

By induction, for $\varepsilon = \pm 1$ we have

$$Au_n + \varepsilon v_n = 2B^{(1-\varepsilon)/2} u_{n+\varepsilon}$$

for all $n \in \mathbb{Z}^+$. Thus

$$\begin{aligned} 2B^{(1-(\frac{\Delta}{p}))/2} u_p &= Au_{p-(\frac{\Delta}{p})} + \left(\frac{\Delta}{p}\right) v_{p-(\frac{\Delta}{p})} \\ &\equiv Au_{p-(\frac{\Delta}{p})} + \left(\frac{\Delta}{p}\right) 2\left(\frac{B}{p}\right) B^{(p-(\frac{\Delta}{p}))/2} \pmod{p^2} \end{aligned}$$

and hence

$$\begin{aligned} &2u_p - AB^{((\frac{\Delta}{p})-1)/2} u_{p-(\frac{\Delta}{p})} \\ &\equiv \left(\frac{\Delta}{p}\right)\left(2\left(\frac{B}{p}\right)\left(B^{(p-1)/2} - \left(\frac{B}{p}\right)\right) + 2\right) \\ &\equiv \left(\frac{\Delta}{p}\right)\left(B^{p-1} - 1 + 2\right) \pmod{p^2}. \end{aligned}$$

So (3.2) is valid. ∎

Lemma 3.3. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. Let m be an integer not divisible by p. Then*

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k+1}}{m^k} \equiv \frac{m-2}{2} \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} - \frac{m}{2} + 2p\delta_{a,1}\left(\frac{-m}{p}\right) \pmod{p^2}. \tag{3.3}$$

Proof. Observe that

$$\begin{aligned} &\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k} + \binom{2k}{k+1}}{m^k} \\ &= \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k+1}{k+1}}{m^k} = \frac{\binom{p^a}{(p^a+1)/2}}{m^{(p^a-1)/2}} + \frac{1}{2} \sum_{k=0}^{(p^a-3)/2} \frac{\binom{2k+2}{k+1}}{m^k} \\ &= \frac{p^a/m^{(p^a-1)/2}}{(p^a+1)/2} \binom{p^a-1}{(p^a-1)/2} + \frac{m}{2} \sum_{k=1}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} \\ &\equiv 2p\delta_{a,1}\left(\frac{-m}{p}\right) + \frac{m}{2} \sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{m^k} - \frac{m}{2} \pmod{p^2} \end{aligned}$$

and hence (3.3) follows. ■

Proof of Theorem 1.2. Set $n = (p^a - 1)/2$. By Lemma 3.3,

$$\sum_{k=0}^{n} \frac{C_k}{m^k} \equiv \left(1 - \frac{m-2}{2}\right) \sum_{k=0}^{n} \frac{\binom{2k}{k}}{m^k} + \frac{m}{2} - 2p\delta_{a,1} \left(\frac{-m}{p}\right) \pmod{p^2}.$$

This proves (1.6). It remains to show (1.5).

By Lemmas 3.1 and 2.1,

$$\begin{aligned} u_{p^a}(4, m) &= \sum_{k=0}^{n} \binom{2n-k}{k} 4^{2n-2k}(-m)^k \\ &= \sum_{k=0}^{n} \binom{2n-k}{2(n-k)} 16^{n-k}(-m)^k = \sum_{k=0}^{n} \binom{n+k}{2k} 16^k (-m)^{n-k} \\ &\equiv \sum_{k=0}^{n} \binom{2k}{k} (-1)^k (-m)^{n-k} = (-m)^n \sum_{k=0}^{n} \frac{\binom{2k}{k}}{m^k} \pmod{p^2}. \end{aligned}$$

Note that

$$(-m)^n = \left((-m)^{(p-1)/2}\right)^{\sum_{s=0}^{a-1} p^s} \equiv \left(\frac{-m}{p}\right)^{\sum_{s=0}^{a-1} p^s} = \left(\frac{-m}{p^a}\right) \pmod{p}$$

and hence

$$\begin{aligned} (-m)^n - \left(\frac{-m}{p^a}\right) &\equiv \left((-m)^n - \left(\frac{-m}{p^a}\right)\right) \frac{(-m)^n + \left(\frac{-m}{p^a}\right)}{2(\frac{-m}{p^a})} \\ &\equiv \frac{(-m)^{p^a-1} - 1}{2} \left(\frac{-m}{p^a}\right) \pmod{p^2}. \end{aligned}$$

Thus

$$(-m)^n \equiv \left(\frac{-m}{p^a}\right) \left(1 + \frac{m^{p^a-1} - 1}{2}\right) \equiv \frac{(\frac{-m}{p^a})}{1 - (m^{p^a-1} - 1)/2} \pmod{p^2}$$

and hence

$$\begin{aligned} \sum_{k=0}^{n} \frac{\binom{2k}{k}}{m^k} &\equiv u_{p^a}(4, m) \left(\frac{-m}{p^a}\right) \left(1 - \frac{m^{p^a-1} - 1}{2}\right) \\ &\equiv u_{p^a}(4, m) \left(\frac{-m}{p^a}\right) \left(1 - \frac{m^{p-1} - 1}{2}\right) \pmod{p^2} \end{aligned}$$

since $m^{p^a-1} \equiv m^{p-1} \pmod{p^2}$ by Euler's theorem. By [17, Lemma 2.3],

$$\begin{aligned}u_{p^a}(4,m) &\equiv \left(\frac{4^2-4m}{p^{a-1}}\right) u_p(4,m) \pmod{p^2}\\ &\equiv \left(\frac{4^2-4m}{p^a}\right) u_1(4,m) = \left(\frac{4-m}{p^a}\right) \pmod{p}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{n}\frac{\binom{2k}{k}}{m^k} &\equiv u_{p^a}(4,m)\left(\frac{-m}{p^a}\right) - u_{p^a}(4,m)\left(\frac{-m}{p^a}\right)\frac{m^{p-1}-1}{2}\\ &\equiv \left(\frac{4-m}{p^{a-1}}\right)\left(\frac{-m}{p^a}\right)u_p(4,m) - \left(\frac{-m(4-m)}{p^a}\right)\frac{m^{p-1}-1}{2}\\ &= \left(\frac{-m}{p}\right)\left(\frac{m(m-4)}{p^{a-1}}\right)u_p(4,m) - \left(\frac{m(m-4)}{p^a}\right)\frac{m^{p-1}-1}{2} \pmod{p^2}.\end{aligned}$$

In view of Lemma 3.2,

$$u_p(4,m) - \left(\frac{4-m}{p}\right)\frac{m^{p-1}-1}{2} \equiv \bar{m}u_{p-(\frac{4-m}{p})}(4,m) + \left(\frac{4-m}{p}\right) \pmod{p^2}.$$

So, by the above, $\sum_{k=0}^{n}\binom{2k}{k}/m^k$ is congruent to

$$\begin{aligned}&\left(\frac{m(m-4)}{p^{a-1}}\right)\left(\frac{-m}{p}\right)\left(\bar{m}u_{p-(\frac{4-m}{p})}(4,m) + \left(\frac{4-m}{p}\right)\right)\\ &= \left(\frac{m(m-4)}{p^a}\right) + \left(\frac{-m}{p}\right)\left(\frac{m(m-4)}{p^{a-1}}\right)\bar{m}u_{p-(\frac{4-m}{p})}(4,m)\end{aligned}$$

modulo p^2. This proves (1.5). We are done. ∎

4. Proofs of Corollaries 1.1-1.2

Proof of Corollary 1.1. Note that $n = p - (\frac{4-8}{p}) \equiv 0 \pmod 4$. The equation $x^2 - 4x + 8 = 0$ has two roots $2 \pm 2i$ where $i = \sqrt{-1}$. Thus

$$u_n(4,8) = \frac{(2+2i)^n - (2-2i)^n}{4i} = \frac{(i(2-2i))^n - (2-2i)^n}{4i} = 0$$

and hence by Theorem 1.2 we have

$$\sum_{k=0}^{(p^a-1)/2}\frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{8(8-4)}{p^a}\right) = \left(\frac{2}{p^a}\right) \pmod{p^2}.$$

Clearly $q = p - (\frac{4-16}{p}) = p - (\frac{p}{3})$ is divisible by 3 and the two roots of the equation $x^2 - 4x + 16 = 0$ are

$$2 + 2\sqrt{-3} = -4\omega^2 \text{ and } 2 - 2\sqrt{-3} = -4\omega,$$

where $\omega = (-1+\sqrt{-3})/2$ is a primitive cubic root of unity. Thus

$$u_q(4,16) = \frac{(-4\omega^2)^q - (-4\omega)^q}{4\sqrt{-3}} = 0$$

since $3 \mid q$. Applying (1.5) with $m = 16$ we get

$$\sum_{k=0}^{(p^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{16(16-4)}{p^a}\right) = \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

The proof of Corollary 1.1 is now complete. ■

Proof of Corollary 1.2. Set $n = (p-1)/2$. Then

$$\sum_{k=1}^{n} \frac{\binom{2k}{k}}{16^k}\left(\frac{1}{2k-1} + \frac{1}{(2k-1)^2}\right)$$
$$= \sum_{k=1}^{n} \frac{2\binom{2k-1}{k}}{16^k} \cdot \frac{2k}{(2k-1)^2} = \frac{1}{4}\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv 0 \pmod{p^2}$$

with the help of [19, (1.4)].

Observe that

$$\sum_{k=1}^{n} \frac{\binom{2k}{k}}{(2k-1)16^k} = \sum_{k=1}^{n} \frac{2\binom{2k-1}{k}}{(2k-1)16^k} = 2\sum_{k=1}^{n} \frac{\binom{2k-2}{k-1}}{k16^k}$$
$$= 2\sum_{j=0}^{n-1} \frac{C_j}{16^{j+1}} = \frac{1}{8}\sum_{k=0}^{n} \frac{C_k}{16^k} - \frac{C_n}{8\times 4^{2n}}.$$

Also,

$$\frac{C_n}{4^{2n}} = \frac{\binom{p-1}{(p-1)/2}}{4^{p-1}(p+1)/2} \equiv (-1)^{(p-1)/2}2(1-p) \pmod{p^2}$$

in view of Morley's congruence ([9])

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2}4^{p-1} \pmod{p^3}.$$

By (1.6) and (1.8),

$$\sum_{k=0}^{n} \frac{C_k}{16^k} \equiv -6\left(\frac{3}{p}\right) + 8 - 2p\left(\frac{-1}{p}\right) \pmod{p^2}.$$

Therefore

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{(2k-1)16^k} \equiv \frac{8-6(\frac{3}{p})-2p(\frac{-1}{p})}{8}-\frac{2(1-p)(\frac{-1}{p})}{8}$$
$$=1-\frac{(\frac{-1}{p})+3(\frac{3}{p})}{4}=1-\left(\frac{-1}{p}\right)\frac{3(\frac{p}{3})+1}{4} \pmod{p^2}$$

and hence

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{(2k-1)^2 16^k} \equiv -\sum_{k=1}^{n}\frac{\binom{2k}{k}}{(2k-1)16^k} \equiv -1+\left(\frac{-1}{p}\right)\frac{3(\frac{p}{3})+1}{4} \pmod{p^2},$$

which yields (1.9) and its equivalent form (1.10). We are done. ■

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