TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 4, pp. 1127-1166, August 2013
DOI: 10.11650/tjm.17.2013.2719
This paper is available online at http://journal.taiwanmathsoc.org.tw

WEIGHTED HARDY SPACES ASSOCIATED WITH OPERATORS SATISFYING REINFORCED OFF-DIAGONAL ESTIMATES

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Abstract. Let L be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the reinforced (p_L, p_L') off-diagonal estimates, where $p_L \in [1, 2)$ and p_L' denotes its conjugate exponent. Assume that $p \in (0, 1]$ and the weight w satisfies the reverse Hölder inequality of order $(p_L'/p)'$. In particular, if the heat kernels of the semigroups $\{e^{-tL}\}_{t>0}$ satisfy the Gaussian upper bounds, then $p_L = 1$ and hence $w \in A_\infty(\mathbb{R}^n)$. In this paper, the authors introduce the weighted Hardy spaces $H^p_{L,\,w}(\mathbb{R}^n)$ associated with the operator L, via the Lusin area function associated with the heat semigroup generated by L. Characterizations of $H^p_{L,\,w}(\mathbb{R}^n)$, in terms of the atom and the molecule, are obtained. As applications, the boundedness of singular integrals such as spectral multipliers, square functions and Riesz transforms on weighted Hardy spaces $H^p_{L,\,w}(\mathbb{R}^n)$ are investigated. Even for the Schrödinger operator $-\Delta + V$ with $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the obtained results in this paper essentially improve the known results by extending the narrow range of the weights into the whole $A_\infty(\mathbb{R}^n)$ weights.

1. INTRODUCTION

Since the famous works on Hardy spaces by Stein and Weiss [43] and Fefferman and Stein [26] were published, the theory of Hardy spaces has played an important role in modern harmonic analysis and has extensive applications in partial differential equations. When studying the boundedness of singular integral operators with smooth

Received December 5, 2012, accepted January 3, 2013.
Communicated by Der-Chen Chang.
2010 *Mathematics Subject Classification*: Primary 42B30; Secondary 42B20, 42B35, 42B25, 42B15, 47B06.
Key words and phrases: Weighted Hardy space, Atom, Molecule, Reinforced off-diagonal estimate, Singular integral, Spectral multiplier, Riesz transform.
Dachun Yang is supported by the National Natural Science Foundation of China (Grant No. 11171027) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003). Sibei Yang is supported by Excellent Doctoral Dissertation Cultivation Fund Project of Beijing Normal University.
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kernel, the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ are good substitutes of $L^p(\mathbb{R}^n)$, for example, the classical Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded on $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ but not on $L^p(\mathbb{R}^n)$. Moreover, a key characterization of the Hardy spaces $H^p(\mathbb{R}^n)$ is their atomic decomposition, which was obtained by Coifman [11] when $n = 1$ and by Later [37] when $n > 1$. Later, Coifman and Weiss [14, 15] used the "atomic method" to extend and develop the theory of Hardy spaces to the far more general setting, the so-called spaces of homogeneous type. However, it is nowadays understood that there are important situations in which the classical Coifman-Weiss theory and the classical Calderón-Zygmund theory are not applicable. For example, the Riesz transform $\nabla L^{-1/2}$ needs not be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $L := -\mathrm{div}(A\nabla)$ is a second order divergence elliptic operator with complex L^∞-coefficients; see, for example, [31, 32]. Hence, to characterize the boundedness of these Riesz transforms, we need some new Hardy spaces.

In the last ten years or so, there are a lot of studies which pay attention to the theory of Hardy spaces associated with operators. Let us give a brief overview of this research direction. In [4, 23], Auscher el al. introduced the theory on Hardy spaces associated with operators L under the assumption of Gaussian upper bounds of the heat kernels associated with the semigroup $\{e^{-tL}\}_{t>0}$. Recently, Auscher, McIntosh and Russ [5] investigated the Hardy spaces associated with Hodge Laplacian on a Riemannian manifold with doubling measure. Moreover, Hofmann and Mayboroda [32] studied the theory of Hardy spaces associated with divergence form elliptic operators L. It is important to notice that in [32], the pointwise estimates on the kernels associated with the semigroup $\{e^{-tL}\}_{t>0}$ are not required. Furthermore, the theory of Hardy spaces associated with nonnegative self-adjoint operators satisfying Davies-Gaffney estimates was investigated in [30]. For further information on this research direction, we refer the reader to [4, 9, 22, 23, 5, 32, 30, 35] and the references therein.

The weighted Hardy space associated with operators therefore is the natural extension of the Hardy space associated with operators. Song and Yan [42] treated the weighted Hardy space $H^1_{L,w}(\mathbb{R}^n)$ associated with Schrödinger operators $L := -\Delta + V$, where the weight $w \in A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n)$ and $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Here and in what follows, $A_p(\mathbb{R}^n)$ with $p \in [1,\infty]$ and $RH_q(\mathbb{R}^n)$ with $q \in (1,\infty]$, respectively, denote the *class of Muckenhoupt weights* and the *reverse Hölder class* (see also Subsection 2.1). Then, the results in [42] were extended by the first author of this paper and Duong [8] in which weighted Hardy spaces $H^p_{L,w}(X)$ associated with nonnegative self-adjoint operators satisfying Davies-Gaffney estimates were investigated, where $p \in (0,1]$, $w \in A_1(X) \cap RH_{2/(2-p)}(X)$ and X is a space of homogeneous type. Moreover, D. Yang and S. Yang [45] studied Musielak-Orlicz-Hardy spaces associated with nonnegative self-adjoint operators. In some circumstances, the Musielak-Orlicz-Hardy spaces in [45] turn out to be the weighted Hardy spaces $H^p_{L,w}(X)$ with $p \in (0,1]$ and $w \in A_\infty(X) \cap RH_{2/(2-p)}(X)$. In other words, the best range for the weight w studied in [42, 8, 45] is $w \in RH_{2/(2-p)}(X)$. Also, it should be pointed out that in the proof of the atomic decomposition theorem of [42], the condition $w \in A_1(\mathbb{R}^n)$ is necessary. Hence, it is natural to ask the following question:

Question. *When can we extend the range of weights w to $A_\infty(\mathbb{R}^n)$?*

On the other hand, an important property of the classical Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is that $L^q(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$ for all $q \in (1, \infty)$. In the setting of (both unweighted and weighted) Hardy spaces associated with operators $H^p_{L,w}(\mathbb{R}^n)$, it was proved, in [4, 9, 22, 23, 5, 32, 30, 35, 42, 8, 45], only that $L^2(\mathbb{R}^n) \cap H^p_{L,w}(\mathbb{R}^n)$ is dense in $H^p_{L,w}(\mathbb{R}^n)$. Recall that in [32, 30, 35, 42, 8, 45] the Hardy space $H^p_{L,w}(\mathbb{R}^n)$ was defined as the completion of $\{f \in L^2(\mathbb{R}^n) : S_L(f) \in L^p_w(\mathbb{R}^n)\}$ in the norm $\|f\|_{H^p_{L,w}(\mathbb{R}^n)} := \|S_L(f)\|_{L^p_w(\mathbb{R}^n)}$, where, for all $x \in \mathbb{R}^n$,

$$S_L(f)(x) := \left\{ \int_0^\infty \int_{|x-y|<t} \left| t^2 L e^{-t^2 L}(f)(y) \right|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2}.$$

A natural question is that *what happen if we replace $L^2(\mathbb{R}^n)$ by $L^q(\mathbb{R}^n)$ with $q \neq 2$.*

The main aim of the present paper is to give answers to the above two questions. In this paper, we *always assume that the operator L is nonnegative self-adjoint and satisfies the reinforced (p_L, p_L') off-diagonal estimates*, where $p_L \in [1, 2)$ and p_L' denotes its conjugate exponent; see Section 3 below for the definition of the reinforced (p_L, p_L') off-diagonal estimates. Let $w \in A_\infty(\mathbb{R}^n)$ and $p \in (0, 1]$. We introduce the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n)$ associated with L via the Lusin area function associated with L. Then, we establish the atomic and the molecular characterizations of $H^p_{L,w}(\mathbb{R}^n)$ when $w \in A_\infty(\mathbb{R}^n) \cap RH_{(p_L'/p)'}(\mathbb{R}^n)$, where $(p_L'/p)'$ denotes the conjugate exponent of p_L'/p. Obviously, the inclusion $RH_{(2/p)'}(\mathbb{R}^n) \subset RH_{(p_L'/p)'}(\mathbb{R}^n)$ holds whenever $p_L \in [1, 2)$. In the particular case when $p_L = 1$ or, equivalently, the Gaussian upper bounds are imposed on the heat kernels of the semigroup e^{-tL}, we can extend all the known results on weighted Hardy spaces $H^p_{L,w}(\mathbb{R}^n)$ to all $w \in A_\infty(\mathbb{R}^n)$. As applications of the atomic and the molecular characterizations of $H^p_{L,w}(\mathbb{R}^n)$, the boundedness of singular integrals such as spectral multipliers, square functions and Riesz transforms on weighted Hardy spaces $H^p_{L,w}(\mathbb{R}^n)$ are investigated. The obtained results in this paper essentially improve the known results in [42, 8, 45] by quite enlarging the range of the weights w. Moreover, we show that $H^p_{L,w}(\mathbb{R}^n)$ does not change if we replace $L^2(\mathbb{R}^n)$ by $L^q(\mathbb{R}^n)$ with $q \in (p_L, p_L')$ in the definition of $H^p_{L,w}(\mathbb{R}^n)$. As a consequence, we see that $L^q(\mathbb{R}^n) \cap H^p_{L,w}(\mathbb{R}^n)$ is dense in $H^p_{L,w}(\mathbb{R}^n)$ whenever $q \in (p_L, p_L')$. These give answers to the above two questions.

The main new ingredient appeared in this paper is the introduction of the notion of the reinforced (p_L, p_L') off-diagonal estimates, which leads us to essentially extend the range of the considered weights. Another innovation of this paper appears in the definition of the atom under the $L^q(\mathbb{R}^n)$ norms with $q \in (2, p'_L)$. Moreover, in the construction of the atomic Hardy spaces $H^p_{L,w,\mathrm{at}}(\mathbb{R}^n)$, the convergent sense of the series in the atomic representation is more flexible than those in previous papers [32, 30, 45]; see Theorem 3.8 below. Precisely, the series in the atomic representation in our construction is required to converge in $L^r(\mathbb{R}^n)$-norm for some $r \in (p_L, p'_L)$, not in $L^2(\mathbb{R}^n)$-norm, disregarding the $L^q(\mathbb{R}^n)$-norm of each atom. This flexibility brings some advantages to obtain the atomic decomposition and the boundedness of singular integrals on Hardy spaces; see Theorem 3.8 and Section 4 below.

The organization of this paper is as follows. In Section 2, we first recall the definition of the weight class $A_\infty(\mathbb{R}^n)$ and some of their properties; and then we address some properties of the weighted tent spaces. In Section 3, we introduce the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n)$ via the Lusin area function associated with the operator L and establish its atomic and molecular characterizations. Section 4 is dedicated to studying the boundedness of some singular integrals such as the square functions, the spectral multipliers and the Riesz transforms on the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n)$. More precisely, in Subsection 4.1, we prove that the spectral multiplier $F(L)$ is bounded on the space $H^p_{L,w}(\mathbb{R}^n)$ with $p \in (0,1]$ and $w \in RH_{(p'_L/p)'}(\mathbb{R}^n)$ (see Theorem 4.2 below). It is worth pointing out that in [8, Theorem 4.9], the $H^p_{L,w}(\mathbb{R}^n)$-boundedness of $F(L)$ was established when $p \in (0,1]$ and $w \in A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n)$. Obviously,

$$A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n) \subset RH_{(p'_L/p)'}(\mathbb{R}^n).$$

Thus, Theorem 4.2 essentially improves [8, Theorem 4.9] (see Remark 4.3 below). In Subsection 4.2, we show that the square function $G_{L,k}$ (see (4.9) below for its definition) with $k \in \mathbb{N}$ is bounded from $H^p_{L,w}(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ when $p \in (0,1]$ and $w \in RH_{(p'_L/p)'}(\mathbb{R}^n)$ which improves [45, Theorem 6.3] in this setting by extending the range of the weight w (see Remark 4.8 below). Finally, in Subsection 4.3, for the Schrödinger operator $L := -\Delta + V$ with $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we first prove that the Riesz transform $\nabla L^{-1/2}$ is bounded from $H^p_{L,w}(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ when $p \in (0,1]$ and $w \in RH_{(p_0/p)'}(\mathbb{R}^n)$, where $p_0 \in (2,\infty)$ satisfies that, for all $r \in (1, p_0)$, $\nabla L^{-1/2}$ is bounded on $L^r(\mathbb{R}^n)$ (see Theorem 4.9 below). We remark that Theorem 4.9 essentially improves [8, Theorem 4.1] and [45, Theorem 7.11] by extending the assumptions $w \in A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n)$ in [8, Theorem 4.1] and $w \in RH_{2/(2-p)}(\mathbb{R}^n)$ in [45, Theorem 7.11] to the assumption $w \in RH_{(p_0/p)'}(\mathbb{R}^n)$ (see Remark 4.10 below). Moreover, we also prove in Subsection 4.3 that $\nabla L^{-1/2}$ is bounded from $H^p_{L,w}(\mathbb{R}^n)$ to the weighted Hardy space $H^p_w(\mathbb{R}^n)$ when $p \in (\frac{n}{n+1}, 1]$ and $w \in A_{q_0}(\mathbb{R}^n) \cap RH_{(p_0/q_0)'}(\mathbb{R}^n)$ with some $q_0 \in [1, \frac{p(n+1)}{n})$ (see Theorem 4.13 below), which essentially improves [42, Theorem 1.1(ii)], [44, Theorem 1.1] and [45, Theorem 7.15] by extending the range of the weight w (see Remark 4.14 below for the details). We would like to emphasize that the results obtained in this paper can be considered as extensions to those in previous works [42, 8, 45].

Finally we make some conventions on notation. Throughout the whole paper, C denotes a *positive geometric constant* which is independent of the main parameters, but may change from line to line. We also use $C(\gamma, \beta, \ldots)$ to denote a *positive constant depending on the indicated parameters* γ, β, $\ldots$. The *symbol* $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The *symbol* $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s. We often just use B for $B(x_B, r_B) := \{x \in \mathbb{R}^n : \ |x - x_B| < r_B\}$. Also given $\lambda > 0$, we write λB for the λ*-dilated ball*, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset \mathbb{R}^n$, we set $S_0(B) := B$ and $S_j(B) := 2^j B \setminus 2^{j-1} B$ for $j \in \mathbb{N}$. For any measurable subset E of $\mathbb{R}^n$, we denote by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$ and by χ_E its *characteristic function*. We also set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\theta := (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1 + \ldots + \theta_n$. For any subsets E, $F \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, let $d(E, F) := \inf_{x \in E, y \in F} |x - y|$ and $\operatorname{dist}(z, E) := \inf_{x \in E} |z - x|$. For $1 \leq q \leq \infty$, we denote by q' its *conjugate exponent*, namely, $1/q + 1/q' = 1$. Finally, we use the notation $⨍_B h(x)dx := \frac{1}{|B|} \int_B h(x)\, dx$.

2. Preliminaries

In this section, we first recall the definition of the weight class $A_\infty(\mathbb{R}^n)$ and some of their properties; and then we address some properties of the weighted tent spaces.

2.1. Muckenhoupt weights

Let $q \in [1, \infty)$. A nonnegative locally integrable function w on $\mathbb{R}^n$ is said to belong to the *Muckenhoupt class* $A_q(\mathbb{R}^n)$, namely, $w \in A_q(\mathbb{R}^n)$, if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$, when $q \in (1, \infty)$,

$$⨍_B w(x)dx \left\{ ⨍_B [w(x)]^{-1/(q-1)} dx \right\}^{q-1} \leq C \tag{2.1}$$

and, when $q = 1$,

$$⨍_B w(x)dx \leq C \operatorname*{ess\,inf}_{x \in B} w(x).$$

Moreover, let $A_\infty(\mathbb{R}^n) := \cup_{q \in [1,\infty)} A_q(\mathbb{R}^n)$. Remark that this kind of weights was first introduced by Muckenhoupt [40]. For the sake of convenience, in what follows, we denote by $w(E)$ the integral $\int_E w(x)dx$ for any measurable set $E \subset \mathbb{R}^n$.

The reverse Hölder classes are defined in the following way. Let $r \in (1, \infty)$. A nonnegative locally integrable function w is said to belong to the *reverse Hölder class* $RH_r(\mathbb{R}^n)$, namely, $w \in RH_r(\mathbb{R}^n)$, if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$,

$$\left\{ ⨍_B [w(x)]^r \, dx \right\}^{1/r} \leq C ⨍_B w(x)dx.$$

Moreover, when $r = \infty$, a nonnegative locally integrable function w is said to belong to the *reverse Hölder class* $RH_\infty(\mathbb{R}^n)$, if there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$ and almost every $x \in B$,

$$w(x) \le C \fint_B w(y)\,dy.$$

Let $w \in A_\infty(\mathbb{R}^n)$ and $p \in (0, \infty)$. The *weighted Lebesgue space* $L^p_w(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right\}^{1/p} < \infty.$$

We recall some properties of the Muckenhoupt classes and the reverse Hölder classes in the following two lemmas (see, for example, [20] for the proofs).

Lemma 2.1. *(i)* $A_1(\mathbb{R}^n) \subset A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$ *for* $1 \le p \le q \le \infty$.

(ii) $RH_\infty(\mathbb{R}^n) \subset RH_q(\mathbb{R}^n) \subset RH_p(\mathbb{R}^n)$ *for* $1 < p \le q \le \infty$.

(iii) If $w \in A_p(\mathbb{R}^n)$ *with* $p \in (1, \infty)$*, then there exists* $q \in (1, p)$ *such that* $w \in A_q(\mathbb{R}^n)$.

(iii) If $w \in RH_q(\mathbb{R}^n)$ *with* $q \in (1, \infty)$*, then there exists* $p \in (q, \infty)$ *such that* $w \in RH_p(\mathbb{R}^n)$.

(iv) $A_\infty(\mathbb{R}^n) = \cup_{p\in[1,\infty)} A_p(\mathbb{R}^n) = \cup_{p\in(1,\infty]} RH_p(\mathbb{R}^n)$.

Lemma 2.2. *Let* $q \in [1, \infty)$ *and* $r \in (1, \infty]$. *Suppose that* $w \in A_q(\mathbb{R}^n) \cap RH_r(\mathbb{R}^n)$. *Then there exists a constant* $C \in (1, \infty)$ *such that, for all balls* $B \subset \mathbb{R}^n$ *and any measurable subset* E *of* B, $C^{-1}(\frac{|E|}{|B|})^q \le \frac{w(E)}{w(B)} \le C(\frac{|E|}{|B|})^{\frac{r-1}{r}}$.

In what follows, for any given $w \in A_\infty(\mathbb{R}^n)$, let

$$\begin{aligned} &q_w := \inf\{q \in [1, \infty) : \ w \in A_q(\mathbb{R}^n)\} \quad \text{and} \\ &r_w := \sup\{r \in (1, \infty] : \ w \in RH_r(\mathbb{R}^n)\}. \end{aligned} \tag{2.2}$$

We remark that if $q_w \in (1, \infty)$, then by Lemma 2.1(iii), we conclude that $w \notin A_{q_w}(\mathbb{R}^n)$. Moreover, there exists $w \notin A_1(\mathbb{R}^n)$ such that $q_w = 1$ (see, for example, [36]). Similarly, if $r_w \in (1, \infty)$, then $w \notin RH_{r_w}(\mathbb{R}^n)$ and there exists $w \notin RH_\infty(\mathbb{R}^n)$ such that $r_w = \infty$ (see, for example, [16]).

2.2. Weighted tent spaces

For simplicity, in what follows we write $\mathbb{R}^{n+1}_+$ instead of $\mathbb{R}^n \times (0, \infty)$. For any given $x \in \mathbb{R}^n$, we let

$$\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ : \ |x - y| < t\}.$$

For any closed set $F \subset \mathbb{R}^n$, we set $\mathcal{R}(F) := \cup_{x\in F}\Gamma(x)$. If O is an open subset of $\mathbb{R}^n$, the tent over O is defined by $\widehat{O} := \{(x,t) \in \mathbb{R}^{n+1}_+ :\ \mathrm{dist}(x, O^{\complement}) \ge t\}$. It is easy to verify that $\widehat{O} = [\mathcal{R}(O^{\complement})]^{\complement}$.

Let $F \subset \mathbb{R}^n$ be a closed set and $O := \mathbb{R}^n \setminus F$. For any fixed $\gamma \in (0,1)$, the set of points with global γ-density with respect to F is defined by

$$F^* := \left\{x \in \mathbb{R}^n :\ \frac{|B(x,r) \cap F|}{|B(x,r)|} \ge \gamma \ \text{ for all } r \in (0,\infty)\right\}. \tag{2.3}$$

The following result is taken from [13, Lemma 2] which is used in the sequel.

Lemma 2.3. *There exist positive constants $\gamma \in (0,1)$ and $C(\gamma)$ so that, for any closed set $F \subset \mathbb{R}^n$ with $|F^{\complement}| < \infty$ and any nonnegative measurable function H on $\mathbb{R}^{n+1}_+$,*

$$\int_{\mathcal{R}(F^*)} H(y,t)t^n dy\,dt \le C(\gamma)\int_F \left\{\int_{\Gamma(x)} H(y,t)dy\,dt\right\} dx.$$

For all measurable functions f on $\mathbb{R}^{n+1}_+$ and $x \in \mathbb{R}^n$, let

$$\mathcal{A}(f)(x) := \left\{\int_{\Gamma(x)} |f(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

For $p \in (0,\infty)$ and $w \in A_\infty(\mathbb{R}^n)$, the *tent space* $T^p_w(\mathbb{R}^{n+1}_+)$ is defined to be the space of all measurable functions f such that $\|f\|_{T^p_w(\mathbb{R}^n)} := \|\mathcal{A}(f)\|_{L^p_w(\mathbb{R}^n)} < \infty$.

Notice that the weighted tent space $T^p_w(\mathbb{R}^{n+1}_+)$ can be considered as an extension of those in [13] when $w \equiv 1$. In this case, we write $T^p(\mathbb{R}^{n+1}_+)$ instead of $T^p_w(\mathbb{R}^{n+1}_+)$. For the tent space $T^p_w(\mathbb{R}^{n+1}_+)$, we have the following simple observation, which is used in what follows.

Remark 2.4. (i) If $\operatorname{supp} f \subset \widehat{B}$ for some ball $B \subset \mathbb{R}^n$, then $\operatorname{supp} \mathcal{A}(f) \subset B$.

(ii) If f is a measurable function on $\mathbb{R}^{n+1}_+$ supported in a compact set $\mathbb{K}$, then there exists a positive constant $C(\mathbb{K},\, p,\, w)$, depending on $\mathbb{K}$, p and w, such that

$$\int_{\mathbb{K}} |f(x,t)|^2 dx\,dt \le C(\mathbb{K},\, p,\, w)\|\mathcal{A}(f)\|^2_{L^p_w(\mathbb{R}^n)}.$$

For the tent space $T^q(\mathbb{R}^{n+1}_+)$ with $q \in (1,\infty)$, we have the following conclusion, which is just [13, Theorem 2].

Theorem 2.5. *Let $q \in (1,\infty)$. Then, the dual of $T^q(\mathbb{R}^{n+1}_+)$ is $T^{q'}(\mathbb{R}^{n+1}_+)$. More precisely, the pairing $\langle f,g\rangle := \int_{\mathbb{R}^{n+1}_+} f(x,t)g(x,t)\frac{dx\,dt}{t}$, realizes $T^{q'}(\mathbb{R}^{n+1}_+)$ as equivalent with the dual of $T^q(\mathbb{R}^{n+1}_+)$.*

Let $p \in (0,1]$ and $w \in A_\infty(\mathbb{R}^n)$. A measurable function a on $\mathbb{R}^{n+1}_+$ is called a $(w,\, p,\, \infty)$-*atom* if there exists a ball $B \subset \mathbb{R}^n$, such that

(i) $\operatorname{supp} a \subset \widehat{B}$;

(ii) for any $q \in (1,\infty)$,

$$\|a\|_{T^q(\mathbb{R}^{n+1}_+)} \le |B|^{\frac{1}{q}}[w(B)]^{-\frac{1}{p}}. \tag{2.4}$$

It is worth noticing that any $(w,\, p,\, \infty)$-atom belongs to $T^p_w(\mathbb{R}^{n+1}_+)$. Indeed, since $w \in A_\infty(\mathbb{R}^n)$, there exists $q \in (1,\infty)$ such that $w \in RH_{q'}(\mathbb{R}^n)$ and $pq > 1$. Then, by Remark 2.4 and Hölder's inequality, we know that

$$\begin{aligned}
\|\mathcal{A}(a)\|_{L^p_w(\mathbb{R}^n)} &= \left\{\int_B \left[\int_{\Gamma(x)} |a(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right]^{p/2} w(x)dx\right\}^{1/p} \\
&\le \left\{\int_B \left[\int_{\Gamma(x)} |a(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right]^{\frac{pq}{2}} dx\right\}^{\frac{1}{pq}} \left\{\int_B [w(x)]^{q'}\right\}^{\frac{1}{pq'}} \\
&\lesssim \|a\|_{T^{pq}(\mathbb{R}^{n+1}_+)}|B|^{\frac{1}{pq'}-\frac{1}{p}}[w(B)]^{1/p} \\
&\lesssim |B|^{\frac{1}{pq}}[w(B)]^{-1/p}|B|^{\frac{1}{pq'}-\frac{1}{p}}[w(B)]^{1/p} \lesssim 1.
\end{aligned}$$

An important result concerning weighted tent spaces is that each function in $T^p_w(\mathbb{R}^{n+1}_+)$ has an atomic decomposition. More precisely, we have the following result.

Theorem 2.6. *Let* $p \in (0,1]$, $w \in A_\infty(\mathbb{R}^n)$ *and* $F \in T^p_w(\mathbb{R}^{n+1}_+)$. *Then, there exist a sequence of* $(w,\, p,\, \infty)$*-atoms,* $\{a_j\}_j$, *and a sequence of numbers,* $\{\lambda_j\}_j \subset \mathbb{C}$, *such that*

$$F = \sum_j \lambda_j a_j \tag{2.5}$$

almost everywhere. Moreover, there exists a positive constant C *such that, for all* $F \in T^p_w(\mathbb{R}^{n+1}_+)$,

$$\left\{\sum_j |\lambda_j|^p\right\}^{1/p} \le C\|F\|_{T^p_w(\mathbb{R}^{n+1}_+)}. \tag{2.6}$$

Furthermore, if $F \in T^p_w(\mathbb{R}^{n+1}_+) \cap T^2(\mathbb{R}^{n+1}_+)$, *then the series in* (2.5) *converges in both* $T^p_w(\mathbb{R}^{n+1}_+)$ *and* $T^2(\mathbb{R}^{n+1}_+)$.

Proof. We exploit some ideas from [13] to our situation (see also [29, 8, 35]).

Let γ be as in Lemma 2.3. For each $k \in \mathbb{Z}$, let $E_k := \{x \in \mathbb{R}^n : \mathcal{A}(F)(x) > 2^k\}$ and $\Omega_k := \{x \in \mathbb{R}^n : \mathcal{M}(\chi_{E_k})(x) > 1 - \gamma\}$, where $\mathcal{M}$ denotes the standard *Hardy-Littlewood maximal function* on $\mathbb{R}^n$, namely, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls $B \ni x$. Then, $E_k \subset \Omega_k$ and, it follows, from the fact that $\mathcal{M}$ is of weak type $(1, 1)$, that $|\Omega_k| \lesssim |E_k|$ for all $k \in \mathbb{Z}$, which, together with $w \in A_\infty(\mathbb{R}^n)$ and Lemmas 2.1(v) and 2.2, implies that

$$w(\Omega_k) \lesssim w(E_k). \tag{2.7}$$

Moreover, it can be showed that $\operatorname{supp} F \subset \cup_{k \in \mathbb{Z}} \widehat{\Omega}_k$.

For each k, due to the Whitney covering lemma (see, for example, [14]), we pick a family $\{\widetilde{B}_k^j\}_j$ of balls and positive constants $1 < c^* < c^{**}$ satisfying the following three conditions:

(i) the family $\{\widetilde{B}_k^j\}_j$ is pairwise disjoint;

(ii) $\Omega_k = \cup_j c^* \widetilde{B}_k^j$;

(iii) $c^{**} \widetilde{B}_k^j \cap (\Omega_k)^\complement \neq \emptyset$.

Taking $c_1 := 4c^{**}$ and setting $B_k^j := c_1 \widetilde{B}_k^j$, then we have $\widehat{\Omega}_k \setminus \widehat{\Omega}_{k+1} \subset \cup_j A_k^j$ with

$$A_k^j := \widehat{B}_k^j \cap (c^* \widetilde{B}_k^j \times (0, \infty)) \cap (\widehat{\Omega}_k \setminus \widehat{\Omega}_{k+1}).$$

Define $a_k^j := 2^{-(k+1)} [w(B_k^j)]^{-1/p} F \chi_{A_k^j}$ and $\lambda_k^j := 2^{(k+1)} [w(B_k^j)]^{1/p}$. Then $F = \sum_{k,\, j} \lambda_k^j a_k^j$ almost everywhere.

For any given $q \in (1, \infty)$, let $h \in T^{q'}(\mathbb{R}_+^{n+1})$ satisfying $\|h\|_{T^{q'}(\mathbb{R}_+^{n+1})} = 1$. Notice that $A_k^j \subset (\widehat{\Omega}_{k+1})^\complement = \mathcal{R}(F_{k+1}^*)$, where $F_{k+1} := (E_{k+1})^\complement$ and F_{k+1}^* is as in (2.3). Then, thanks to Lemma 2.3, Hölder's inequality and $\operatorname{supp} \mathcal{A}(a_k^j) \subset B_k^j$, we conclude that

$$\begin{aligned}
|\langle a_k^j, h\rangle| &\leq \int_{\mathbb{R}_+^{n+1}} \left| \left(a_k^j \chi_{A_k^j} \right)(y, t) h(y, t) \right| \frac{dy\, dt}{t} \\
&\leq \int_{F_{k+1}} \int_{\Gamma(x)} \left| a_k^j(y, t) h(y, t) \right| \frac{dy\, dt}{t^{n+1}} dx \lesssim \int_{F_{k+1}} \mathcal{A}(a_k^j)(x) \mathcal{A}(h)(x) dx \\
&\lesssim 2^{-(k+1)} [w(B_k^j)]^{-1/p} \left\{ \int_{F_{k+1} \cap B_k^j} |A(F)(x)|^q dx \right\}^{1/q} \lesssim |B_k^j|^{1/q} [w(B_k^j)]^{-1/p},
\end{aligned}$$

which implies that, for any given $q \in (1, \infty)$, $\|a_k^j\|_{T^q(\mathbb{R}_+^{n+1})} \lesssim |B_k^j|^{1/q} [w(B_k^j)]^{-1/p}$. As a consequence, we see that a_k^j is a multiple of a $(w,\, p,\, \infty)$-atom.

Furthermore, from the definition of λ_k^j and Lemma 2.2, we deduce that

$$\sum_{k,j}|\lambda_k^j|^p=\sum_{k,\,j}2^{p(k+1)}w(B_k^j)\lesssim\sum_{k,\,j}2^{p(k+1)}w(\widetilde{B}_k^j).$$

By this, the above properties (i) and (ii), and (2.7), we know that

$$\begin{aligned}\sum_{k,\,j}|\lambda_k^j|^p&\lesssim\sum_{k}2^{p(k+1)}w(\Omega_k)\lesssim\sum_{k}2^{p(k+1)}w(E_k)\\&\lesssim\sum_{k}2^{p(k+1)}w\left(\left\{x\in\mathbb{R}^n:\ \mathcal{A}(F)(x)>2^k\right\}\right)\\&\lesssim\|\mathcal{A}(F)\|^p_{L^p_w(\mathbb{R}^n)}\lesssim\|F\|^p_{T^p_w(\mathbb{R}^{n+1}_+)}.\end{aligned}$$

Moreover, similar to the proof of [35, Proposition 3.1], we further know that, if $F\in T^p_w(\mathbb{R}^{n+1}_+)\cap T^2(\mathbb{R}^{n+1}_+)$, then (2.5) holds true in both $T^p_w(\mathbb{R}^{n+1}_+)$ and $T^2(\mathbb{R}^{n+1}_+)$. This finishes the proof of Theorem 2.6. ∎

Let $T^p_{w,\,c}(\mathbb{R}^{n+1}_+)$ and $T^q_c(\mathbb{R}^{n+1}_+)$ denote, respectively, the *sets of all functions in* $T^p_w(\mathbb{R}^{n+1}_+)$ *and* $T^q(\mathbb{R}^{n+1}_+)$ *with compact support*, where $p,\,q\in(0,\infty)$. The following result plays an important role in the sequel.

Lemma 2.7. *Let* $w\in A_\infty(\mathbb{R}^n)$ *and* $p\in(0,1]$. *Then,* $T^p_{w,\,c}(\mathbb{R}^{n+1}_+)\subset T^2_c(\mathbb{R}^{n+1}_+)$ *as sets.*

Proof. We first observe that [13, (1.3)] says that

$$T^q_c(\mathbb{R}^{n+1}_+)\subset T^2_c(\mathbb{R}^{n+1}_+)\tag{2.8}$$

holds for all $q\in(0,\infty)$. Since $w\in A_\infty(\mathbb{R}^n)$, we can pick $r\in(0,p)$ such that $w\in A_{p/r}(\mathbb{R}^n)$.

Let $f\in T^p_{w,\,c}(\mathbb{R}^{n+1}_+)$ with $\operatorname{supp} f\subset K$ for some compact set K. Assume that B is the ball satisfying $K\subset\widehat{B}$. Then it follows, from Lemma 2.4(i), that $\operatorname{supp}\mathcal{A}(f)\subset B$. Thus, by this, Hölder's inequality, $w\in A_{p/r}(\mathbb{R}^n)$ and (2.1), we see that

$$\begin{aligned}\|f\|^r_{T^r(\mathbb{R}^{n+1}_+)}&=\|\mathcal{A}(f)\|^r_{L^r(\mathbb{R}^n)}=\int_B|\mathcal{A}(f)(x)|^r\,dx\\&=\int_B|\mathcal{A}(f)(x)|^r[w(x)]^{r/p}[w(x)]^{-r/p}\,dx\\&\le\left\{\int_B|\mathcal{A}(f)(x)|^pw(x)\,dx\right\}^{r/p}\left\{\int_B[w(x)]^{(-r/p)(p/r)'}\,dx\right\}^{\frac{1}{(p/r)'}}\\&\lesssim\|f\|^r_{T^p_w(\mathbb{R}^{n+1}_+)}\frac{|B|}{[w(B)]^{r/p}}<\infty,\end{aligned}$$

which, together with (2.8), implies that $T^p_{w,\,c}(\mathbb{R}^{n+1}_+)\subset T^r_c(\mathbb{R}^{n+1}_+)\subset T^2_c(\mathbb{R}^{n+1}_+)$. This finishes the proof of Lemma 2.7. ∎

3. Weighted Hardy Spaces Associated with Operators

In this section, we introduce the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n)$ via the Lusin area function associated with the operator L and then establish its atomic and molecular characterizations.

Throughout the whole paper, we always suppose that the considered operator L satisfies the following assumptions:

Assumption (H1). *L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$.*

Assumption (H2). *There exists a constant $p_L \in [1, 2)$ so that the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the reinforced (p_L, p_L') off-diagonal estimates, namely, for all r, $q \in (p_L, p_L')$ with $r \le q$, there exist two positive constants C and c such that*

$$\|e^{-tL}f\|_{L^q(F)} \le Ct^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}e^{-\frac{[d(E,F)]^2}{ct}}\|f\|_{L^r(E)} \tag{3.1}$$

holds true for every closed sets E, $F \subset \mathbb{R}^n$, $t \in (0,\infty)$, $f \in L^r(E)$ and $\operatorname{supp} f \subset E$, where $d(E,F) := \inf\{|x-y| :\ x \in E,\ y \in F\}$ and $\|f\|_{L^r(E)} := \left\{\int_E |f(x)|^r\,dx\right\}^{1/r}$.

Remark 3.1. The notion of the off-diagonal estimates (or the so called Davies-Gaffney estimates) of the semigroup $\{e^{-tL}\}_{t>0}$ are first introduced by Gaffney [27] and Davies [19], which serves as good substitutes of the Gaussian upper bound of the associated heat kernel; see also [6, 3] and their references. The reinforced off-diagonal estimate requires that the off-diagonal estimates hold for all the associated exponents p and q in some interval of $[1, \infty]$, which are stronger than the off-diagonal estimates. We also point out that an assumption similar to the reinforced off-diagonal estimate which required the off-diagonal estimates satisfied for all p, $q \in (p_-(L),\ p_+(L))$ with $p \le q$ has also been given out in [10]. More precisely, let $(p_-(L),\ p_+(L))$ be the range of exponents $p \in [1, \infty]$ such that the semigroup $\{e^{-tL}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Then [10, Assumption $(\mathcal{L})_4$] required that for all $p_-(L) < p \le q < p_+(L)$, $\{e^{-tL}\}_{t>0}$ satisfies (3.1) with $t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\frac{[d(E,F)]^2}{ct}}$ replaced by $t^{-\frac{n}{2k}(\frac{1}{p}-\frac{1}{q})}e^{-\frac{[d(E,F)]^{2k/(2k-1)}}{ct^{1/(2k-1)}}}$, where $k \in \mathbb{N}$.

According to [5], we define

$$H^2(\mathbb{R}^n) := H^2_L(\mathbb{R}^n) := \overline{\mathcal{R}(L)} := \overline{\{Lu \in L^2(\mathbb{R}^n) :\ u \in \mathcal{D}(L)\}},$$

where $\mathcal{D}(L)$ is the *domain* of L.

It is well known that $L^2(\mathbb{R}^n) = H^2(\mathbb{R}^n) \oplus \mathcal{N}(L)$, where $\mathcal{N}(L)$ denotes the *kernel* of L, and the summation is orthogonal. Moreover, it was proved in [30] that in our situation $H^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Lusin area function* $S_L(f)$ of f is defined by

$$S_L(f)(x) = \left\{ \int_{\Gamma(x)} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

For the Lusin area function S_L, we have the following useful result.

Proposition 3.2. *The operator S_L, initially defined on $L^2(\mathbb{R}^n)$, can be extended to a bounded operator on $L^q(\mathbb{R}^n)$ for all $q \in (p_L, p_L')$, where p_L is as in* (3.1).

Proof. The proof is similar to that for vertical square function in [1]. Hence, we omit the details here. ■

Now we introduce the weighted Hardy space $H^{p,q}_{L,w}(\mathbb{R}^n)$ associated with L, via the Lusin area function S_L.

Definition 3.3. Let $w \in A_\infty(\mathbb{R}^n)$, $p \in (0,1]$ and $q \in (p_L, p_L')$, where p_L is as in (3.1). The *weighted Hardy space* $H^{p,q}_{L,w}(\mathbb{R}^n)$ is defined as the completion of $\{f \in L^q(\mathbb{R}^n) :\ S_L(f) \in L^p_w(\mathbb{R}^n)\}$ in the *norm* $\|f\|_{H^{p,q}_{L,w}(\mathbb{R}^n)} := \|S_L(f)\|_{L^p_w(\mathbb{R}^n)} < \infty$.

For the weighted Hardy space $H^{p,q}_{L,w}(\mathbb{R}^n)$, we have the following result.

Theorem 3.4. *Let $p \in (0,1]$ and $w \in RH_{(p_L'/p)'}(\mathbb{R}^n)$, where p_L is as in* (3.1). *Then $H^{p,2}_{L,w}(\mathbb{R}^n)$ and $H^{p,s}_{L,w}(\mathbb{R}^n)$ coincide with equivalent norms, whenever $s \in (p_L, p_L')$.*

The proof of Theorem 3.4 is given in Subsection 3.2 below.

It is worth pointing out that Theorem 3.4 enables us to define the space $H^p_{L,w}(\mathbb{R}^n)$, for $p \in (0,1]$ and $w \in RH_{(p_L'/p)'}(\mathbb{R}^n)$, to be any one of the spaces $H^{p,q}_{L,w}(\mathbb{R}^n)$ for $q \in (p_L, p_L')$.

3.1. Atomic and molecular characterizations of $H^p_{L,w}(\mathbb{R}^n)$

In this subsection, we establish the atomic and molecular characterizations of $H^p_{L,w}(\mathbb{R}^n)$. We begin with the notions of $(p,\,q,\,M,\,w)$-atoms and $(p,\,q,\,M,\,w,\,\epsilon)$-molecules associated with the operator L.

Definition 3.5. Let $w \in A_\infty(\mathbb{R}^n)$, $p \in (0,1]$, $q \in (0,\infty)$ and $M \in \mathbb{N}$. A function $a \in L^q(\mathbb{R}^n)$ is called a $(p,\,q,\,M,\,w)$-*atom* associated with the operator L if there exists a function $b \in \mathcal{D}(L^M)$, the *domain* of L^M, and a ball $B \subset \mathbb{R}^n$ such that

(i) $a = L^M b$;

(ii) $\operatorname{supp} L^k b \subset B,\ k \in \{0,\,\ldots,\,M\}$;

(iii) $\|(r_B^2 L)^k b\|_{L^q(\mathbb{R}^n)} \le r_B^{2M} |B|^{1/q} [w(B)]^{-1/p},\ k \in \{0,\,\ldots,\,M\}$.

We remark that the above definition of L-adapted atom is rather standard, which first appeared in [30] in the unweighted case and [42] in the weighted case. Now, let f be a function on $\mathbb{R}^n$, f is said to belong to the *set* $\mathbb{H}^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$ if it can be written as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{3.2}$$

where $\{\lambda_j\}_{j=1}^{\infty} \in \ell^p$, each a_j is a $(p,\,q,\,M,\,w)$-atom, and the summation converges in $L^r(\mathbb{R}^n)$ for some $r \in (p_L, p_L')$. The *space* $H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{H}^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$ in the *norm*

$$\|f\|_{H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)} := \inf\left\{\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p} :\ f = \sum_{j=1}^{\infty}\lambda_j a_j\right\},$$

where the infimum is taken over all decompositions of f as in (3.2).

Definition 3.6. Let $w \in A_\infty(\mathbb{R}^n)$, $p \in (0,1]$, $M \in \mathbb{N}$, $\epsilon \in (0,\infty)$ and $q \in (0,\infty)$. A function $m \in L^q(\mathbb{R}^n)$ is called a $(p,\,q,\,M,\,w,\,\epsilon)$-*molecule* associated with the operator L, if there exists a function $b \in \mathcal{D}(L^M)$ and a ball $B \subset \mathbb{R}^n$ such that

(i) $m = L^M b$;

(ii) $\|(r_B^2 L)^k b\|_{L^q(S_j(B))} \le 2^{-j\epsilon} r_B^{2M} |2^j B|^{1/q} [w(2^j B)]^{-1/p}, k \in \{0,\,\ldots,\,M\}$ and $j \in \mathbb{Z}_+$.

Moreover, the *space* $H^{p,q,M,\epsilon}_{L,\,w,\,\mathrm{mol}}(\mathbb{R}^n)$ is defined as $H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$ with $(p,\,q,\,M,\,w)$-atoms replaced by $(p,\,q,\,M,\,w,\,\epsilon)$-molecules.

For $(p,\,q,\,M,\,w)$-atoms and $(p,\,q,\,M,\,w,\,\epsilon)$-molecules, we have the following observation.

Remark 3.7. (i) If $q_1,\,q_2 \in (r_w',\infty)$ with $q_1 \ge q_2$, then any $(p,\,q_1,\,M,\,w)$-atom is also a $(p,\,q_2,\,M,\,w)$-atom.

(ii) Let $p,\,q,\,M,\,w$ and ϵ be as in Definition 3.6. If a is a $(p,\,q,\,M,\,w)$-atom related to the ball B, then it is also a $(p,\,q,\,M,w,\,\epsilon)$-molecule related to the same ball B.

We are ready to state the main results of this section.

Theorem 3.8. *Let $p \in (0,1]$ and $w \in RH_{(p_L'/p)'}(\mathbb{R}^n)$, where p_L is as in* (3.1). *Then, the spaces $H^p_{L,\,w}(\mathbb{R}^n) = H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$ with equivalent norms whenever $q \in [2,\infty)\cap(pr_w', p_L')$ and $M \in \mathbb{N}$ with $M > \frac{n}{2}(\frac{q_w}{p} - \frac{1}{2})$, where q_w and r_w are as in* (2.2). *Furthermore, in this case, the series in* (3.2) *converges in $H^p_{L,\,w}(\mathbb{R}^n)$.*

Theorem 3.9. *Let p, w, q and M be as in Theorem 3.8 and $\epsilon \in (n, \infty)$. Then, the spaces $H^p_{L,w}(\mathbb{R}^n) = H^{p,q,M,\epsilon}_{L,w,\mathrm{mol}}(\mathbb{R}^n)$ with equivalent norms.*

The proofs of Theorems 3.8 and 3.9 are given in Subsection 3.2 below.

Remark 3.10. Observe that when the operator L has the kernel p_t satisfying the Gaussian upper bound estimate or, equivalently, $p_L' = \infty$, the condition of the weights w in Theorem 3.8 is just $w \in A_\infty(\mathbb{R}^n)$. This answers the question mentioned in the introduction.

3.2. Proofs of Theorems 3.4, 3.8 and 3.9

In this subsection, we give the proofs of Theorems 3.4, 3.8 and 3.9. Before going into details, we need to recall some notation and results from [30]. Let $K_{\cos(t\sqrt{L})}$ be the integral kernel of the operator $\cos(t\sqrt{L})$. By [30, Proposition 3.4] (see also [18] and related references), we know that there exists a positive constant c_0 such that

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subset \mathcal{D}_t := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \ |x-y| \le c_0 t\}. \tag{3.3}$$

We now recall a useful result which is just [30, Lemma 3.5].

Lemma 3.11. *Let $\varphi \in C_c^\infty(\mathbb{R})$ be even and $\operatorname{supp}\varphi \subset (-c_0^{-1}, c_0^{-1})$, where c_0 is as in* (3.3)*. Let Φ denote the Fourier transform of φ. Then, for all $k \in \mathbb{N}$ and $t \in (0,\infty)$,*

$$\operatorname{supp} K_{(t^2L)^k\Phi(t\sqrt{L})} \subset \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \ |x-y| \le t\}.$$

Moreover, the following lemma gives self-improving properties of the reinforced (p_L, p_L') off-diagonal estimates.

Lemma 3.12. *Let L satisfy Assumptions* (H1) *and* (H2)*, and p_L be as in* (3.1)*. Then, for every $k \in \mathbb{N}$, the family $\{(tL)^k e^{-tL}\}_{t>0}$ also satisfies the reinforced (p_L, p_L') off-diagonal estimates.*

Proof. The proof of this lemma is very standard. However, for the completeness, we sketch its proof here.

Fix $\theta \in (0, \pi/2)$. By the Cauchy integral formula, it suffices to show that there exist positive constants C and c such that, for all closed sets E and F of $\mathbb{R}^n$, $p_L < r \le q < p_L'$, $f \in L^r(E)$ with $\operatorname{supp} f \subset E$, $t \in (0,\infty)$ and $z \in S_\theta := \{z \in \mathbb{C} : \ |\arg z| < \theta\}$,

$$\|e^{-zL} f\|_{L^q(F)} \le C|z|^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})} \exp\left\{-\frac{[d(E,F)]^2}{c|z|}\right\} \|f\|_{L^r(E)}. \tag{3.4}$$

Notice that if (3.4) holds for such r and q, (3.4) also holds for $\widetilde{r} \le \widetilde{q}$ with $p_L < r \le \widetilde{r} \le \widetilde{q} \le q < p_L'$. Hence, we need only to prove (3.4) for $r \le 2 \le q$.

We now assume that $z = 2s + it$ with $s \in (0, \infty)$ and $t \in \mathbb{R}$. Then by $z \in S_\theta$, we conclude that $s \approx |z|$. Moreover, it is easy to see that $e^{-zL} = e^{-sL}e^{-itL}e^{-sL}$. Therefore, the reinforced (p_L, p_L') off-diagonal property of $\{e^{-tL}\}_{t>0}$ gives that, for all r, $q \in (p_L, p_L')$ with $r \le q$,

$$\begin{aligned}
\|e^{-zL}f\|_{L^q(F)} &= \|e^{-sL}e^{-itL}e^{-sL}f\|_{L^q(F)}\\
&\le \|e^{-sL}\|_{L^2\to L^q}\|e^{-itL}\|_{L^2\to L^2}\|e^{-sL}f\|_{L^2(F)}\\
&\lesssim s^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}s^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\exp\left\{-\frac{[d(E,F)]^2}{cs}\right\}\|f\|_{L^r(E)}\\
&\lesssim |z|^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\exp\left\{-\frac{[d(E,F)]^2}{c|z|}\right\}\|f\|_{L^r(E)},
\end{aligned}$$

which proves (3.4), and hence completes the proof of Lemma 3.12. ■

In what follows, denote by $f \in L^2_c(\mathbb{R}^{n+1}_+)$ the *set of all functions in* $L^2(\mathbb{R}^{n+1}_+)$ *with compact support*. Let Φ be as in Lemma 3.11. Then, for all $f \in L^2_c(\mathbb{R}^{n+1}_+)$ and $x \in \mathbb{R}^n$, define

$$\pi_{\Phi,L,M}(f)(x) := c_{\Phi,M}\int_0^\infty (t^2L)^{M+1}\Phi(t\sqrt{L})(f(\cdot,t))(x)\frac{dt}{t},$$

where $c_{\Phi,M}$ is a constant such that

$$1 = c_{\Phi,M}\int_0^\infty t^{2(M+1)}\Phi(t)t^2e^{-t^2}\frac{dt}{t}.$$

For any $N \in \mathbb{N}$, let

$$\widetilde{O_N} := \left\{(x,t) \in \mathbb{R}^{n+1}_+ :\ |x| < N \text{ and } N^{-1} < t < N\right\}.$$

Then, by the $L^2(\mathbb{R}^n)$-functional calculus associated with L (see, for example, [39]), we see that, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
f(x) &= \pi_{\Phi,L,M}\left(t^2Le^{-t^2L}f\right)(x)\\
&= c_{\Phi,M}\int_0^\infty (t^2L)^{M+1}\Phi(t\sqrt{L})\left(\left(t^2Le^{-t^2L}f\right)(\cdot,t)\right)(x)\frac{dt}{t}\\
&= \lim_{N\to\infty} c_{\Phi,M}\int_0^\infty (t^2L)^{M+1}\Phi(t\sqrt{L})\left(\left(t^2Le^{-t^2L}f\right)(\cdot,t)\chi_{\widetilde{O_N}}\right)(x)\frac{dt}{t},
\end{aligned} \tag{3.5}$$

where the integral converges in $L^2(\mathbb{R}^n)$.

To prove Theorems 3.4, 3.8 and 3.9, we need the following key lemmas.

Lemma 3.13. *Let* p, w, q, M *be as in Theorem 3.8 and* ϵ *as in Theorem 3.9. Then, there exists a positive constant* C *such that*

(i) *for every* $(p,\, q,\, M,\, w)$*-atom* a *related to the ball* B*,* $\|S_L(a)\|_{L^p_w(\mathbb{R}^n)} \le C$*;*

(ii) *for every* $(p,\, q,\, M,\, w,\, \epsilon)$*-molecule* m *related to the ball* B*,* $\|S_L(m)\|_{L^p_w(\mathbb{R}^n)} \le C$*.*

Proof. (i) By the hypothesis of p, w and q, we know that $w \in RH_{(q/p)'}(\mathbb{R}^n)$. Let a be a $(p,\, q,\, M,\, w)$-atom related to the ball B. Then we have

$$\|S_L(a)\|^p_{L^p_w(\mathbb{R}^n)} = \sum_{j=0}^{\infty} \int_{S_j(B)} |S_L(a)(x)|^p w(x)\, dx =: \sum_{j=0}^{\infty} \mathrm{I}_j.$$

When $j \in \{0,\, 1,\, 2\}$, by Hölder's inequality, Proposition 3.2, $w \in RH_{(q/p)'}(\mathbb{R}^n)$ and Lemma 2.2, we see that

$$\begin{aligned}\mathrm{I}_j &\le \|S_L a\|^p_{L^q(\mathbb{R}^n)} \left\{\int_{S_j(B)} [w(x)]^{(q/p)'} dx\right\}^{\frac{1}{(q/p)'}} \lesssim |2^j B|^{-p/q} w(2^j B) \|a\|^p_{L^q(\mathbb{R}^n)} \\ &\lesssim |2^j B|^{-p/q} w(2^j B) |B|^{p/q} [w(B)]^{-1} \lesssim 1.\end{aligned}$$

When $j \in \mathbb{N}$ with $j \ge 3$, from Hölder's inequality and $w \in RH_{(q/p)'}(\mathbb{R}^n)$, it follows that

$$\begin{aligned}\mathrm{I}_j &\le \|S_L(a)\|^p_{L^q(S_j(B))} \left\{\int_{S_j(B)} [w(x)]^{(q/p)'} dx\right\}^{\frac{1}{(q/p)'}} \\ &\lesssim \|S_L(a)\|^p_{L^q(S_j(B))} |2^j B|^{-p/q} w(2^j B).\end{aligned}$$

To estimate $\|S_L(a)\|^p_{L^q(S_j(B))}$, we write

$$\begin{aligned}&\|S_L(a)\|^q_{L^q(S_j(B))} \\ &= \int_{S_j(B)} \left\{\left[\int_0^{\frac{d(x,x_B)}{4}} + \int_{\frac{d(x,x_B)}{4}}^{\infty}\right] \int_{B(x,t)} \left|t^2 L e^{-t^2 L} a(y)\right|^2 \frac{dy\, dt}{t^{n+1}}\right\}^{q/2} dx \\ &\lesssim \int_{S_j(B)} \left\{\int_0^{\frac{d(x,x_B)}{4}} \int_{B(x,t)} \left|(t^2 L)^{M+1} e^{-t^2 L} b(y)\right|^2 \frac{dy\, dt}{t^{n+4M+1}}\right\}^{q/2} dx \\ &\quad + \int_{S_j(B)} \left\{\int_{\frac{d(x,x_B)}{4}}^{\infty} \int_{B(x,t)} \cdots\right\}^{q/2} dx =: \mathrm{II}_j + \mathrm{III}_j,\end{aligned}$$

where b satisfies $a = L^M b$. Let $F_j(B) := \{y \in \mathbb{R}^n :\ |x-y| < \frac{d(x,x_B)}{4}$ for some $x \in S_j(B)\}$. Then $d(B, F_j(B)) \ge 2^{j-2} r_B$. By $M > \frac{n}{2}(\frac{q_w}{p} - \frac{1}{2})$ and the definition of q_w, we know that there exists $\widetilde{q} \in (q_w, \infty)$ such that $w \in A_{\widetilde{q}}(\mathbb{R}^n)$ and $M > \frac{n}{2}(\frac{\widetilde{q}}{p} - \frac{1}{2})$.

Moreover, by Assumption (H2) and Hölder's inequality, together with Lemma 2.2, we conclude that

$$\begin{aligned}
\mathrm{II}_j &\le \int_{S_j(B)}\left\{\int_0^{2^j r_B}\int_{F_j(B)}\left|(t^2L)^{M+1}e^{-t^2L}b(y)\right|^2\frac{dy\,dt}{t^{n+4M+1}}\right\}^{q/2}dx\\
&\lesssim \|b\|_{L^2(B)}^q\int_{S_j(B)}\left\{\int_0^{2^j r_B}e^{-\frac{[d(B,F_j(B))]^2}{ct^2}}\frac{dy\,dt}{t^{n+4M+1}}\right\}^{q/2}dx\\
&\lesssim r_B^{2qM}|B|^{q/2}[w(B)]^{-q/p}\int_{S_j(B)}\left\{\int_0^{2^j r_B}\left(\frac{t}{2^j r_B}\right)^{n+4M+1}\frac{dy\,dt}{t^{n+4M+1}}\right\}^{q/2}dx\\
&\lesssim r_B^{2qM}|B|^{q/2}[w(B)]^{-q/p}|2^jB|(2^jr_B)^{-2qM}|2^jB|^{-q/2}\\
&\lesssim 2^{-jq(2M+n/2-n\widetilde{q}/p)}|2^jB|[w(2^jB)]^{-q/p}.
\end{aligned}$$

Similarly, for the term III_j, we have

$$\begin{aligned}
\mathrm{III}_j &\le \int_{S_j(B)}\left\{\int_{2^{j-3}r_B}^{\infty}\int_{\mathbb{R}^n}\left|(t^2L)^{M+1}e^{-t^2L}b(y)\right|^2\frac{dy\,dt}{t^{n+4M+1}}\right\}^{q/2}dx\\
&\lesssim \|b\|_{L^2(B)}^q\int_{S_j(B)}\left\{\int_{2^{j-3}r_B}^{\infty}\frac{dt}{t^{n+4M+1}}\right\}^{q/2}dx\\
&\lesssim 2^{-jq(2M+n/2-n\widetilde{q}/p)}|2^jB|[w(2^jB)]^{-q/p}.
\end{aligned}$$

Combining the above estimates of II_j and III_j, by $M>\frac{n}{2}(\frac{\widetilde{q}}{p}-\frac{1}{2})$, we know that

$$\|S_La\|_{L_w^p(\mathbb{R}^n)}^p=\sum_{j=0}^{2}\mathrm{I}_j+\sum_{j=3}^{\infty}\mathrm{I}_j\lesssim 1+\sum_{j=3}^{\infty}2^{-jp(2M+n/2-n\widetilde{q}/p)}\lesssim 1.$$

(ii) The proof of (ii) is similar to that of (i). The main difference is that the support of the $(p,\,q,\,M,\,w,\,\epsilon)$-molecule is not the ball B. However, we can overcome this difficulty by decomposing $\mathbb{R}^n$ into annuli associated with the ball B. We omit the details here. ■

Lemma 3.14. *Let p, w, q and M be as in Theorem 3.8. Then,*

(*i*) *the operator $\pi_{\Phi,L,M}$, initially defined on $T_c^2(\mathbb{R}_+^{n+1})$, extends to a bounded linear operator from $T^2(\mathbb{R}_+^{n+1})$ to $L^2(\mathbb{R}^n)$;*

(*ii*) *the operator $\pi_{\Phi,L,M}$, initially defined on $T_{w,c}^p(\mathbb{R}_+^{n+1})$, extends to a bounded linear operator from $T_w^p(\mathbb{R}_+^{n+1})$ to $H_{L,w}^{p,2}(\mathbb{R}^n)$.*

Proof. (i) For the proof of (i), we refer to the proof of [34, Proposition 4.1(i)].

(ii) Let $f \in T^p_{w,c}(\mathbb{R}^{n+1}_+)$. Then by Lemma 2.7, we know that $f \in T^2_c(\mathbb{R}^{n+1}_+)$. This, together Theorem 2.6 and (i), implies that

$$\pi_{\Phi,L,M}(f) = \sum_{j=1}^{\infty} \lambda_j \pi_{\Phi,L,M}(a_j)$$

in $L^2(\mathbb{R}^n)$, where $\{\lambda_j\}_j$ and $\{a_j\}_j$ satisfy (2.5) and (2.6), which, together with the $L^2(\mathbb{R}^n)$-boundedness of S_L, implies that

$$S_L(\pi_{\Phi,L,M} f)(x) \le \sum_{j=1}^{\infty} |\lambda_j| S_L(\pi_{\Phi,L,M}(a_j))(x)$$

for almost every $x \in \mathbb{R}^n$. From this and Lemma 3.13, it follows that, to show (ii), we only need to prove that $\pi_{\Phi,L,M}(a_j)$ is a constant multiple of a $(p,\, q,\, M,\, w)$-atom for each j.

Indeed, we have $\pi_{\Phi,L,M}(a_j) = L^M b_j$, where

$$b_j := c_{\Phi,M} \int_0^\infty t^{2M} t^2 L\Phi(t\sqrt{L})(a_j(\cdot,t))\frac{dt}{t}.$$

Notice that, for each j, there exists some ball B_j such that $\operatorname{supp} a_j \subset \widehat{B_j}$. Therefore, by Lemma 3.11, we see that $\operatorname{supp}(L^k b_j) \subset B_j$ for all $k \in \{0, \ldots, M\}$. Moreover, for any $h \in L^2(\mathbb{R}^n) \cap L^{q'}(\mathbb{R}^n)$ supported in B_j, from Hölder's inequality and Theorem 2.5, we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} (r^2_{B_j} L)^k b_j(x) h(x)\, dx\right| \\
&= r^{2k}_{B_j} \left|\int_{\mathbb{R}^n}\int_0^\infty a_j(y,t) t^{2M+2} L^{k+1}\Phi(t\sqrt{L})h(y) dy\, \frac{dt}{t}\right| \\
&= r^{2k}_{B_j} \left|\int_{\mathbb{R}^n}\int_0^{r_{B_j}} a_j(y,t) t^{2M+2} L^{k+1}\Phi(t\sqrt{L})h(y)\, dy\, \frac{dt}{t}\right| \\
&\le r^{2M}_{B_j} \int_{\mathbb{R}^n}\int_0^\infty |a_j(y,t)| \left|(t^2 L)^{k+1}\Phi(t\sqrt{L})h(y)\right| dy\, \frac{dt}{t} \\
&\le r^{2M}_{B_j} \int_{\mathbb{R}^n} \left\{\int_{\Gamma(x)} |a_j(y,t)| \left|(t^2L)^{k+1}\Phi(t\sqrt{L})h(y)\right| \frac{dydt}{t^{n+1}}\right\} dx \\
&\le r^{2M}_{B_j} \int_{\mathbb{R}^n} \mathcal{A}(a_j)(x)\widetilde{S}^k_L(h)(x)\, dx \\
&\le r^{2M}_{B_j} \|\mathcal{A}(a_j)\|_{L^q(\mathbb{R}^n)} \left\|\widetilde{S}^k_L(h)\right\|_{L^{q'}(\mathbb{R}^n)} \\
&\lesssim r^{2M}_{B_j} |B_j|^{1/q} [w(B_j)]^{-1/p} \|h\|_{L^{q'}(\mathbb{R}^n)},
\end{aligned}$$

where in the last inequality, we used the fact that the operator

$$\widetilde{S}^k_L(g)(x) := \left\{\int_{\Gamma(x)} \left|(t^2L)^{k+1}\Phi(t\sqrt{L})(g)(y)\right|^2 \frac{dy\, dt}{t^{n+1}}\right\}^{1/2} \tag{3.6}$$

is bounded on $L^r(\mathbb{R}^n)$ for all $r \in (p_L, p_L')$ (see Lemma 5.3 below), which implies that

$$\left\|(r_{B_j}^2 L)^k b_j\right\|_{L^q(\mathbb{R}^n)} \lesssim r_{B_j}^{2M} |B_j|^{1/q} [w(B_j)]^{-1/p},$$

and hence $\pi_{\Phi,L,M}(a_j)$ is a constant multiple of a (p, q, M, w)-atom. This finishes the proof of Lemma 3.14. ■

Lemma 3.15. *Let p and w be as in Theorem 3.8. Then $H_{L,w}^{p,s}(\mathbb{R}^n) \subset H_{L,w}^{p,2}(\mathbb{R}^n)$ whenever $s \in (p_L, p_L')$, where p_L is as in* (3.1).

Proof. Let $s \in (p_L, p_L')$ and $f \in H_{L,w}^{p,s}(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$. Then by the definition of $H_{L,w}^{p,s}(\mathbb{R}^n)$, we see that $t^2 L e^{-t^2 L} f \in T_w^p(\mathbb{R}_+^{n+1})$. For each $N \in \mathbb{Z}_+$ and all $x \in \mathbb{R}^n$, we define

$$\begin{aligned} f_N(x) &:= \pi_{\Phi,L,M}\left(t^2 L e^{-t^2 L} f \chi_{\widetilde{O_N}}\right) \\ &= c_{\Phi,M} \int_0^\infty (t^2 L)^{M+1} \Phi(t\sqrt{L}) \left(\left(t^2 L e^{-t^2 L} f\right)(\cdot,t)\chi_{\widetilde{O_N}}\right)(x)\frac{dt}{t}. \end{aligned}$$

By Remark 2.4(ii), we know that $t^2 L e^{-t^2 L} f \chi_{\widetilde{O_N}} \in T^2(\mathbb{R}_+^{n+1}) \cap T_w^p(\mathbb{R}_+^{n+1})$, which, together with Lemma 3.14, implies that $f_N \in H_{L,w}^{p,2}(\mathbb{R}^n)$. Moreover, it follows, from $t^2 L e^{-t^2 L} f \in T_w^p(\mathbb{R}_+^{n+1})$, that

$$\|S_L(f_N - f)\|_{L_w^p(\mathbb{R}^n)} \lesssim \left\|t^2 L e^{-t^2 L} f \chi_{(\widetilde{O_N})^\complement}\right\|_{T_w^p(\mathbb{R}_+^{n+1})} \to 0,$$

as $N \to \infty$. This allows us to conclude that $H_{L,w}^{p,s}(\mathbb{R}^n) \subset H_{L,w}^{p,2}(\mathbb{R}^n)$. ■

Now we prove Theorems 3.4, 3.8 and 3.9 by using Lemmas 3.13, 3.14 and 3.15.

Proofs of Theorems 3.4, 3.8 and 3.9. Thanks to Lemma 3.15, the following three steps suffice to prove these theorems.

Step 1. $H_{L,w}^{p,2}(\mathbb{R}^n) = H_{L,w,\mathrm{at}}^{p,q,M}(\mathbb{R}^n)$ *with equivalent norms.*

Step 2. $H_{L,w}^{p,2}(\mathbb{R}^n) \subset H_{L,w}^{p,s}(\mathbb{R}^n)$ *for all* $s \in (p_L, p_L')$.

Step 3. $H_{L,w}^{p,2}(\mathbb{R}^n) = H_{L,w,\mathrm{mol}}^{p,q,M,\epsilon}(\mathbb{R}^n)$ *with equivalent norms.*

Proof of Step 1. We first prove that $H_{L,w,\mathrm{at}}^{p,q,M}(\mathbb{R}^n) \subset H_{L,w}^{p,2}(\mathbb{R}^n)$ and the inclusion is continuous. Indeed, by their definitions, it is sufficient to show that, for all $f = \sum_{j=1}^\infty \lambda_j a_j$ as in (3.2), where the summation converges in $L^r(\mathbb{R}^n)$ for some $r \in (p_L, p_L')$,

$$\|f\|_{H_{L,w}^{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{H_{L,w,\mathrm{at}}^{p,q,M}(\mathbb{R}^n)}.$$

By the $L^r(\mathbb{R}^n)$-boundedness of S_L, we see that, for each $k \in \mathbb{N}$, $S_L(\sum_{j=1}^k \lambda_j a_j - f)(x) \le \sum_{j=k+1}^\infty |\lambda_j| S_L(a_j)(x)$ for almost every $x \in \mathbb{R}^n$, which, together with Lemma 3.13 and $f \in H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$, implies that $\sum_{j=1}^k \lambda_j a_j - f \in H^{p,\,2}_{L,\,w}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Moreover,

$$\|f\|_{H^{p,\,2}_{L,\,w}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)}$$

and the series $\sum_{j=1}^k \lambda_j a_j$ converges to f in $H^{p,\,2}_{L,\,w}(\mathbb{R}^n)$ as $k \to \infty$.

Conversely, to prove that $H^{p,\,2}_{L,\,w}(\mathbb{R}^n) \subset H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$, by their definitions, it is sufficient to show that, for any $f \in L^2(\mathbb{R}^n) \cap H^{p,\,2}_{L,\,w}(\mathbb{R}^n)$,

$$\|f\|_{H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,\,2}_{L,\,w}(\mathbb{R}^n)}.$$

Indeed, from the $L^2(\mathbb{R}^n)$-boundedness of S_L and the definition of $H^{p,\,2}_{L,\,w}(\mathbb{R}^n)$, it follows that $t^2Le^{-t^2L} \in T^2(\mathbb{R}^{n+1}_+) \cap T^p_w(\mathbb{R}^{n+1}_+)$. Then, by Theorem 2.6, the proof of Lemma 3.14 and (3.5), we see that $f = \pi_{\Phi,\,L,\,M}(t^2Le^{-t^2L}f) \in H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)$. Moreover,

$$\|f\|_{H^{p,\,q,\,M}_{L,\,w,\,\mathrm{at}}(\mathbb{R}^n)} \lesssim \left\|t^2Le^{-t^2L}f\right\|_{T^p_w(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{H^{p,\,2}_{L,\,w}(\mathbb{R}^n)},$$

which ends the proof of Step 1.

Proof of Step 2. For any $f \in L^2(\mathbb{R}^n) \cap H^{p,\,2}_{L,\,w}(\mathbb{R}^n)$, by Step 1, we know that

$$f = \sum_{j=1}^\infty \lambda_j a_j,$$

where $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p$, a_j for each $j \in \mathbb{N}$ is a $(p,\,q,\,M,\,w)$-atom for some $q \in (s, p'_L)$, and the summation converges in $H^{p,\,2}_{L,\,w}(\mathbb{R}^n)$. From the definition of $(p,\,q,\,M,\,w)$-atoms and $q > s$, it follows that, for each $j \in \mathbb{N}$, a_j is also a $(p,\,s,\,M,\,w)$-atom, and hence $\sum_{j=1}^N \lambda_j a_j \in L^s(\mathbb{R}^n)$ for all $N \in \mathbb{N}$, which implies that $f \in H^{p,\,s}_{L,\,w}(\mathbb{R}^n)$ and hence $H^{p,\,2}_{L,\,w}(\mathbb{R}^n) \subset H^{p,\,s}_{L,\,w}(\mathbb{R}^n)$. This finishes the proof of Step 2.

Proof of Step 3. The proof of Step 3 is similar to that of Step 1. We omit the details here and hence complete the proofs of Theorems 3.4, 3.8 and 3.9. ∎

Furthermore, the proofs of Theorems 3.4, 3.8 and 3.9 give the following interesting conclusion whose proof is similar to that of Step 2. We omit the details here again.

Corollary 3.16. *Let L satisfy* (H1) *and* (H2), $p \in (0, 1]$ *and* $w \in RH_{(p'_L/p)'}(\mathbb{R}^n)$, *where p_L is as in* (3.1). *Then, for all $q \in (p_L, p'_L)$, the space $L^q(\mathbb{R}^n) \cap H^p_{L,\,w}(\mathbb{R}^n)$ is dense in $H^p_{L,\,w}(\mathbb{R}^n)$.*

Remark 3.17. Moreover, it is worth pointing out that when L has the kernel satisfying the Gaussian estimate, Corollary 3.16 implies that $L^q(\mathbb{R}^n) \cap H^p_{L,w}(\mathbb{R}^n)$ is dense in $H^p_{L,w}(\mathbb{R}^n)$ whenever $w \in A_\infty(\mathbb{R}^n)$ and $q \in (1,\infty)$.

4. Some Applications

In this section, we study the boundedness of some singular integrals on the weighted Hardy spaces $H^p_{L,w}(\mathbb{R}^n)$. Before going into details, we need the following result.

Lemma 4.1. *Let $p \in (0,1]$, $q \in (p_L, p'_L)$, $w \in RH_{(q/p)'}(\mathbb{R}^n)$ and $M > \frac{n}{2}(\frac{q_w}{p} - \frac{1}{2})$, where p_L and q_w are, respectively, as in* (3.1) *and* (2.2)*.*

(i) *Suppose that T is a linear operator (or nonnegative sublinear operator), which is bounded on $L^r(\mathbb{R}^n)$ for some $r \in (p_L, p'_L)$. If there exists a positive constant C such that, for all $(p,\, q,\, M,\, w)$-atoms a, $\|Ta\|_{L^p_w(\mathbb{R}^n)} \le C$, then T extends to a bounded operator from $H^p_{L,w}(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$.*

(ii) *Suppose that T is a linear operator which is bounded on $L^r(\mathbb{R}^n)$ for some $r \in (p_L, p'_L)$. If there exists a positive constant C such that, for all $(p,\, q,\, M,\, w)$-atoms a,*

$$\|Ta\|_{H^p_{L,w}(\mathbb{R}^n)} \le C,$$

then T extends to a bounded operator on $H^p_{L,w}(\mathbb{R}^n)$.

Since the proof of Lemma 4.1 is quite standard, we omit the details here; see, for example, [35, Lemma 5.1].

4.1. Spectral multiplier theorem on $H^p_{L,w}(\mathbb{R}^n)$

Let L satisfy Assumptions (H1) and (H2), and $E(\lambda)$ be the spectral resolution of L. For any bounded Borel function $F:\ [0,\infty) \to \mathbb{C}$, by using the spectral theorem, it is well known that the *operator*

$$F(L) := \int_0^\infty F(\lambda)\, dE(\lambda)$$

is well defined and bounded on $L^2(\mathbb{R}^n)$. Let ϕ be a nonnegative C_c^∞ function on $\mathbb{R}$ such that

$$\operatorname{supp}\phi \subset (1/4, 1) \text{ and } \sum_{l\in\mathbb{Z}} \phi(2^{-l}\lambda) = 1 \text{ for all } \lambda \in (0,\infty). \tag{4.1}$$

Then the main result of this subsection is the following conclusion.

Theorem 4.2. *Let L be an operator satisfying Assumptions* (H1) *and* (H2)*, $p \in (0,1]$ and $w \in RH_{(p'_L/p)'}(\mathbb{R}^n)$, where p_L is as in* (3.1)*. Suppose that $s \in (n(\frac{q_w}{p} - \frac{1}{r_0}), \infty)$ with q_w as in* (2.2) *and $r_0 := \max\{pr'_w, 2\}$. Then for any Borel function F on $\mathbb{R}$ such that $\sup_{t>0} \|\phi\delta_t F\|_{W^\infty_s(\mathbb{R})} < \infty$, where, ϕ is as in* (4.1)*, $\delta_t F(\lambda) := F(t\lambda)$ for all $t \in (0,\infty)$ and $\lambda \in \mathbb{R}$, and $\|F\|_{W^q_s(\mathbb{R})} := \|(I - d^2/dx^2)^{s/2} F\|_{L^q(\mathbb{R})}$ with $q \in (1,\infty]$, the operator $F(L)$ is bounded on $H^p_{L,w}(\mathbb{R}^n)$.*

Remark 4.3. Let p, L and F be as in Theorem 4.2. It was proved in [8, Theorem 4.9] that the operator $F(L)$ is bounded on $H^p_{L,w}(\mathbb{R}^n)$ for $w \in A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n)$. Moreover, by $p'_L \in (2, \infty)$, we know that $(p'_L/p)' < (2/p)' = 2/(2-p)$, which, together with (ii) and (v) of Lemma 2.1, implies that $A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n) \subset RH_{(p'_L/p)'}(\mathbb{R}^n)$. Thus, Theorem 4.2 essentially improves [8, Theorem 4.9].

To prove Theorem 4.2, we need the following technical lemmas.

Lemma 4.4. *Let $R \in (0, \infty)$ and F be a bounded Borel function with* $\operatorname{supp} F \subset [R/4, R]$. *Assume that p_L is as in* (3.1). *Then for any $p \in (2, p'_L)$, there exists a positive constant C such that, for all balls $B \subset \mathbb{R}^n$, $f \in L^2(B)$ and $j \in \mathbb{Z}_+$,*

$$\left\|F(\sqrt{L})f\right\|_{L^q(S_j(B))} \le CR^{n(\frac{1}{2}-\frac{1}{q})}\|f\|_{L^2(B)}\|F\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. For all $\lambda \in \mathbb{R}$, let $G(\lambda) := e^{\lambda^2/R^2}F(\lambda)$. Then by the functional calculus of L, we know that $F(\sqrt{L}) = G(\sqrt{L})e^{-\frac{1}{R^2}L}$. Thus, for all $f \in L^2(B)$,

$$\begin{aligned}
&\left\|F(\sqrt{L})f\right\|_{L^q(S_j(B))}\\
&\le \left\|G(\sqrt{L})e^{-\frac{1}{R^2}L}f\right\|_{L^q(S_j(B))} \lesssim \left\|G(\sqrt{L})e^{-\frac{1}{R^2}L}\right\|_{L^2\to L^q}\|f\|_{L^2(B)}\\
&\lesssim \left\|G(\sqrt{L})\right\|_{L^2\to L^2}\left\|e^{-\frac{1}{R^2}L}\right\|_{L^2\to L^q}\|f\|_{L^2(B)}\\
&\lesssim R^{n(\frac{1}{2}-\frac{1}{q})}\|G\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(B)} \sim R^{n(\frac{1}{2}-\frac{1}{q})}\|F\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(B)},
\end{aligned}$$

which completes the proof of Lemma 4.4. ■

Lemma 4.5. *Let p_L be as in* (3.1) *and $q \in [2, p'_L)$. Then there exist two positive constants C and c such that, for all closed sets E, $F \subset \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$ with* $\operatorname{supp} f \subset E$, *and $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$,*

$$\left\|e^{-zL}f\right\|_{L^q(F)} \le C(|z|\cos\theta)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{q})}\exp\left\{-\frac{[d(E,F)]^2}{c|z|}\cos\theta\right\}\|f\|_{L^2(E)},$$

where $\theta := \arg z$.

The proof of Lemma 4.5 depends on a Phragmén-Lindelöf type theorem (see, for example, [41, Lemma 6.18]), which extends the estimates for the semigroup on real times to complex times. For more details, we refer to the proof of (3.8) in [30] and [19, 7].

Lemma 4.6. *Let R, $s \in (0, \infty)$ and $q \in [2, p'_L)$, where p'_L is as in* (3.1). *For any $\epsilon \in (0, \infty)$, there exists a positive constant $C := C(\epsilon, s)$, depending on ϵ and s, such* *that, for all balls* $B := B(x_B, r_B) \subset \mathbb{R}^n$*,* $j \in \mathbb{N}$ *with* $j \geq 3$*,* $f \in L^2(B)$ *and bounded Borel functions* F *on* $\mathbb{R}$ *supported in* $[R/4, R]$*,*

$$\left\|F(\sqrt{L})f\right\|_{L^q(S_j(B))} \leq C\frac{R^{n(\frac{1}{2}-\frac{1}{q})}}{(2^j r_B R)^s}\|\delta_R F\|_{W^\infty_{s+\epsilon}(\mathbb{R})}\|f\|_{L^2(B)}. \tag{4.2}$$

Proof. Using the Fourier inversion transform formula and the functional calculus of L, we have

$$G(L/R^2)e^{-\frac{1}{R^2}L} = c\int_{\mathbb{R}} e^{-\frac{1-i\tau}{R^2}L}\widehat{G}(\tau)d\tau,$$

where the function G is defined by $G(\cdot) := [\delta_R F](\sqrt{\cdot})e^{\cdot}$, c is a positive constant and $\widehat{G}$ denotes the *Fourier transform* of G. Thus,

$$F(\sqrt{L})f = c\int_{\mathbb{R}} \widehat{G}(\tau)e^{-\frac{1-i\tau}{R^2}L}f d\tau.$$

Applying Lemma 4.5, we see that, for all $f \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B$ and $j \in \mathbb{N}$ with $j \geq 3$,

$$\begin{aligned}
&\left\|F(\sqrt{L})f\right\|_{L^q(S_j(B))}\\
&\lesssim \int_{\mathbb{R}}\left|\widehat{G}(\tau)\right|\left\|e^{-\frac{1-i\tau}{R^2}L}f\right\|_{L^q(S_j(B))}d\tau\\
&\lesssim R^{n(\frac{1}{2}-\frac{1}{q})}\int_{\mathbb{R}}\left|\widehat{G}(\tau)\right|\exp\left\{-c\frac{(2^j r_B R)^2}{(1+\tau^2)}\right\}d\tau\|f\|_{L^2(B)}\\
&\lesssim R^{n(\frac{1}{2}-\frac{1}{q})}\|f\|_{L^2(B)}\int_{\mathbb{R}}\left|\widehat{G}(\tau)\right|\frac{(1+\tau^2)^{s/2}}{(2^j r_B R)^s}d\tau\\
&\lesssim \frac{R^{n(\frac{1}{2}-\frac{1}{q})}}{(2^j r_B R)^s}\|f\|_{L^2(B)}\left\{\int_{\mathbb{R}}|\widehat{G}(\tau)|^2(1+\tau^2)^{s+\epsilon+1/2}d\tau\right\}^{1/2}\\
&\quad\times\left\{\int_{\mathbb{R}}(1+\tau^2)^{-\epsilon-1/2}d\tau\right\}^{1/2} \lesssim \frac{R^{n(\frac{1}{2}-\frac{1}{q})}}{(2^j r_B R)^s}\|G\|_{W^2_{s+\epsilon+1/2}(\mathbb{R})}\|f\|_{L^2(B)}.
\end{aligned}$$

Moreover, $\operatorname{supp} F \subset [R/4, R]$ implies that

$$\|G\|_{W^2_{s+\epsilon+1/2}(\mathbb{R})} \lesssim \|\delta_R F\|_{W^2_{s+\epsilon+1/2}(\mathbb{R})} \lesssim \|\delta_R F\|_{W^\infty_{s+\epsilon+1/2}(\mathbb{R})},$$

and hence

$$\left\|F(\sqrt{L})f\right\|_{L^q(S_j(B))} \lesssim \frac{R^{n(\frac{1}{2}-\frac{1}{q})}}{(2^j r_B R)^s}\|\delta_R F\|_{W^\infty_{s+\epsilon+1/2}(\mathbb{R})}\|f\|_{L^2(B)}. \tag{4.3}$$

To replace $W^{\infty}_{s+\epsilon+1/2}(\mathbb{R})$ by $W^{\infty}_{s+\epsilon}(\mathbb{R})$ on the right-hand side of (4.3), we use the interpolation arguments as in [38, 21] (see also [7]). Since the proof is very similar to that in [21, 1]. We omit details here. This finishes the proof of Lemma 4.6. ∎

Now we prove Theorem 4.2 by using Lemmas 4.4 through 4.6.

Proof of Theorem 4.2. By Lemma 2.1(iv), we know that there exists $q \in [r_0, p_L')$ such that $w \in RH_{(q/p)'}(\mathbb{R}^n)$ and $s > n(\frac{q_w}{p} - \frac{1}{q})$. Since the condition $\sup_{t>0} \|\eta\delta_t F\|_{W^{\infty}_s(\mathbb{R})} < \infty$ is invariant under the change of variable $\lambda \mapsto \sqrt{\lambda}$ and independent of the choice of η, the $H^p_{L,w}(\mathbb{R}^n)$-boundednesses of $F(L)$ and $F(\sqrt{L})$ are equivalent. Thus, instead of proving the $H^p_{L,w}(\mathbb{R}^n)$-boundedness of $F(L)$, we show that $F(\sqrt{L})$ is bounded on $H^p_{L,w}(\mathbb{R}^n)$. Due to Theorem 3.9, it suffices to prove that there exists $\epsilon \in (0,\infty)$ such that, for any $(p,\,q,\,2M,\,w)$-atom $a = L^{2M}b$ with $M \in \mathbb{N}$ and $M > \frac{n}{2}(\frac{q_w}{p} - \frac{1}{2})$, the function

$$\widetilde{a} := F(\sqrt{L})a = L^M[F(\sqrt{L})L^M b]$$

is a multiple of a $(p,\,q,\,M,\,w,\,\epsilon)$-molecule associated with the ball B. To this end, it suffices to prove that, for all $k \in \mathbb{Z}_+$ and $l \in \{0,\,\ldots,\,M\}$,

$$\left\|(r_B^2 L)^l F(\sqrt{L})L^M b\right\|_{L^q(S_k(B))} \lesssim 2^{-k\epsilon} r_B^{2M} |2^k B|^{1/q} [w(2^k B)]^{-1/p}. \tag{4.4}$$

When $k \in \{0,\,1,\,2\}$, by the $L^q(\mathbb{R}^n)$-boundedness of $F(\sqrt{L})$ with $q \in (p_L, p_L')$ (see [24]), we know that, for all $l \in \{0,\,\ldots,\,M\}$,

$$\begin{aligned}&\left\|(r_B^2 L)^l F(\sqrt{L})L^M b\right\|_{L^q(S_k(B))}\\ &\quad\lesssim \left\|(r_B^2 L)^l L^M b\right\|_{L^q(S_k(B))} \lesssim 2^{-k\epsilon} r_B^{2M} |2^k B|^{1/q} [w(2^k B)]^{-1/p}.\end{aligned}$$

Now we prove (4.4) for all $k \in \mathbb{N}$ with $k \geq 3$. To do this, using the argument as in [8, 25], we fix a function $\phi \in C_c^{\infty}(\frac{1}{4}, 1)$ such that, for all $\lambda \in (0,\infty)$,

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}\lambda) = 1.$$

Let j_0 be the smallest integer such that $2^{j_0} r_B \geq 1$. Then, for all $l \in \{0,\,\ldots,\,M\}$, we have

$$\begin{aligned}(r_B^2 L)^l F(\sqrt{L})\widetilde{b} &= r_B^{2l} \sum_{j\geq j_0} \phi(2^{-j}\sqrt{L})F(\sqrt{L})L^{l+M} b\\ &\quad+ r_B^{2l} \sum_{j<j_0} \phi(2^{-j}\sqrt{L})L^M F(\sqrt{L})L^l b,\end{aligned} \tag{4.5}$$

where $\widetilde{b} := L^M b$.

Let $b_1 := L^{l+M} b$, $b_2 := L^l b$ and

$$F_j(\lambda) := \begin{cases} F(\lambda)\phi(2^{-j}\lambda), & j \geq j_0, \\ F(\lambda)(2^{-j}\lambda)^{2M}\phi(2^{-j}\lambda), & j < j_0. \end{cases}$$

Then, by Hölder's inequality and the definitions of b_1 and b_2, we see that

$$\|b_1\|_{L^2(B)} \leq r_B^{2M-2l}|B|^{1/2}[w(B)]^{-1/p} \text{ and } \|b_2\|_{L^2(B)} \leq r_B^{4M-2l}|B|^{1/2}[w(B)]^{-1/p}.$$

Moreover, we can rewrite (4.5) as follows:

$$(r_B^2 L)^l F(\sqrt{L})\widetilde{b} = r_B^{2l} \sum_{j \geq j_0} F_j(\sqrt{L}) b_1 + r_B^{2l} 2^{2jM} \sum_{j<j_0} F_j(\sqrt{L}) b_2, \tag{4.6}$$

which implies that, for all $k \in \mathbb{N}$ with $k \geq 3$,

$$\begin{aligned} &\left\|(r_B^2 L)^l F(\sqrt{L})\widetilde{b}\right\|_{L^q(S_k(B))} \\ &\leq r_B^{2l} \sum_{j \geq j_0} \left\|F_j(\sqrt{L}) b_1\right\|_{L^q(S_k(B))} + r_B^{2l} 2^{2jM} \sum_{j<j_0} \left\|F_j(\sqrt{L}) b_2\right\|_{L^q(S_k(B))}. \end{aligned}$$

Take $\widetilde{s} \in (n[\frac{q_w}{p} - \frac{1}{q}], s)$ and $\epsilon \in (0, \widetilde{s} - n[\frac{q_w}{p} - \frac{1}{q}])$. By the definition of q_w, we know that there exists $\widetilde{q} \in (q_w, \infty)$ such that $\widetilde{s} > n[\frac{\widetilde{q}}{p} - \frac{1}{q}]$, $\epsilon < \widetilde{s} - n[\frac{\widetilde{q}}{p} - \frac{1}{q}]$ and $w \in A_{\widetilde{q}}(\mathbb{R}^n)$. We first estimate $\|F_j(\sqrt{L})b_1\|_{L^q(S_k(B))}$ for all $j \geq j_0$. Since $\operatorname{supp} F_j \subset [R/4, R]$ with $R := 2^j$, from Lemma 4.6, it follows that, for all $k \in \mathbb{N}$ with $k \geq 3$,

$$\begin{aligned} &\left\|F_j(\sqrt{L}) b_1\right\|_{L^q(S_k(B))} \\ &\leq 2^{jn(1/2-1/q)}\|b_1\|_{L^2(B)}(2^{j+k}r_B)^{-\widetilde{s}}\,\|\delta_{2^j} F_j\|_{W_s^\infty(\mathbb{R})} \\ &\lesssim r_B^{2M-2l} 2^{jn(1/2-1/q)}(2^{j+k}r_B)^{-\widetilde{s}}|B|^{1/2}[w(B)]^{-1/p}\,\|\phi\delta_{2^j} F\|_{W_s^\infty(\mathbb{R})} \\ &\lesssim r_B^{2M-2l} 2^{-k\widetilde{s}}|B|^{1/q}[w(B)]^{-1/p}(2^j r_B)^{-\widetilde{s}+n(1/2-1/q)}, \end{aligned} \tag{4.7}$$

which implies that, for all $k \in \mathbb{N}$ with $k \geq 3$,

$$\begin{aligned} r_B^{2l} \sum_{j \geq j_0} \left\|F_j(\sqrt{L}) b_1\right\|_{L^q(S_k(B))} &\lesssim 2^{-k\widetilde{s}} r_B^{2M}|B|^{1/q}[w(B)]^{-1/p} \\ &\lesssim 2^{-k(\widetilde{s}+\frac{n}{q}-\frac{n\widetilde{q}}{p})} r_B^{2M}|2^k B|^{1/q}[w(2^k B)]^{-1/p} \\ &\lesssim 2^{-k\epsilon} r_B^{2M}|2^k B|^{1/q}[w(2^k B)]^{-1/p}. \end{aligned} \tag{4.8}$$

For $j \in \mathbb{Z}$ with $j < j_0$, repeating the argument above, we see that, for all $k \in \mathbb{N}$ with $k \geq 3$,

$$\begin{aligned}
& r_B^{2l} 2^{2jM} \sum_{j<j_0} \left\| F_j(\sqrt{L}) b_2 \right\|_{L^q(S_k(B))} \\
&\lesssim \sum_{j<j_0} 2^{-k\widetilde{s}} (2^j r_B)^{2M-\widetilde{s}+n(1/2-1/q)} r_B^{2M} |B|^{1/q} [w(B)]^{-1/p} \\
&\lesssim 2^{-k\epsilon} r_B^{2M} |B|^{1/q} [w(B)]^{-1/p} \\
&\lesssim 2^{-k(\widetilde{s}+\frac{n}{q}-\frac{n\widetilde{q}}{p})} r_B^{2M} |2^k B|^{1/q} [w(2^k B)]^{-1/p} \\
&\lesssim 2^{-k\epsilon} r_B^{2M} |2^k B|^{1/q} [w(2^k B)]^{-1/p},
\end{aligned}$$

which, together with (4.6) and (4.8), implies that (4.4) holds true for all $k \in \mathbb{N}$ with $k \geq 3$. Thus, $\widetilde{a} := F(\sqrt{L})a$ is a multiple of a $(p,\, q,\, w,\, \epsilon)$-molecule, which completes the proof of Theorem 4.2. ■

4.2. Square functions

Let L satisfy Assumptions (H1) and (H2), and $k \in \mathbb{N}$. For all functions $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the *square function* $G_{k,\,L}$ by

$$G_{k,\,L}(f)(x) := \left\{ \int_0^\infty \left| (t^2 L)^k e^{-t^2 L} f(x) \right|^2 \frac{dt}{t} \right\}^{1/2}. \tag{4.9}$$

It is well known that for every $k \in \mathbb{N}$, $G_{k,\,L}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (p_L, p_L')$ (see, for example, [1]).

The main result of this subsection is the following.

Theorem 4.7. *Let L satisfy Assumptions* (H1) *and* (H2)*, and $k \in \mathbb{N}$. Then, for any $p \in (0,1]$ and $w \in RH_{(p_L'/p)'}(\mathbb{R}^n)$, where p_L is as in* (3.1)*, $G_{k,\,L}$ is bounded from $H^p_{L,\,w}(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$.*

Proof. Let $T := G_{k,\,L}$. By [32, Theorem 3.3], we know that, for any closed subsets $E,\, F \subset \mathbb{R}^n$ with $d(E,F) > 0$, $f \in L^2(E)$ with $\operatorname{supp} f \subset E$, $M \in \mathbb{N}$ and $t \in (0,\infty)$,

$$\left\| T(I - e^{-tL})^M f \right\|_{L^2(F)} \lesssim \left\{ \frac{t}{[d(E,F)]^2} \right\}^M \|f\|_{L^2(E)}$$

and

$$\left\| T(tLe^{-tL})^M f \right\|_{L^2(F)} \lesssim \left\{ \frac{t}{[d(E,F)]^2} \right\}^M \|f\|_{L^2(E)}.$$

From Assumptions (H1) and (H2), we deduce that, for any $q \in [2, p_L')$, $(I - e^{-tL})^M$ and $(tLe^{-tL})^M$ are bounded on $L^q(\mathbb{R}^n)$. Thus, $T(I - e^{-tL})^M$ and $T(tLe^{-tL})^M$ are also bounded on $L^q(\mathbb{R}^n)$ for all $q \in [2, p'_L)$. This, together with the interpolation, implies that, for all closed sets E, $F \subset \mathbb{R}^n$ with $d(E, F) > 0$, $f \in L^r(E)$ with $\operatorname{supp} f \subset E$ and $r \in [2, p'_L)$, and $t \in (0, \infty)$,

$$\left\|T(I - e^{-tL})^M f\right\|_{L^r(F)} \lesssim \left\{\frac{t}{[d(E, F)]^2}\right\}^M \|f\|_{L^r(E)} \tag{4.10}$$

and

$$\left\|T(tLe^{-tL})^M f\right\|_{L^r(F)} \lesssim \left\{\frac{t}{[d(E, F)]^2}\right\}^M \|f\|_{L^r(E)}. \tag{4.11}$$

It is worth pointing out that the exponents of $\frac{t}{[d(E,F)]^2}$ in (4.10) and (4.11) may not be equal. However, without loss of generality, for simplicity, we may assume that these two exponents are equal.

By Lemma 2.1(iv), we see that there exists $q \in [2, p'_L)$ such that $w \in RH_{(q/p)'}(\mathbb{R}^n)$. To end the proof of Theorem 4.7, due to Lemma 4.1, we need to prove that, for all $(p,\, q,\, M,\, w)$-atoms a with $M \in \mathbb{N}$ and $M > \frac{n}{2}(\frac{q_w}{p} - \frac{1}{2})$, $\|T(a)\|_{L^p_w(\mathbb{R}^n)} \lesssim 1$.

Let a be a $(p,\, q,\, M,\, w)$-atom associated with a ball $B := B(x_B, r_B)$. We write

$$\begin{aligned}\|Ta\|^p_{L^p_w(\mathbb{R}^n)} \leq &\int_{\mathbb{R}^n} \left|T\left(\left[I - e^{r_B^2 L}\right]^M a\right)(x)\right|^p w(x)\, dx\\ &+ \int_{\mathbb{R}^n} \left|T\left(\left[I - (I - e^{r_B^2 L})^M\right] L^M b\right)(x)\right|^p w(x)\, dx =: \mathrm{I} + \mathrm{II},\end{aligned}$$

where $a = L^M b$.

The remainder of the proof is standard; see, for example, [8, 32]. For the sake of completeness, we sketch the proof here.

For the term I, by Hölder's inequality and the fact that $w \in RH_{(q/p)'}(\mathbb{R}^n)$, we conclude that

$$\begin{aligned}\mathrm{I} &\leq \sum_{k=0}^{\infty} \int_{S_k(B)} \left|T\left(\left[I - e^{r_B^2 L}\right]^M a\right)(x)\right|^p w(x)\, dx\\ &\leq \sum_{k=0}^{\infty} \left\|T\left(\left[I - e^{r_B^2 L}\right]^M a\right)\right\|^p_{L^q(S_k(B))} |2^k B|^{-q/p} w(2^k B) =: \sum_{k=0}^{\infty} \mathrm{I}_k.\end{aligned} \tag{4.12}$$

When $k \in \{0,\, 1,\, 2\}$, from the $L^q(\mathbb{R}^n)$-boundedness of $T(I - e^{r_B^2 L})^M$ and Lemma 2.2, it follows that

$$\mathrm{I}_k \lesssim \|a\|^p_{L^q(B)} |2^k B|^{-q/p} w(2^k B) \lesssim |B|^{p/q} [w(B)]^{-1} |2^k B|^{-q/p} w(2^k B) \lesssim 1. \tag{4.13}$$

By $M > \frac{n}{2}(\frac{q_w}{p} - \frac{1}{2})$ and the definition of q_w, we know that there exists $\widetilde{q} \in (q_w, \infty)$ such that $w \in A_{\widetilde{q}}(\mathbb{R}^n)$ and $M > \frac{n}{2}(\frac{\widetilde{q}}{p} - \frac{1}{2})$. When $k \in \mathbb{N}$ and $k \geq 3$, by (4.10), we see that

$$\left\|T\left(\left[I-e^{r_B^2 L}\right]^M a\right)\right\|_{L^q(S_k(B))} \lesssim 2^{-2Mk}\|a\|_{L^q(B)} \lesssim 2^{-2Mk}|B|^{1/q}[w(B)]^{-1/p},$$

which, together with Lemma 2.2, implies that

$$\mathrm{I}_k \lesssim 2^{-2Mpk}|B|^{p/q}[w(B)]^{-1}|2^k B|^{-p/q}w(2^k B) \lesssim 2^{-k(2Mp+np/q-n\widetilde{q})}.$$

From this, the fact that $M > \frac{n}{2}(\frac{\widetilde{q}}{p}-\frac{1}{q})$, (4.12) and (4.13), it follows that $\mathrm{I} \le \sum_{k=0}^{\infty} \mathrm{I}_k \lesssim 1$.

For the term II, the same argument as above gives

$$\mathrm{II} \le \sum_{k=0}^{\infty} \left\|T\left(\left[I-(I-e^{r_B^2 L})^M\right] L^M b\right)\right\|^p_{L^q(S_k(B))} |2^k B|^{-p/q}w(2^k B) =: \sum_{k=0}^{\infty} \mathrm{II}_k.$$

Moreover, we have

$$I-(I-e^{r_B^2 L})^M = \sum_{l=1}^{M} c_l e^{-lr_B^2 L},$$

where $c_l = (-1)^{l+1}\frac{M!}{(M-l)!l!}$. Therefore,

$$\begin{aligned}
&\mathrm{II}_k\\
&\lesssim \sup_{1\le l\le M} \left\|Te^{-lr_B^2 L}L^M b\right\|^p_{L^q(S_k(B))} |2^k B|^{-p/q}w(2^k B)\\
&\lesssim \sup_{1\le l\le M} \left\|T\left(\frac{l}{M}r_B^2 L e^{-\frac{l}{M}r_B^2 L}\right)^M (r_B^{-2}L^{-1})^M L^M b\right\|_{L^q(S_k(B))} |2^k B|^{-p/q}w(2^k B).
\end{aligned}$$

At this point, by the same argument as in the estimate I_k, we also conclude that $\mathrm{II} \lesssim 1$, which completes the proof of Theorem 4.7. ■

Remark 4.8. Let p, L and $G_{1,L}$ be as in Theorem 4.7. It was proved in [45, Theorem 6.3] that $G_{1,L}$ is bounded from $H^p_{L,w}(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ when $w \in RH_{2/(2-p)}(\mathbb{R}^n)$. From $p'_L \in (2,\infty)$, it follows that $(p'_L/p)' < (2/p)' = 2/(2-p)$ and hence $RH_{2/(2-p)}(\mathbb{R}^n) \subset RH_{(p'_L/p)'}(\mathbb{R}^n)$. Thus, Theorem 4.7 improves [45, Theorem 6.3] when p, L and $G_{1,L}$ are as in Theorem 4.7.

4.3. Riesz transforms associated with Schrödinger operators

Let $L := -\Delta + V$, where $-\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator on $\mathbb{R}^n$ and $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

It is well known that the kernels $\{p_t\}_{t>0}$ associated with the semigroup $\{e^{-tL}\}_{t>0}$ satisfy the *Gaussian upper bounds estimates*, namely, for almost every x, $y \in \mathbb{R}^n$ and all $t \in (0,\infty)$,

$$0 \le p_t(x,y) \le \frac{1}{(4\pi t)^{n/2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\}.$$

It is easy to see that L satisfies Assumptions (H1) and (H2) with $p_L = 1$.

We consider the Riesz transform associated with L defined by

$$\nabla L^{-1/2} f := \frac{1}{2\sqrt{\pi}} \int_0^\infty \nabla e^{-tL} f \frac{dt}{\sqrt{t}}$$

for all $f \in L^2(\mathbb{R}^n)$. It was proved in [17] that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$ and of weak type $(1,1)$. Thus, by interpolation, $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,2]$. Moreover, if $V \in A_\infty(\mathbb{R}^n)$, then there exists some $p_0 \in (2,\infty)$ such that $\nabla L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, p_0)$; see [2].

In this subsection, we concern the boundedness of $\nabla L^{-1/2}$ on the weighted Hardy space $H^p_{L,w}(\mathbb{R}^n)$. Our first main results are formulated by the following theorem.

Theorem 4.9. *Let $L := -\Delta + V$, where $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that $\nabla L^{-1/2}$ is bounded on $L^r(\mathbb{R}^n)$ for all $r \in (1, p_0)$ with some $p_0 \in (2,\infty)$. Then, for any $p \in (0,1]$ and $w \in RH_{(p_0/p)'}(\mathbb{R}^n)$, $\nabla L^{-1/2}$ is bounded from $H^p_{L,w}(\mathbb{R}^n)$ into $L^p_w(\mathbb{R}^n)$.*

Remark 4.10. In comparison with the results in [8, 45], the range of weights w in Theorem 4.9 is larger than those in [8, Theorem 4.1] and [45, Theorem 7.11]. More precisely, let p and L be as in Theorem 4.9. It was proved, respectively, in [8, Theorem 4.1] and [45, Theorem 7.11] that the operator $\nabla L^{-1/2}$ is bounded from $H^p_{L,w}(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ for $w \in A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n)$ and $w \in RH_{2/(2-p)}(\mathbb{R}^n)$. From the assumption $p_0 \in (2,\infty)$, we deduce that $(p_0/p)' < (2/p)' = 2/(2-p)$ and hence $A_1(\mathbb{R}^n) \cap RH_{2/(2-p)}(\mathbb{R}^n) \subset RH_{2/(2-p)}(\mathbb{R}^n) \subset RH_{(p_0/p)'}(\mathbb{R}^n)$.

Proof of Theorem 4.9. By an argument similar to that of Theorem 4.7, it is sufficient to show that, for any $p \in (1, p_0)$, $M \in \mathbb{N}$, all closed sets E, $F \subset \mathbb{R}^n$ with $d(E,F) > 0$, $f \in L^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subset E$, and $t \in (0,\infty)$,

$$\left\| \nabla L^{-1/2} \left(I - e^{-tL}\right)^M f \right\|_{L^p(F)} \lesssim \left\{ \frac{t}{[d(E,F)]^2} \right\}^M \|f\|_{L^p(E)} \tag{4.14}$$

and

$$\left\| \nabla L^{-1/2} \left(tLe^{-tL}\right)^M f \right\|_{L^p(F)} \lesssim \left\{ \frac{t}{[d(E,F)]^2} \right\}^M \|f\|_{L^p(E)}. \tag{4.15}$$

It is well known that, for any $M \in \mathbb{N}$, all closed sets E, $F \subset \mathbb{R}^n$ and $t \in (0,\infty)$,

$$\left\| \nabla L^{-1/2} (I - e^{-tL})^M f \right\|_{L^2(F)} \lesssim \left\{ \frac{t}{[d(E,F)]^2} \right\}^M \|f\|_{L^2(E)}$$

and

$$\left\| \nabla L^{-1/2} (tLe^{-tL})^M f \right\|_{L^2(F)} \lesssim \left\{ \frac{t}{[d(E,F)]^2} \right\}^M \|f\|_{L^2(E)}$$

(see, for example, [32, 8]).

From $w \in RH_{(p_0/p)'}(\mathbb{R}^n)$ and Lemma 2.1(iv), it follows that there exists $q \in (1, p_0)$ such that $w \in RH_{(q/p)'}(\mathbb{R}^n)$. Moreover, by the assumption that $\nabla L^{-1/2}$ is bounded on $L^r(\mathbb{R}^n)$ for all $r \in (1, p_0)$, we know that $\nabla L^{-1/2}$ is bounded on $L^q(\mathbb{R}^n)$, which, together with the facts that $(I - e^{-tL})^M$ and $(tLe^{-tL})^M$ are bounded on $L^r(\mathbb{R}^n)$ for all $r \in (1, \infty)$, implies that $\nabla L^{-1/2}(I - e^{-tL})^M$ and $\nabla L^{-1/2}(tLe^{-tL})^M$ are bounded on $L^q(\mathbb{R}^n)$. At this stage, using the interpolation, we see that (4.14) and (4.15) hold true. This finishes the proof of Theorem 4.9. ■

Before going into the next result, we would like to recall the classical weighted Hardy space $H^p_w(\mathbb{R}^n)$. In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a non-zero function satisfying the following properties: $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$ and

$$\int_0^\infty \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} = 1$$

for all $\xi \neq 0$. For all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, let $\psi_t(x) := t^{-n}\psi(x/t)$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the *Lusin area function* $S_\psi(f)$ by

$$S_\psi(f)(x) := \left\{\int_{\Gamma(x)} |\psi_t * f(y)| \frac{dy\, dt}{t^{n+1}}\right\}^{1/2}.$$

Then for $p \in (0, 1]$ and $w \in A_\infty(\mathbb{R}^n)$, an $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the *weighted Hardy space* $H^p_w(\mathbb{R}^n)$, if $S_\psi(f) \in L^p_w(\mathbb{R}^n)$; moreover, define $\|f\|_{H^p_w(\mathbb{R}^n)} := \|S_\psi(f)\|_{L^p_w(\mathbb{R}^n)}$.

It is interesting that the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ can be characterized in terms of weighted atoms. Let us review the definition of $(w,\, p,\, q,\, s)$-atoms.

Definition 4.11. Let $p \in (0, 1]$, $q \in [1, \infty]$ with $q > p$ and $w \in A_q(\mathbb{R}^n)$. Assume that $s \in \mathbb{Z}$ satisfies $s \geq \lfloor n(q_w/p - 1)\rfloor$, where q_w is as in (2.2). A function a is called a $(w,\, p,\, q,\, s)$*-atom* associated with the ball B, if the following hold:

(i) $\operatorname{supp} a \subset B$;

(ii) $\|a\|_{L^q_w(\mathbb{R}^n)} \leq [w(B)]^{1/q - 1/p}$;

(iii) for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq s$, $\int_{\mathbb{R}^n} a(x) x^\alpha\, dx = 0$.

The *atomic weighted Hardy space* $H^{p,\,q,\,s}_w(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_j \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$ satisfies $\sum_j |\lambda_j|^p < \infty$ and $\{a_j\}_j$ is a sequence of $(w,\, p,\, q,\, s)$-atoms; moreover, the norm of f is defined by $\|f\|_{H^{p,\,q,\,s}_w(\mathbb{R}^n)} := \inf\{(\sum_j |\lambda_j|^p)^{1/p}\}$, where the infimum is taken over all possible decompositions of f as above.

Recall that for the classical weighted Hardy space $H^p_w(\mathbb{R}^n)$, we have the following atomic characterization (see, for example, [28]).

Lemma 4.12. *Let p, q, s and w be as in Definition 4.11. Then the spaces $H^p_w(\mathbb{R}^n)$ and $H^{p,\,q,\,s}_w(\mathbb{R}^n)$ coincide with equivalent norms.*

Now we state another main result of this subsection.

Theorem 4.13. *Let $L := -\Delta + V$, where $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Assume that $\nabla L^{-1/2}$ is bounded on $L^r(\mathbb{R}^n)$ for all $r \in (1, p_0)$ with some $p_0 \in (2, \infty)$. Then, for any $p \in (\frac{n}{n+1}, 1]$ and $w \in A_{q_0}(\mathbb{R}^n) \cap RH_{(p_0/q_0)'}(\mathbb{R}^n)$ with any $q_0 \in [1, \frac{p(n+1)}{n})$, $\nabla L^{-1/2}$ is bounded from $H^p_{L,\,w}(\mathbb{R}^n)$ into $H^p_w(\mathbb{R}^n)$.*

Remark 4.14. Let L be as in Theorem 4.13. In [42, Theorem 1.1(ii)], Song and Yan proved that $\nabla L^{-1/2}$ is bounded from $H^1_{L,\,w}(\mathbb{R}^n)$ into $H^1_w(\mathbb{R}^n)$ for $w \in A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n)$. Then, Wang [44, Theorem 1.1] proved that $\nabla L^{-1/2}$ is bounded from $H^p_{L,\,w}(\mathbb{R}^n)$ into $H^p_w(\mathbb{R}^n)$ for $w \in A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n)$ and $p \in (\frac{n}{n+1}, 1]$. Moreover, it was proved in [45, Theorem 7.15] that $\nabla L^{-1/2}$ is bounded from $H^p_{L,\,w}(\mathbb{R}^n)$ to $H^p_w(\mathbb{R}^n)$ when $p \in (\frac{n}{n+1}, 1]$ and $w \in A_{q_0}(\mathbb{R}^n) \cap RH_{(2/q_0)'}(\mathbb{R}^n)$ with some $q_0 \in [1, \frac{p(n+1)}{n})$. From the assumption $p_0 \in (2, \infty)$, it follows that $(p_0/q_0)' < (2/q_0)' \le 2$ when $q_0 \in [1, \frac{p(n+1)}{n})$, which, together with (ii) and (v) of Lemma 2.1, implies that

$$A_1(\mathbb{R}^n) \cap RH_2(\mathbb{R}^n) \subset A_{q_0}(\mathbb{R}^n) \cap RH_{(2/q_0)'}(\mathbb{R}^n) \subset A_{q_0}(\mathbb{R}^n) \cap RH_{(p_0/q_0)'}(\mathbb{R}^n).$$

Thus, Theorem 4.13 essentially improves these results in [44, 42, 45].

To prove Theorem 4.13, we need a variant notion of $(p,\,q,\,w,\,\epsilon)$-molecules.

Definition 4.15. Let $p \in (0, 1]$, $q \in [1, \infty]$ with $q > p$, $w \in A_q(\mathbb{R}^n)$ and $\epsilon \in (n, \infty)$. A function $m \in L^q(\mathbb{R}^n)$ is called a $(p,\,q,\,w,\,\epsilon)$-*molecule* associated with the ball B if the following hold:

(i) for any $j \in \mathbb{Z}_+$, $\|m\|_{L^q(S_j(B))} \le 2^{-j\epsilon}|2^jB|^{1/q}[w(2^jB)]^{-1/p}$;

(ii) $\int_{\mathbb{R}^n} m(x)\,dx = 0$.

We have the following conclusion.

Proposition 4.16. *Let $p \in (\frac{n}{n+1}, 1]$ and $q \in [2, \infty]$. If $w \in A_{q_0}(\mathbb{R}^n) \cap RH_{(q/q_0)'}(\mathbb{R}^n)$ with any $q_0 \in [1, \frac{p(n+1)}{n})$, then there exists a positive constant C such that, for all $(p,\,q,\,w,\,\epsilon)$-molecules m with $\epsilon \in (n, \infty)$, $\|m\|_{H^p_w(\mathbb{R}^n)} \le C$.*

Proof. Let $\epsilon \in (n, \infty)$ and m be a $(p,\,q,\,w,\,\epsilon)$-molecule associated with the ball B. To prove this result, we follow the structure as in [12] (see also [45]). For completeness, we sketch the proof here.

For each $j \in \mathbb{Z}_+$, let $\alpha_j := \int_{S_j(B)} m(x)\,dx$ and $\chi_j := \frac{1}{|S_j(B)|}\chi_{S_j(B)}$. Then, for each $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, define

$$M_j(x) := m(x)\chi_{S_j(B)}(x) - \alpha_j \chi_j(x).$$

Moreover, for each $j \in \mathbb{Z}_+$, let $N_j = \sum_{k=j}^{\infty} \alpha_k$. Then we have

$$m = \sum_{j=0}^{\infty} M_j + \sum_{j=0}^{\infty} N_{j+1}(\chi_{j+1} - \chi_j) =: \sum_{j=0}^{\infty} M_j + \sum_{j=0}^{\infty} P_j.$$

For each $j \in \mathbb{Z}_+$, it is easy to see that $\int_{\mathbb{R}^n} M_j(x)\,dx = 0$, $\operatorname{supp} M_j \subset \widetilde{B}_j := 2^j B$ and

$$\|M_j\|_{L^q(\mathbb{R}^n)} \le 2\|m\|_{L^q(S_j(B))} \lesssim 2^{-j\epsilon}|B|^{1/q}[w(B)]^{-1/p},$$

which, together with Hölder's inequality, $w \in RH_{(q/q_0)'}(\mathbb{R}^n)$ and Lemma 2.2, implies that

$$\begin{aligned} \|M_j\|_{L^{q_0}_w(\mathbb{R}^n)} &\le \|M_j\|_{L^q(\mathbb{R}^n)} \left\{ \int_{S_j(B)} [w(x)]^{(q/q_0)'}\,dx \right\}^{\frac{1}{q_0 (q/q_0)'}} \\ &\lesssim 2^{-j\epsilon}[w(B)]^{1/q_0 - 1/p}. \end{aligned} \tag{4.16}$$

Therefore, M_j is a multiple of a $(w,\, p,\, q_0,\, 0)$-atom.

Moreover, we also have, for each $j \in \mathbb{Z}_+$, $\int_{\mathbb{R}^n} P_j(x)\,dx = 0$ and $\operatorname{supp} P_j \subset B_j^* := 2^{j+1}B$. Furthermore,

$$\|P_j\|_{L^q(\mathbb{R}^n)} \le |N_{j+1}|(|2^j B|^{1/q-1} + |2^{j+1}B|^{1/q-1}) \lesssim |N_{j+1}||B_j^*|^{1/q-1}.$$

Moreover, by Hölder's inequality and $\epsilon \in (n, \infty)$, we conclude that

$$\begin{aligned} |N_{j+1}| &\le \sum_{k\ge j} \int_{S_k(B)} |m(x)|\,dx \le \sum_{k \ge j} |2^k B|^{1-1/q} \|m\|_{L^q(S_k(B))} \\ &\le \sum_{k\ge j} 2^{-k\epsilon}|2^k B|[w(2^k B)]^{-1/p} \le 2^{-j\epsilon} \sum_{k \ge j} 2^{-(k-j)(\epsilon - n)} |2^j B| [w(B_j^*)]^{-1/p} \\ &\lesssim 2^{-j\epsilon}|B_j^*|[w(B_j^*)]^{-1/p}. \end{aligned}$$

Repeating the estimates in (4.16), we also see that, for each $j \in \mathbb{Z}_+$, P_j is a multiple of a $(w,\, p,\, q_0,\, 0)$-atom. Moreover, since the condition $q_0 \in [1, \frac{p(n+1)}{n})$ implies $\lfloor n(\frac{q_0}{p} - 1)\rfloor = 0$, as a direct consequence of Theorem 4.12, $m \in H^p_w(\mathbb{R}^n)$. Moreover, from the above proof, we easily deduce that $\|m\|_{H^p_w(\mathbb{R}^n)} \lesssim 1$, which completes the proof of Proposition 4.16. ∎

Proof of Theorem 4.13. By [30, Lemma 6.2] and the argument used in Theorem 4.7, we know that, for any $p \in (1, p_0)$, $M \in \mathbb{N}$, all closed sets $E, F \subset \mathbb{R}^n$ with $d(E, F) > 0$, $f \in L^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subset E$, and $t \in (0, \infty)$,

$$\left\|\sqrt{t}\nabla e^{-tL}(f)\right\|_{L^p(F)} \lesssim \left\{\frac{t}{[d(E,F)]^2}\right\}^M \|f\|_{L^p(E)}. \tag{4.17}$$

Let $w \in A_{q_0}(\mathbb{R}^n) \cap RH_{(p_0/q_0)'}(\mathbb{R}^n)$. From Lemma 2.1(iv), we deduce that there exists $q \in (1, p_0)$ such that $w \in A_{q_0}(\mathbb{R}^n) \cap RH_{(q/q_0)'}(\mathbb{R}^n)$. Let a be a $(p,\, q,\, M,\, w)$-atom with $2M + n/q - nq_0/p > n$. By Proposition 4.16, it suffices to show that $\nabla L^{-1/2}(a)$ is a $(p,\, q,\, w,\, \epsilon)$-molecule with $\epsilon \in (n, \infty)$.

Indeed, using an argument as in [30, Theorem 8.6], we have

$$\int_{\mathbb{R}^n} \nabla L^{-1/2}(a)(x)\, dx = 0. \tag{4.18}$$

Thus, in order to prove that $\nabla L^{-1/2}(a)$ is a $(p,\, q,\, w,\, \epsilon)$-molecule, we need to show that $\nabla L^{-1/2}(a)$ satisfies Definition 4.15(i).

When $j \in \{0, \ldots, 3\}$, by the $L^q(\mathbb{R}^n)$-boundedness of $\nabla L^{-1/2}$, we see that

$$\begin{aligned} &\left\|\nabla L^{-1/2}(a)\right\|_{L^q(S_j(B))} \\ &\quad \lesssim \|a\|_{L^q(\mathbb{R}^n)} \lesssim |B|^{1/q}[w(B)]^{1/p} \lesssim 2^{-j\epsilon}|2^jB|^{1/q}[w(2^jB)]^{1/p}. \end{aligned} \tag{4.19}$$

When $j \in \mathbb{N}$ with $j \geq 4$, we write

$$\begin{aligned} &\left\|\nabla L^{-1/2}(a)\right\|_{L^q(S_j(B))} \\ &\quad \lesssim \left\|\int_0^{r_B^2} \sqrt{t}\nabla e^{-tL} a \frac{dt}{t}\right\|_{L^q(S_j(B))} + \left\|\int_{r_B^2}^\infty \sqrt{t}\nabla e^{-tL} a \frac{dt}{t}\right\|_{L^q(S_j(B))} =: \mathrm{I}+\mathrm{II}. \end{aligned} \tag{4.20}$$

For the term I, from Minkowski's inequality, (4.17) and Lemma 2.2, it follows that

$$\begin{aligned} \mathrm{I} &\leq \int_0^{r_B^2} \left\|\sqrt{t}\nabla e^{-tL} a\right\|_{L^q(S_j(B))} \frac{dt}{t} \lesssim \|a\|_{L^q(\mathbb{R}^n)} \int_0^{r_B^2} \frac{t^M}{(2^j r_B)^{2M}} \frac{dt}{t} \\ &\lesssim 2^{-2Mj}|B|^{1/q}[w(B)]^{-1/p} \lesssim 2^{-j(2M+n/q-nq_0/p)}|2^jB|^{1/q}[w(2^jB)]^{-1/p}. \end{aligned} \tag{4.21}$$

For II, by Minkowski's inequality and the semigroup property of $\{e^{-tL}\}_{t>0}$, we know that

$$\begin{aligned} \mathrm{II} &= \left\|\int_{r_B^2}^\infty \sqrt{t}\nabla e^{-tL/2}(tL)^M e^{-tL/2}(b) \frac{dt}{t^{M+1}}\right\|_{L^q(S_j(B))} \\ &\leq \int_{r_B^2}^\infty \left\|\sqrt{t}\nabla e^{-tL/2}(tL)^M e^{-tL/2}(b)\right\|_{L^q(S_j(B))} \frac{dt}{t^{M+1}}. \end{aligned} \tag{4.22}$$

Notice that $\sqrt{t}\nabla e^{-tL/2}(tL)^M e^{-tL/2}$ is bounded on $L^q(\mathbb{R}^n)$. This, together with (4.22) and Lemma 2.2, implies that

$$\begin{aligned}\mathrm{II} &\lesssim \|b\|_{L^q(\mathbb{R}^n)} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+1}} \lesssim 2^{-j2M}|B|^{1/q}[w(B)]^{-1/p}\\ &\lesssim 2^{-j(2M+n/q-nq_0/p)}|2^jB|^{1/q}[w(2^jB)]^{-1/p}.\end{aligned} \tag{4.23}$$

Then by (4.18) through (4.23), we know that $\nabla L^{-1/2}(a)$ is a multiple of a $(p,\, q,\, w,\, \epsilon)$-molecule with $\epsilon \in (n, \infty)$, which completes the proof of Theorem 4.13. ■

5. Appendix

In this appendix, we prove that the square function $\widetilde{S}_L^k$ defined as in (3.6) is bounded on $L^p(\mathbb{R}^n)$ with $p \in (p_L, p_L')$, where p_L is as in (3.1).

Through this appendix, we always assume that L satisfies Assumptions (H1) and (H2).

To begin with, we first recall the definition of Hardy spaces associated with the operator L, introduced in [30]. For $p \in [1, \infty)$, the *Hardy space* $H_L^p(\mathbb{R}^n)$ is defined as the completion of $\{f \in L^2(\mathbb{R}^n) : S_L f \in L^p(\mathbb{R}^n)\}$ in the *norm* $\|f\|_{H_L^p(\mathbb{R}^n)} := \|S_L f\|_{L^p(\mathbb{R}^n)}$, where S_L is the Lusin area function defined as in Section 3.

Remark 5.1. By an argument similar to that used in [33, Section 9], we know that $H_L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all $p \in (p_L, p_L')$.

Now we recall the notion of $(1,\, 2,\, M)$-atoms associated with the operator L.

Definition 5.2. Let $p \in (0, 1]$ and $M \in \mathbb{N}$. A function $a \in L^2(\mathbb{R}^n)$ is called a $(1,\, 2,\, M)$-*atom* associated with the operator L, if there exists a function b which belongs to $D(L^M)$, the *domain* of L^M, and a ball $B \subset \mathbb{R}^n$ such that

(i) $a = L^M b$;

(ii) $\operatorname{supp}(L^k b) \subset B,\ k \in \{0, \ldots, M\}$;

(iii) $\|(r_B^2 L)^k b\|_{L^2(B)} \le r_B^{2M}|B|^{-1/2},\ k \in \{0, \ldots, M\}$.

In this section, we establish the following useful result.

Lemma 5.3. *Let L satisfy Assumptions* (H1) *and* (H2) *and $k \in \mathbb{N}$. Assume that Φ is as in Lemma 3.11. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the square function $\widetilde{S}_L^k(f)$ by*

$$\widetilde{S}_L^k(f)(x) := \left\{ \int_{\Gamma(x)} \left|(t^2L)^{k+1}\Phi(t\sqrt{L})(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

Then, for all $p \in (p_L, p_L')$, $\widetilde{S}_L^k$ is bounded on $L^p(\mathbb{R}^n)$, where p_L is as in (3.1).

Before going into the proof of this lemma, we first recall the following useful estimate.

Let $\psi : \mathbb{C} \to \mathbb{C}$ satisfy that there exist positive constants C and s such that, for all $z \in \mathbb{C}$,

$$|\psi(z)| \le C\frac{|z|^s}{(1+|z|)^{2s}}.$$

Then there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n)$,

$$\int_0^\infty \int_{\mathbb{R}^n} |\psi(tL)f(y)|^2 \frac{dy\,dt}{t} \le C\|f\|^2_{L^2(\mathbb{R}^n)}; \tag{5.1}$$

see [30, p. 17].

Proof of Lemma 5.3. We first notice that by (5.1), we are easy to know that $\widetilde{S}^k_L$ is bounded on $L^2(\mathbb{R}^n)$. Now we claim that $\widetilde{S}^k_L$ is bounded from $H^1_L(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. To this end, by [30, Lemma 4.3 and Theorem 4.14], it suffices to show that, for all $(1,\,2,\,M)$-atoms a associated with the ball $B := B(x_B, r_B)$, $\|\widetilde{S}^k_L(a)\|_{L^1(\mathbb{R}^n)} \lesssim 1$.

Indeed, we have

$$\left\|\widetilde{S}^k_L(a)\right\|_{L^1(\mathbb{R}^n)} = \sum_{j\in\mathbb{Z}_+} \left\|\widetilde{S}^k_L(a)\right\|_{L^1(S_j(B))}. \tag{5.2}$$

When $j \in \{0,\,\ldots,\,4\}$, it follows, from the $L^2(\mathbb{R}^n)$-boundedness of $\widetilde{S}^k_L$ and Hölder's inequality, that

$$\left\|\widetilde{S}^k_L(a)\right\|_{L^1(S_j(B))} \lesssim \left\|\widetilde{S}^k_L(a)\right\|_{L^2(S_j(B))} |B|^{1/2} \lesssim 1. \tag{5.3}$$

When $j \in \mathbb{N}$ with $j \ge 5$, by Hölder's inequality, we see that

$$\left\|\widetilde{S}^k_L(a)\right\|_{L^1(S_j(B))} \lesssim \left\|\widetilde{S}^k_L(a)\right\|_{L^2(S_j(B))} |2^jB|^{1/2}. \tag{5.4}$$

Furthermore, we write

$$\begin{aligned}
&\left\|\widetilde{S}^k_L(a)\right\|^2_{L^2(S_j(B))}\\
&= \int_{S_j(B)} \int_0^{|x-x_B|/4} \int_{B(x,t)} \left|(t^2L)^{k+1}\Phi(t\sqrt{L})(a)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\,dx \\
&\quad + \int_{S_j(B)}\int_{|x-x_B|/4}^\infty\int_{B(x,t)} \left|(t^2L)^{k+M+1}\Phi(t\sqrt{L})(b)(y)\right|^2 \frac{dy\,dt}{t^{4M+n+1}}\,dx =: \mathrm{I}_1+\mathrm{I}_2,
\end{aligned} \tag{5.5}$$

where $a = L^M b$.

From $\operatorname{supp} a \subset B$, $j \geq 5$ and Lemma 3.11, it follows that, for all $t \in (0, |x - x_B|/4)$, $\operatorname{supp} K_{(t^2L)^{k+1}\Phi(t\sqrt{L})} \cap \operatorname{supp} a = \emptyset$ and hence $\mathrm{I}_1 = 0$.

Now we deal with I_2. By Fubini's theorem and (5.1), we see that

$$\begin{aligned}
\mathrm{I}_2 &\leq \frac{1}{(2^j r_B)^{4M}} \int_{S_j(B)} \int_{|x-x_B|/4}^{\infty} \int_{B(x,t)} \left|(t^2L)^{k+M+1}\Phi(t\sqrt{L})(b)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\,dx \\
&\leq \frac{1}{(2^j r_B)^{4M}} \int_{\mathbb{R}^n} \int_0^{\infty} \int_{B(x,t)} \left|(t^2L)^{k+M+1}\Phi(t\sqrt{L})(b)(y)\right|^2 \frac{dy\,dt}{t^{n+1}} dx \\
&\leq \frac{1}{(2^j r_B)^{4M}} \int_0^{\infty} \int_{\mathbb{R}^n} \left|(t^2L)^{k+M+1}\Phi(t\sqrt{L})b(y)\right|^2 \frac{dy\,dt}{t} \\
&\lesssim \frac{1}{(2^j r_B)^{4M}} \|b\|_{L^2(\mathbb{R}^n)}^2 \lesssim 2^{-4Mj}|B|^{-1},
\end{aligned}$$

which, together with $\mathrm{I}_1 = 0$, (5.2), (5.3), (5.4) and (5.5), implies that

$$\sum_{j\in\mathbb{Z}_+} \left\|\widetilde{S}_L^k(a)\right\|_{L^1(S_j(B))} \lesssim \sum_{j\in\mathbb{Z}_+} 2^{-j(2M-n/2)} \lesssim 1$$

as long as $M > n/4$. Thus, $\widetilde{S}_L^k$ is bounded from $H_L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. At this stage, using the interpolation in [30, Theorem 9.7] and Remark 5.1, we conclude the $L^p(\mathbb{R}^n)$-boundedness of $\widetilde{S}_L^k$ for $p \in (p_L, 2]$.

To prove the $L^p(\mathbb{R}^n)$-boundedness of $\widetilde{S}_L^k$ for $p \in (2, p_L')$, we borrow some ideas from [12]. Let $h \in L^{(p/2)'}(\mathbb{R}^n)$. Then by Fubini's theorem, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\left[\widetilde{S}_L^k(f)(x)\right]^2 h(x)\right| dx \\
&= \int_0^{\infty} \int_{\mathbb{R}^n} \left|(t^2L)^{k+M+1}\Phi(t\sqrt{L})f(y)\right|^2 \left\{\frac{1}{t^n}\int_{B(y,t)} |h(x)|\,dx\right\} \frac{dy\,dt}{t}.
\end{aligned}$$

For each $k \in \mathbb{Z}$, let

$$E_k := \left\{(y,t) \in \mathbb{R}^n \times (0,\infty) :\ 2^k < \frac{1}{t^n}\int_{B(y,t)} |h(x)|\,dx \leq 2^{k+1}\right\}.$$

Obviously, if $(y,t) \in E_k$, then $\mathcal{M}(h)(y) > 2^k$ and

$$(t^2L)^{k+M+1}\Phi(t\sqrt{L})f(y) = (t^2L)^{k+M+1}\Phi(t\sqrt{L})\left(f\chi_{\{x\in\mathbb{R}^n:\ \mathcal{M}(h)(x)>2^k\}}\right)(y),$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function on $\mathbb{R}^n$. From this and (5.1), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \left[\widetilde{S}_L^k(f)(x) \right]^2 h(x) \right| dx \\
&\le \sum_{k\in\mathbb{Z}} 2^{k+1} \int_{E_k} \left| (t^2L)^{k+M+1} \Phi(t\sqrt{L}) f(y) \right|^2 \frac{dy\,dt}{t} \\
&\le \sum_{k\in\mathbb{Z}} 2^{k+1} \int_{E_k} \left| (t^2L)^{k+M+1} \Phi(t\sqrt{L}) \left(f\chi_{\{x\in\mathbb{R}^n:\ \mathcal{M}(h)(x)>2^k\}} \right)(y) \right|^2 \frac{dy\,dt}{t} \\
&\le \sum_{k\in\mathbb{Z}} 2^{k+1} \int_0^\infty \int_{\mathbb{R}^n} \left| (t^2L)^{k+M+1} \Phi(t\sqrt{L}) \left(f\chi_{\{x\in\mathbb{R}^n:\ \mathcal{M}(h)(x)>2^k\}} \right)(y) \right|^2 \frac{dy\,dt}{t} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^k \int_{\mathbb{R}^n} |f(y)|^2 \chi_{\{x\in\mathbb{R}^n:\ \mathcal{M}(h)(x)>2^k\}}(y)\, dy,
\end{aligned}$$

which, together with Hölder's inequality, implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} \left| \left[\widetilde{S}_L^k(f)(x) \right]^2 h(x) \right| dx &\le \sum_{k\in\mathbb{Z}} 2^{k+1} \int_{E_k} \left| (t^2L)^{k+M+1} \Phi(t\sqrt{L}) f(y) \right|^2 \frac{dy\,dt}{t} \\
&\lesssim \|f\|^2_{L^p(\mathbb{R}^n)} \sum_{k\in\mathbb{Z}} 2^k \left| \left\{ x\in\mathbb{R}^n :\ \mathcal{M}(h)(x) > 2^k \right\} \right|^{\frac{1}{(p/2)'}} \\
&\lesssim \|f\|^2_{L^p(\mathbb{R}^n)} \|\mathcal{M}(h)\|_{L^{(p/2)'}(\mathbb{R}^n)} \lesssim \|f\|^2_{L^p(\mathbb{R}^n)} \|h\|_{L^{(p/2)'}(\mathbb{R}^n)}.
\end{aligned}$$

This further implies that $\|[\widetilde{S}_L^k(f)]^2\|_{L^{p/2}(\mathbb{R}^n)} \lesssim \|f\|^2_{L^p(\mathbb{R}^n)}$ or, equivalently, $\|\widetilde{S}_L^k(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, which completes the proof of Lemma 5.3. ■

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