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EXPONENTIAL INTEGRABILITY FOR LOGARITHMIC POTENTIALS OF FUNCTIONS IN GENERALIZED LEBESGUE SPACES $L(\log L)^{q(\cdot)}$ OVER NON-DOUBLING MEASURE SPACES

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Abstract. In this paper, we are concerned with exponential integrability for logarithmic potentials of functions in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}$ over non-doubling measure spaces. Here q satisfies the loglog-Hölder condition.

1. Introduction

The properties of the logarithmic potentials were studied by some authors (see e.g. [7, 8, 9, 10, 12]). Our aim in this paper is to establish exponential integrability for logarithmic potentials of functions in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}$ over non-doubling measure spaces, as an extension of [11, Theorem 8.1] in the Euclidean setting.

We denote by (X, d, μ) a metric measure spaces, where X is a bounded set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_X = \sup\{d(x, y) : x, y \in X\}$. We assume that $0 < d_X < \infty$,

$$\mu(\{x\}) = 0$$

for $x \in X$ and $\mu(B(x, r)) > 0$ for $x \in X$ and $r > 0$ for simplicity. In the present paper, we do not postulate on μ the "so called" doubling condition. Recall that a Radon measure μ is said to be doubling if there exists a constant $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in \operatorname{supp}(\mu)(= X)$ and $r > 0$. Otherwise μ is said to be non-doubling. Assume that there exist positive constants K_0 and s such that, for all balls $B(x, r)$ with center $x \in X$ and of radius $0 < r < d_X$,

$$\mu(B(x, r)) \leq K_0 r^s \tag{1.1}$$

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(see e.g. [1, 5] and [6]).

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. For a survey, see [3] and [4].

In this paper, following Cruz-Uribe and Fiorenza [2], we consider a variable exponent $q(\cdot) : X \to [0,1)$ such that

$$|q(x) - q(y)| \le \frac{C_q}{\log(e + \log(e + 1/d(x,y)))} \qquad \text{for all } x, y \in X \tag{1.2}$$

with a constant $C_q \ge 0$.

Define the norm by

$$\|f\|_{L(\log L)^{q(\cdot)}(X)} = \inf\left\{\lambda > 0 : \int_X \left|\frac{f(x)}{\lambda}\right| \left(\log\left(e + \left|\frac{f(x)}{\lambda}\right|\right)\right)^{q(x)} d\mu(x) \le 1\right\}$$

and denote by $L(\log L)^{q(\cdot)}(X)$ the space of all measurable functions f on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} < \infty$.

We define the logarithmic potential for a locally integrable function f on X by

$$Lf(x) = \int_X \left(\log^+(1/d(x,y))\right) f(y)\, d\mu(y),$$

where $\log^+ r = \max\{0, \log r\}$. Here it is natural to assume that

$$\int_X (\log(e + d(x_0, y)))|f(y)|\, d\mu(y) < \infty \tag{1.3}$$

for some $x_0 \in X$ since this implies

$$\left|\int_X (\log(1/d(x,y)))f(y)\, d\mu(y)\right| < \infty$$

for μ-a.e. in X (see [7, Lemma 1] and [9, Theorem 6.1, Chapter 2]).

In [11], we studied exponential integrability for logarithmic potentials of functions in $L(\log L)^{q(\cdot)}(\mathbf{R}^N)$ in the Euclidean setting. Our main aim in the present paper is to establish exponential integrability for Lf in generalized Lebesgue spaces $L(\log L)^{q(\cdot)}(X)$ over non-doubling measure spaces, as an extension of [11, Theorem 8.1].

Theorem 1.1. *There exist constants $c_1, c_2 > 0$ such that*

$$\int_X \exp\left((c_1 Lf(x))^{1/(1-q(x))}\right) d\mu(x) \le c_2$$

for all nonnegative measurable functions f on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$.

Corollary 1.2. *There exists a constant* $c_3 > 0$ *such that*

$$\int_X \left\{ \exp\left((c_3 Lf(x))^{1/(1-q(x))} \right) - 1 \right\} d\mu(x) \le 1$$

for all nonnegative measurable functions f *on* X *with* $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$.

Our strategy is to give an estimate of Lf by use of a logarithmic type potential

$$\int_X \mu(B(x,4r))^{-1}(\log(e+1/r))^{-\beta} f(y)(\log(e+f(y)))^{q(y)}\, d\mu(y)$$

with $\beta > 1$, which plays a role of maximal functions.

The sharpness of the exponent will be discussed in Section 4.

In the final section, we show the continuity for logarithmic potentials of functions in $L^{p(\cdot)}(\log L)^{r(\cdot)}(X)$ over non-doubling measure spaces, as an extension of [11, Theorem 8.4] and [9, Theorem 9.1, Section 5.9] (see Section 5 for the definition of $L^{p(\cdot)}(\log L)^{r(\cdot)}(X)$). For related results, see [12].

2. Preliminary Lemmas

Throughout this paper, let C denote various positive constants independent of the variables in question.

To prove Theorem 1.1, we estimate Lf by the logarithmic potential

$$J = \int_X \rho_{-\beta}(d(x,y)) g(y)\, d\mu(y),$$

where $\rho_{-\beta}(r) = \mu(B(x,4r))^{-1}(\log(e+1/r))^{-\beta}$ with $\beta > 1$ and $g(y) = f(y)(\log(e+f(y)))^{q(y)}$.

Lemma 2.1. *Let* f *be a nonnegative measurable function on* X *with* $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. *Then there is a constant* $C > 0$ *such that*

$$F \equiv \int_{B(x,\delta)} \rho_{-\beta}(d(x,y)) f(y)\, d\mu(y) \le CJ \left\{ (\log(e+J))^{-q(x)} + (\log(e+1/\delta))^{-q(x)} \right\}$$

for all $x \in X$ *and* $0 < \delta < d_X$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. We have for $k > 0$

$$\begin{aligned} F &\le k \int_{B(x,d_X)} \rho_{-\beta}(d(x,y))\, d\mu(y) \\ &\quad + \int_{B(x,\delta)} \rho_{-\beta}(d(x,y)) f(y) \left(\frac{\log(e+f(y))}{\log(e+k)} \right)^{q(y)} d\mu(y). \end{aligned}$$

Since $\beta > 1$, we have

$$\begin{aligned}
&\int_{B(x,d_X)} \rho_{-\beta}(d(x,y))\, d\mu(y) \\
&= \sum_{j=1}^{\infty} \int_{X\cap(B(x,2^{-j+1}d_X)\setminus B(x,2^{-j}d_X))} \mu(B(x,4d(x,y)))^{-1}(\log(e+1/d(x,y)))^{-\beta}\, d\mu(y) \\
&\le \sum_{j=1}^{\infty} \int_{X\cap(B(x,2^{-j+1}d_X)\setminus B(x,2^{-j}d_X))} \mu(B(x,2^{-j+2}d_X))^{-1}(\log(e+1/(2^{-j+1}d_X)))^{-\beta} d\mu(y) \\
&\le \sum_{j=1}^{\infty} (\log(e+1/(2^{-j+1}d_X)))^{-\beta} \\
&\le C.
\end{aligned}$$

If $J \le \delta^{-1}$, then we set $k = J(\log(e+J))^{-q(x)}$. Since $\delta \le J^{-1}$, we see from (1.2) that

$$(\log(e+k))^{-q(y)} \le C(\log(e+J))^{-q(x)}$$

for $y \in B(x,\delta)$. Consequently it follows that

$$F \le CJ(\log(e+J))^{-q(x)}.$$

If $J > \delta^{-1}$, then we set $k = \delta^{-1}(\log(e+1/\delta))^{-q(x)}$ and obtain

$$\begin{aligned}
F &\le C\left\{\delta^{-1}(\log(e+1/\delta))^{-q(x)} + J(\log(e+1/\delta))^{-q(x)}\right\} \\
&\le CJ(\log(e+1/\delta))^{-q(x)}.
\end{aligned}$$

Now the result follows. ■

Lemma 2.2. *Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. Then there is a constant $C > 0$ such that*

$$\int_{X\setminus B(x,\delta)} \log^{+}(1/d(x,y)) f(y)\, d\mu(y) \le C(\log(e+1/\delta))^{-q(x)+1}$$

for all $x \in X$ and $0 < \delta < d_X$.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \le 1$. Let $0 < \gamma < s$, where s is a constant appearing in (1.1). For $y \in X \setminus B(x,\delta)$ and $0 < \delta < d_X$, set $N(x,y) = d(x,y)^{-\gamma}$. Let j_0 be the smallest integer such that

$2^{j_0}\delta \geq d_X$. We have by (1.1)

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))N(x,y)d\mu(y)\\
&= \sum_{j=1}^{j_0} \int_{X\cap(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta))} \log^+(1/d(x,y))N(x,y)\,d\mu(y)\\
&\leq \sum_{j=1}^{j_0} \int_{X\cap(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta))} \log^+(1/(2^{j-1}\delta))(2^{j-1}\delta)^{-\gamma}d\mu(y)\\
&\leq C\sum_{j=1}^{j_0} \log^+(1/(2^{j-1}\delta))(2^{j-1}\delta)^{s-\gamma}\\
&\leq C
\end{aligned}$$

since $\gamma < s$. Hence, we see from (1.2) that

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\,d\mu(y)\\
&\leq \int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))N(x,y)\,d\mu(y)\\
&\quad + \int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\left(\frac{\log(e+f(y))}{\log(e+N(x,y))}\right)^{q(y)}\,d\mu(y)\\
&\leq C\left\{1+\int_{X\setminus B(x,\delta)}(\log(e+1/d(x,y)))^{-q(y)+1}g(y)\,d\mu(y)\right\}\\
&\leq C\left\{1+(\log(e+1/\delta))^{-q(x)+1}\int_{X\setminus B(x,\delta)} g(y)\,d\mu(y)\right\}\\
&\leq C(\log(e+1/\delta))^{-q(x)+1},
\end{aligned}$$

where $g(y) = f(y)(\log(e+f(y)))^{q(y)}$. Thus this lemma is proved. ∎

3. Proof of Theorem 1.1

Let f be a nonnegative measurable function on X with $\|f\|_{L(\log L)^{q(\cdot)}(X)} \leq 1$. For $x \in X$ and $0 < \delta < d_X$, write

$$\begin{aligned}
Lf(x) &= \int_{B(x,\delta)} \log^+(1/d(x,y))f(y)\,d\mu(y) + \int_{X\setminus B(x,\delta)} \log^+(1/d(x,y))f(y)\,d\mu(y)\\
&= I_1 + I_2.
\end{aligned}$$

For $\beta > 1$, we infer from Lemma 2.1 and (1.1) that

$$\begin{aligned} I_1 &\le C\delta^s(\log(e+1/\delta))^{1+\beta}\int_{B(x,\delta)}\rho_{-\beta}(d(x,y))f(y)\,d\mu(y) \\ &\le C\delta^s(\log(e+1/\delta))^{1+\beta}J\left\{(\log(e+1/\delta))^{-q(x)}+(\log(e+J))^{-q(x)}\right\}. \end{aligned}$$

Hence, in view of Lemma 2.2, we find

$$\begin{aligned} Lf(x) \le C\Big[&\delta^s(\log(e+1/\delta))^{1+\beta}J\left\{(\log(e+1/\delta))^{-q(x)}+(\log(e+J))^{-q(x)}\right\} \\ &+(\log(e+1/\delta))^{-q(x)+1}\Big]. \end{aligned}$$

Now, considering $\delta = \min\{d_X, J^{-1/s}(\log(e+J))^{-\beta/s}\}$, we find

$$Lf(x) \le C(\log(e+J))^{-q(x)+1}.$$

Hence

$$\exp\left((c_1 Lf(x))^{1/(1-q(x))}\right) \le e+J.$$

By using Fubini's theorem, we obtain

$$\begin{aligned} &\int_X \exp\left((c_1 Lf(x))^{1/(1-q(x))}\right)\,d\mu(x) \\ &\le \int_X (e+J)\,d\mu(x) \\ &\le \int_X g(y)\left(\int_X \frac{(\log(e+1/d(x,y)))^{-\beta}}{\mu(B(x,4d(x,y)))}d\mu(x)\right)\,d\mu(y)+C \\ &\le \int_X g(y)\left(\sum_{j=1}^{\infty}\int_{X\cap(B(y,2^{-j+1}d_X)\setminus B(y,2^{-j}d_X))}\frac{(\log(e+1/d(x,y)))^{-\beta}}{\mu(B(y,2d(x,y)))}\,d\mu(x)\right)d\mu(y)+C \\ &\le \int_X g(y)\left(\sum_{j=1}^{\infty}\int_{X\cap(B(y,2^{-j+1}d_X)\setminus B(y,2^{-j}d_X))}\frac{(\log(e+1/(2^{-j+1}d_X)))^{-\beta}}{\mu(B(y,2^{-j+1}d_X))}\,d\mu(x)\right)d\mu(y)+C \\ &\le \int_X g(y)\left(\sum_{j=1}^{\infty}(\log(e+1/(2^{-j+1}d_X)))^{-\beta}\right)\,d\mu(y)+C \\ &\le c_2, \end{aligned}$$

since $\beta > 1$. This completes the proof of the theorem. ∎

4. Sharpness

Let $X = B(0,1) \subset \mathbf{R}^N$ and $q(\cdot) = q$. For $\delta > 0$, consider the function

$$u(x) = \int_{B(0,1)} \log^+(1/|x-y|) f(y)\, dy$$

with

$$f(y) = |y|^{-N} (\log(e/|y|))^{\delta - 2} \qquad \text{for } y \in B(0,1).$$

Then f satisfies

$$\int_{B(0,1)} f(y) \left(\log(e + f(y))\right)^q dy < \infty \tag{4.1}$$

if and only if $\delta - 1 + q < 0$. We see that

$$u(x) \geq C \int_{\{y \in B(0,1/2) : |y| > |x|\}} \log^+(1/|y|) f(y)\, dy \geq C(\log(e/|x|))^\delta$$

for $|x| < 1/2$. Hence, if $\beta\delta > 1$, then

$$\int_{B(0,1)} \exp\left(u(x)^\beta\right) dx = \infty. \tag{4.2}$$

If $\beta > 1/(1-q)$, then we can choose δ such that

$$1/\beta < \delta < 1 - q.$$

In this case, both (4.1) and (4.2) hold. This implies that the exponent $1/(1-q)$ in Theorem 1.1 is sharp.

5. Continuity

In this section, we consider variable exponents $p(\cdot) : X \to [1, \infty)$ and $r(\cdot) : X \to (-\infty, \infty)$ such that

$$-\infty < \inf_{x \in X} r(x) \leq \sup_{x \in X} r(x) < \infty. \tag{5.1}$$

Define the norm by

$$\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} = \inf\left\{\lambda > 0 : \int_X \left|\frac{f(x)}{\lambda}\right|^{p(x)} \left(\log\left(e + \left|\frac{f(x)}{\lambda}\right|\right)\right)^{r(x)} d\mu(x) \leq 1\right\}$$

and denote by $L^{p(\cdot)}(\log L)^{r(\cdot)}(X)$ the space of all measurable functions f on X with $\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} < \infty$.

Theorem 5.1. (cf. [9, Theorem 9.1, Section 5.9]). *Let $p(\cdot)$ and $r(\cdot)$ be two variable exponents on X satisfying (5.1) such that*

$$p(x) > 1 \qquad \text{or} \qquad r(x) \geq 1$$

for all $x \in X$. If f is a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} < \infty$, then Lf is continuous on X.

Proof. Let f be a nonnegative measurable function on X with $\|f\|_{L^{p(\cdot)}(\log L)^{r(\cdot)}(X)} < \infty$. Then note that

$$\int_X f(y)(\log(e + f(y)))\, d\mu(y) < \infty.$$

Hence, it follows from [7, Theorem 1] that Lf is continuous on X by (1.3). ∎

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