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THE GROWTH OF THE SOLUTIONS OF CERTAIN TYPE OF DIFFERENCE EQUATIONS

Xiaoguang Qi, Yong Liu and Lianzhong Yang

Abstract. In this paper, we investigate the growth of meromorphic solutions of the equation: $f(z+n)+\sum_{j=0}^{n-1}\{P_j(e^{A(z)})+Q_j(e^{-A(z)})\}f(z+j)=0$, where $A(z)$, $P_j(z)$ and $Q_j(z)$ are polynomials in z. This article extends earlier results by Li et al [7, 15].

1. Introduction

In this paper, we will assume that the reader is familiar with the fundamental results and the standard notations of the value distribution theory of meromorphic functions(e.g. see [11, 24]). In addition, we denote by $\sigma(f)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ the order, the exponent of convergence of zeros and poles of $f(z)$, respectively.

The foundation of the theory of complex difference equations was laid by Batchelder [1], Nӧrlund [17], and Whittaker [20] in the early twentieth century. Later on, Shimomura [19] and Yanagihara [21, 22, 23] investigated nonlinear complex difference equations from the viewpoint of Nevanlinna theory. Recently, difference counterparts of Nevanlinna theory have been established. The key result is the difference analogue of the lemma on the logarithmic derivative obtained by Halburd-Korhonen [10] and Chiang-Feng [7], independently. Hence, there has been an increasing renewed interest in complex difference equations and difference analogues of Nevanlinna theory, some new results can be seen in [2, 5, 6, 12, 13, 18].

In a recent paper [15], Li et al. obtained results concerning the growth of solutions of the following difference equation.

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Theorem A. *Suppose that $f(z)$ is a nonconstant entire solution of the difference equation*

$$f(z+\eta)-a(z)=e^{P(z)}(f(z)-a(z)), \tag{1.1}$$

where $a(z)$ is an entire function such that $\delta(a)<\delta(f)$, $P(z)$ is a nonconstant polynomial. If $\lambda(f-a)<\delta(f)$, then $\delta(f)=\deg\{P(z)\}+1$.

In fact, equation (1.1) can be changed into the following equation as $a(z)$ is a periodic function with the period η:

$$F(z+\eta)-e^{P(z)}F(z)=0.$$

where $F(z)=f(z)-a(z)$. Noting Theorem A and the above equation, a natural question is: what will happen if $e^{P(z)}$ is replaced with a polynomial of exponential functions. Another reason that we consider this question is that we find a counterexample related to the following theorem:

Theorem B. [7, Theorem 9.2]. *Let $A_0(z), \ldots, A_n(z)$ be entire functions such that there exists an integer $l (0\le l\le n)$ that satisfies*

$$\sigma(A_l)>\max\{\sigma(A_j)\}, \quad 0\le l\le n \quad and \quad j\ne l. \tag{1.2}$$

If $f(z)$ is a meromorphic solution of the difference equation

$$A_n(z)y(z+n)+\cdots+A_1(z)y(z+1)+A_0(z)y(z)=0,$$

then $\sigma(f)\ge\sigma(A_l)+1$.

Example C. $f(z)=e^{z^2}$ is a solution of the difference equation

$$f(z+2)+(e^z+e^{-z})f(z+1)-(e^{4z+4}+e^{3z+1}+e^{z+1})f(z)=0. \tag{1.3}$$

Denote $P_0(\zeta)=-e^4\zeta^4-e\zeta^3-e\zeta$ and $Q_0(\zeta^{-1})=0$; $P_1(\zeta)=\zeta$ and $Q_1(\zeta^{-1})=\zeta^{-1}$. Clearly, the coefficients $P_1(e^z)+Q_1(e^{-z})=e^z+e^{-z}$ and $P_0(e^z)+Q_0(e^{-z})=-(e^{4z+4}+e^{3z+1}+e^{z+1})$ of (1.3) are transcendental entire functions which do not satisfy (1.2). Furthermore, we see deg$P_0>$ degP_1, and $\sigma(f)=\lambda(f-a)=2$ for every nonzero value $a\in\mathbb{C}$.

Due to above considerations, we investigate the following difference equation:

$$f(z+n)+\sum_{j=0}^{n-1}\{P_j(e^{A(z)})+Q_j(e^{-A(z)})\}f(z+j)=0, \tag{1.4}$$

where $P_j(z)$ and $Q_j(z)$ $(j=0, 1, \ldots, n-1)$ are polynomials in z, $A(z)$ is a polynomial of degree k. We obtain the following results.

Theorem 1.1. *Let $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \dots, n-1)$ be polynomials, $A(z) = a_k z^k + a_{k-1} z^{k-1} + \dots + a_0$, ($a_k \neq 0$) be a nonconstant polynomial. If*

$$\deg(P_0) > \deg(P_j) \quad \textit{or} \quad \deg(Q_0) > \deg(Q_j), \quad j = 1, \dots, n-1.$$

Then, each nontrivial meromorphic solution $f(z)$ with finite order of the equation (1.4) satisfies $\sigma(f) = \lambda(f - a) \geq k + 1$, and so f assumes every nonzero complex value $a \in \mathbb{C}$ infinitely often.

Theorem 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. If $f(z)$ is a nontrivial entire solution with finite order of the equation (1.4) that satisfies $\lambda(f) \leq k$, then $\sigma(f) = k + 1$.*

Remark. Example C shows that Theorem 1.1 is sharp. It is also shown that the conclusion both in Theorem 1.1 and Theorem 1.2 may occur.

2. Some Lemmas

Lemma 2.1. [7, Theorem 8.1]. *Let $f(z)$ be a meromorphic function with finite order σ, η be a nonzero complex number, and $\varepsilon > 0$ be given real constants. Then there exits a subset $E \subset (1, \infty)$ of finite logarithmic measure, for all $|z| = r \notin [0, 1] \cup E$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left| \frac{f(z+\eta)}{f(z)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.2. [10, Theorem 3.2]. *Let $w(z)$ be a nonconstant finite order meromorphic solution of $P(z, w) = 0$, where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a(z)$ satisfying $T(r, a) = S(r, w)$, then*

$$m\left(r, \frac{1}{w - a}\right) = S(r, w).$$

Lemma 2.3. [9, Lemma 5]. *Let $g : (0, +\infty) \to R$, $h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ hold for all $r > r_0$.*

Lemma 2.4. [8, Theorem 2.1]. *Let $f(z)$ be a meromorphic function with finite order σ, $\eta \in C$. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of $|z| = r$ of finite logarithmic measure, such that*

$$\frac{f(z+\eta)}{f(z)} = \exp\{\eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})\},$$

holds for $r \notin [0, 1] \cup E$. If $\lambda < 1$, $\beta = \max\{\sigma - 2,\ 2\lambda - 2\}$; and if $\lambda \geq 1$, $\beta = \max\{\sigma - 2,\ \lambda - 1\}$, where $\lambda = \max\{\lambda(f), \lambda(\frac{1}{f})\}$.

Lemma 2.5. [3, Lemma 3.2]. *Let $f(z)$ be a meromorphic function with finite order σ, then for any given $\varepsilon > 0$, there exists a set $E \subset (1, \infty)$ of finite linear measure, such that for all $|z| = r \notin [0, 1] \cup E$, and r sufficiently large,*

$$\exp\{-r^{\sigma+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\sigma+\varepsilon}\}.$$

Using the similar proof as that of Remark 1 of [4], we can obtain the following result:

Lemma 2.6. *Suppose that $f(z)$ is a transcendental entire function with finite order σ, and a set $E \subset (1, \infty)$ has a finite logarithmic measure. Then there exists a sequence of points such that $r_k \notin E$, and for any given $\varepsilon > 0$, as r_k sufficiently large, we have*

$$r_k^{\sigma-\varepsilon} < v(r_k, f) < r_k^{\sigma+\varepsilon},$$

where $v(r, f)$ is the central index of $f(z)$.

Lemma 2.7. [16]. *Let*

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where n is a positive integer and $a_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given $0 < \varepsilon < \frac{\pi}{4n}$, consider $2n$ open angles:

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon, \quad j = 0, \ldots, 2n-1.$$

Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$, when $z \in S_j$ and j is even,

$$Re\{Q(z)\} > \alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n,$$

when $z \in S_j$ and j is odd,

$$Re\{Q(z)\} < -\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n.$$

3. Proof of Theorem 1.1

Suppose that $j = 0, 1, \ldots, n-1$ and

$$P_j(z) = a_{j\,p_j} z^{p_j} + a_{j\,p_{j-1}} z^{p_{j-1}} + \ldots + a_{j\,0},$$

$$Q_j(z) = b_{j\,q_j} z^{q_j} + b_{j\,q_{j-1}} z^{q_{j-1}} + \ldots + b_{j\,0}.$$

Assume that $f(z) \not\equiv 0$ is a solution of the equation (1.4) such that $\sigma(f) = \sigma < \infty$. From Lemma 2.1, we get that, for any given $\varepsilon > 0$, there exits a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$,

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left|\frac{f(z+i)}{f(z)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}, \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Case 1. If $\deg(P_0) > \deg(P_j)(j = 1, 2, \ldots, n-1)$, then we take a suitable z such that $a_k z^k = |a_k| r^k$. Combining (1.4) and (3.1), we have that for all sufficiently large r and $r \notin [0,1] \cup E$, that

$$\begin{aligned}
\left|P_0(e^A) + Q_0(e^{-A})\right| &= |a_{0p_0}|\, e^{p_0 r^k |a_k|}(1 + o(1)) \\
&\leq \left|\frac{f(z+n)}{f(z)}\right| + |P_{n-1}(e^A) + Q_{n-1}(e^{-A})| \left|\frac{f(z+n-1)}{f(z)}\right| + \cdots \\
&\quad + |P_1(e^A) + Q_1(e^{-A})| \left|\frac{f(z+1)}{f(z)}\right| \\
&\leq \exp\{r^{\sigma-1+\varepsilon}\} + |a_{n-1p_{n-1}}| e^{p_{n-1} r^k |a_k|} \exp\{r^{\sigma-1+\varepsilon}\}(1 + o(1)) \\
&\quad + \cdots + |a_{1p_1}| e^{p_1 r^k |a_k|} \exp\{r^{\sigma-1+\varepsilon}\}(1 + o(1)) \\
&\leq nM \exp\{r^{\sigma-1+\varepsilon}\} e^{\max\{p_1, \ldots, p_{n-1}\} r^k |a_k|}(1 + o(1)),
\end{aligned}$$

and $M = \max\{|a_{n-1p_{n-1}}|, \ldots, |a_{1p_1}|, 1\}$. Since $p_0 > \max\{p_1, \ldots, p_{n-1}\} = N$, we have

$$\frac{|a_{0p_0}|}{nM} e^{(p_0 - N)|a_k| r^k}(1 + o(1)) \leq e^{r^{\sigma-1+\varepsilon}}. \tag{3.2}$$

By Lemma 2.3 and (3.2), we have that $\sigma - 1 + \varepsilon \geq k$, which implies $\sigma(f) \geq k + 1$.

Case 2. If $\deg Q_0 > \deg Q_j$, then taking a suitable z such that $a_k z^k = -|a_k| r^k$. Using the similar arguments mentioned above, we also get $\sigma(f) \geq k + 1$.

In the following, we prove that $\sigma(f) = \lambda(f - a) \geq k + 1$, where $a \in \mathbb{C} \backslash \{0\}$. Let

$$P(z, f) = f(z+n) + \sum_{j=0}^{n-1} [P_j(e^{A(z)}) + Q_j(e^{-A(z)})] f(z+j).$$

Clearly,

$$\begin{aligned}
P(z,a) &= a[1 + P_{n-1}(e^{A(z)}) + Q_{n-1}(e^{-A(z)}) + \cdots + P_0(e^{A(z)}) + Q_0(e^{-A(z)})] \\
&\not\equiv 0.
\end{aligned} \tag{3.3}$$

By (3.3) and Lemma 2.2, it follows that

$$m\left(r, \frac{1}{f-a}\right) = S(r, f),$$

thus

$$N\left(r, \frac{1}{f-a}\right) = T(r, f) + S(r, f),$$

and we get that $\lambda(f - a) = \sigma(f) \geq k + 1$, completing the proof of Theorem 1.1.

4. Proof of Theorem 1.2

By Lemma 2.4 and the condition that $\lambda(f) \leq k$, we know that there exists a set $E_1 \in (1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, where r is sufficiently large, we have

$$\frac{f(z+j)}{f(z)} = \exp\{j\frac{f'(z)}{f(z)} + o(r^{\sigma-1-\varepsilon})\}, \quad j = 1, \ldots, n, \tag{4.1}$$

for any given $0 < \varepsilon < \frac{1}{2}$. From Wiman-Valiron theory, there exists a set $E_2 \subset (0, \infty)$ of finite logarithmic measure such that

$$\frac{f'(z)}{f(z)} = (1 + o(1))\frac{v(r, f)}{z} \tag{4.2}$$

for $|z| = r \notin E_2$, as $r \to \infty$.

Thus, by (1.4), (4.1) and (4.2), we see

$$\sum_{j=1}^{n} \frac{P_j(e^{A(z)}) + Q_j(e^{-A(z)})}{P_0(e^{A(z)}) + Q_0(e^{-A(z)})} \exp\{j\frac{v(r, f)}{z}(1 + o(1)) + o(r^{\sigma-1-\varepsilon})\} = -1, \tag{4.3}$$

where $P_n(e^{A(z)}) + Q_n(e^{-A(z)}) = 1$.

Let
$$H(z) = \frac{P_j(e^{A(z)}) + Q_j(e^{-A(z)})}{P_0(e^{A(z)}) + Q_0(e^{-A(z)})},$$
then we conclude that $\sigma(H) = k$. It follows from Lemma 2.5 that there exists a set $E_3 \subset (1, \infty)$ of finite linear measure, such that for all $|z| = r \notin [0, 1] \cup E_3$ and r sufficiently large

$$\exp\{-r^{k+\varepsilon}\} \leq \left|\frac{P_j(e^{A(z)}) + Q_j(e^{-A(z)})}{P_0(e^{A(z)}) + Q_0(e^{-A(z)})}\right| \leq \exp\{r^{k+\varepsilon}\}, \quad j = 1, \ldots, n. \tag{4.4}$$

In the following, we set $E = E_1 \cup (E_2 \cup E_3)$. By Lemma 2.6, there exists a sequence of points such that $r_m \notin E$, for any given $0 < \varepsilon < \frac{1}{2}$, as r_m sufficiently large, we have

$$r_m^{\sigma-\varepsilon} < v(r_m, f) < r_m^{\sigma+\varepsilon}. \tag{4.5}$$

In addition, we obtain that

$$Re\left\{\frac{v(r_m, f)}{z_m}\right\} = Re\left\{\frac{v(r_m, f)\bar{z_m}}{r_m^2}\right\} = \frac{v(r_m, f)Re\{z_m\}}{r_m^2}. \tag{4.6}$$

From Lemma 2.7, for r_m sufficiently large, we get

$$Re\{z_m\} < -\beta_m r_m \quad or \quad Re\{z_m\} > \beta_m r_m,$$

where $\beta_m > 0$ is a constant. We discuss the following two cases:

Case 1. Suppose first that $Re\{z_m\} < -\beta_m r_m$, the by (4.4)–(4.6), we get

$$\begin{aligned}
&\left|\frac{P_j(e^{A(z_m)}) + Q_j(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{j\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right| \\
&\le \exp\{-j\beta_m r_m^{\sigma-1+\varepsilon}(1+o(1)) + r_m^{k+\varepsilon}\} \\
&\le \exp\{-\beta_m r_m^{\sigma-1+\varepsilon}(1+o(1)) + r_m^{k+\varepsilon}\}.
\end{aligned}$$

This, together with (4.3), yields

$$\begin{aligned}
1 &= \left|\sum_{j=1}^{n} \frac{P_j(e^{A(z_m)}) + Q_j(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{j\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right| \\
&\le \sum_{j=1}^{n} \left|\frac{P_j(e^{A(z_m)}) + Q_j(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{j\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right| \\
&\le n\exp\{-\beta_m r_m^{\sigma-1+\varepsilon}(1+o(1)) + r_m^{k+\varepsilon}\}.
\end{aligned}$$

Thus, we have $\sigma - 1 + \varepsilon \le k + \varepsilon$, that is, $\sigma \le k+1$. By Theorem 1.1, we have $\sigma(f) = k+1$.

Case 2. Suppose that $Re\{z_m\} > \beta_m r_m$. In this case, we prove the theorem by contradiction. Now we assume that $\sigma(f) > k+1$. Then, take $0 < \varepsilon < \min\{\frac{1}{2}, \frac{\sigma-k-1}{2}\}$. From (4.4)–(4.6), by calculating carefully, we obtain

$$\begin{aligned}
&\left|\frac{P_j(e^{A(z_m)}) + Q_j(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{j\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right| \\
&= o\left(\left|\frac{P_n(e^{A(z_m)}) + Q_n(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{n\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right|\right),
\end{aligned}$$

for $j = 1, \ldots, n-1$. This, together with (4.3)–(4.6), yields that

$$\begin{aligned}
1 &= \left|\sum_{j=1}^{n} \frac{P_j(e^{A(z_m)}) + Q_j(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{j\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right| \\
&= \left|\frac{P_n(e^{A(z_m)}) + Q_n(e^{-A(z_m)})}{P_0(e^{A(z_m)}) + Q_0(e^{-A(z_m)})} \exp\{n\frac{v(r_m, f)}{z_m}(1+o(1)) + o(r_m^{\sigma-1-\varepsilon})\}\right|(1+o(1)) \\
&\ge \exp\{n\beta_m r_m^{\sigma-1-\varepsilon}(1+o(1)) - r_m^{k+\varepsilon}\}.
\end{aligned}$$

Hence, $\sigma - 1 - \varepsilon \leq k + \varepsilon$, that is, $\sigma \leq k + 1$ which contradicts the assumption that $\sigma(f) > k + 1$. Thus $\sigma(f) \leq k + 1$, by Theorem 1.1 again, we have $\sigma(f) = k + 1$. This proves Theorem 1.2.

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Xiaoguang Qi
University of Jinan
School of Mathematics
Jinan, Shandong 250022
P. R. China
E-mail: xiaogqi@gmail.com
xiaogqi@mail.sdu.edu.cn

Yong Liu
Department of Mathematics
Shaoxing College of Arts and Sciences
Shaoxing, Zhejiang 312000
P. R. China
E-mail: liuyongsdu@aliyun.com

Lianzhong Yang
Shandong University
School of Mathematics
Jinan, Shandong 250100
P. R. China
E-mail: lzyang@sdu.edu.cn