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MAXIMIZATION AND MINIMIZATION PROBLEMS RELATED TO A p-LAPLACIAN EQUATION ON A MULTIPLY CONNECTED DOMAIN

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Abstract. In this paper we investigate maximization and minimization problems related to a p-Laplacian equation on a multiply connected domain in $\mathbb{R}^2$, where the admissible set is a rearrangement class of a fixed function. We prove existence and representation of the maximizers and existence, uniqueness and representation of the minimizer.

1. INTRODUCTION

Let Ω be a nonempty, bounded, connected open set in $\mathbb{R}^2$ whose boundary is a disjoint union of simple closed curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ of class C^2, and suppose Γ_0 encloses Ω. Let $1 < p < \infty$, we denote the conjugate of p by $p' = \frac{p}{p-1}$. We consider the following boundary value problem

$$(1) \qquad \begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ u = \text{constant} & \text{on } \Gamma_i, \quad i = 1, \ldots, n, \\ -\int_{\Gamma_i} |\nabla u|^{p-2} \nabla u \cdot \mathbf{n} \, \mathrm{d}s = \gamma_i & \text{for } \ i = 1, \ldots, n, \end{cases}$$

where $f \in L^{p'}(\Omega)$, $\mathbf{n}$ is the unit outer normal to $\partial\Omega$, boundary of Ω, and $\gamma_1, \ldots, \gamma_n$ are real numbers. When $p = 2$, G. R. Burton in [4, Appendix] has proved that the problem (1) has exactly one solution. By similar method, we show that (1) still has a unique solution when $1 < p < \infty$. For each $f \in L^{p'}(\Omega)$ we denote the unique solution of (1) by u_f. For application of (1), when $p = 2$, to fluids dynamics (vorticity) see [4]. Our interest in this paper are in the maximization and minimization of the quantity $\frac{1}{p} \int_\Omega |\nabla u_f|^p \, \mathrm{d}x$, the kinetic energy, as f varies in a rearrangement class of a fixed function in $L^{p'}(\Omega)$; see the next section for precise definition of rearrangement of

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functions. Rearrangement optimization problems have been investigated in recent years by many authors, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. However, the boundary value problem in the one which is discussed here is interesting and very different from others.

2. Rearrangement

Let E be a (Lebesgue) measurable set in $\mathbb{R}^N$. Real measurable functions f and g on E are *rearrangements* of each other whenever

$$\mathcal{L}_N(\{x \in \Omega : f(x) \geq \alpha\}) = \mathcal{L}_N(\{x \in \Omega : g(x) \geq \alpha\}), \quad \text{for all } \alpha \in \mathbb{R},$$

where $\mathcal{L}_N$ denotes the N-dimensional Lebesgue measure. It is well known that if $f \in L^r(E)$, $1 \leq r \leq \infty$, and g be a rearrangement of f, then $g \in L^r(E)$ and in fact $\|f\|_r = \|g\|_r$, where $\|.\|_r$ denotes the standard norm on $L^r(E)$. We denote the rearrangement class of f by $\mathcal{R}(f)$ which comprises all functions which are rearrangements of f. The readers can see [2, 3] for more results about rearrangements of functions.

We now collect some useful lemmas to be applied later.

Lemma 2.1. ([3]). *Let $p > 1$ and $f_0 \in L^p(E)$. Then*

(i) $\overline{\mathcal{R}(f_0)}$ *, the weak closure of $\mathcal{R}(f_0)$ in $L^p(E)$, is compact with respect to $L^{p'}$-topology, weak topology, on $L^p(E)$.*

(ii) $\overline{\mathcal{R}(f_0)}$ *is convex.*

Lemma 2.2. ([3]). *Let $f_0 : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be two measurable functions. If every level set of g has measure zero then there exists an increasing function ξ such that $\xi(g) \in \mathcal{R}(f_0)$. Furthermore there exists a decreasing function η such that $\eta(g) \in \mathcal{R}(f_0)$.*

Lemma 2.3. ([3]). *Let $p > 1$, $f_0 \in L^p(E)$ and $g \in L^{p'}(E)$.*

(i) *If there is an increasing function ξ such that $\xi(g) \in \mathcal{R}(f_0)$ then*

$$\int_E fg \,\mathrm{d}x \leq \int_E \xi(g)g \,\mathrm{d}x, \quad \textit{for all } f \in \overline{\mathcal{R}(f_0)},$$

and $\xi(g)$ is the unique maximizer relative to $\overline{\mathcal{R}(f_0)}$.

(ii) *If there is a decreasing function η such that $\eta(g) \in \mathcal{R}(f_0)$ then*

$$\int_E fg \,\mathrm{d}x \geq \int_E \eta(g)g \,\mathrm{d}x, \quad \textit{for all } f \in \overline{\mathcal{R}(f_0)},$$

and $\eta(g)$ is the unique minimizer relative to $\overline{\mathcal{R}(f_0)}$.

Lemma 2.4. ([2]). *Let $1 \leq r \leq \infty$ and s be the conjugate exponent of r. Let $g \in L^r(E)$ and $\Psi : L^r(E) \to \mathbb{R}$ be convex.*

(i) *Suppose that Ψ is sequentially continuous in the L^s-topology on $L^r(E)$. Then Ψ attains a maximum value relative to $\mathcal{R}(g)$.*

(ii) *Suppose that Ψ is strictly convex, that g^* is a maximizer for Ψ relative to $\mathcal{R}(g)$ and that w is a member of sub-gradient of Ψ at g^*. Then $g^* = \xi(w)$ almost everywhere in E for some increasing function ξ.*

3. Existence and Uniqueness

In this section we prove the existence and uniqueness for the boundary value problem (1).

Theorem 3.1. *Let $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, $1 < p < \infty$, and $f \in L^{p'}(\Omega)$. Then the boundary value problem (1) has a unique solution.*

Proof. Let $\Omega_0, \Omega_1, \ldots, \Omega_n$ be the regions enclosed by $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$. Let

$$W = \{w \in W^{1,p}(\Omega) \mid w = 0 \text{ on } \Gamma_0 \text{ and } w = \text{constant on } \Gamma_i,\ i = 1, \ldots, n\}.$$

If $w \in W$, then we denote the value of w on Γ_i by $(w)_i$ for $i = 1, \ldots, n$. Define

$$J(w) := \frac{1}{p}\int_\Omega |\nabla w|^p \, \mathrm{d}x - \int_\Omega fw \, \mathrm{d}x + \sum_{i=1}^n \gamma_i (w)_i, \quad w \in W.$$

By the trace embedding $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ we infer that W is a closed linear subspace of $W^{1,p}(\Omega)$, and W comprises the restrictions to Ω of elements of $W_0^{1,p}(\Omega_0)$ that are constant on Ω_i, $i = 1, \ldots, n$. We consider the equivalent norm for $W_0^{1,p}(\Omega_0)$ that is defined as follows

$$\|w\| = \left(\int_{\Omega_0} |\nabla w|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

It is well known that the function $x \to |x|^p$, $x \in \mathbb{R}^N$, is strictly convex. From this, it is easy to deduce that J is strictly convex. We know that J is differentiable on W with

$$J'(w)v = \int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x + \sum_{i=1}^n \gamma_i (v)_i.$$

Moreover,

$$J(w) \geq \frac{1}{p}\|w\|^p - \|f\|_{p'}\|w\|_p + \sum_{i=1}^n \gamma_i (w)_i \geq \frac{1}{p}\|w\|^p - C\|w\| + \sum_{i=1}^n \gamma_i (w)_i,$$

for some $C > 0$. Thus J is coercive because $p > 1$. Therefore J has a unique global minimizer. We know that $u \in W$ is a critical point of J whenever $J'(u)v = 0$, for all $v \in W$. Hence

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x + \sum_{i=1}^n \gamma_i (v)_i = 0, \quad \text{for all } v \in W. \tag{2}$$

Let Lipschitz functions $g^1, \ldots, g^n \in W$ be chosen to satisfy the boundary conditions $(g^j)_i = \delta_{ij}$, $i, j = 1, \ldots, n$. Then (2) is equivalent to

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x = 0, \quad \text{for all } v \in W_0^{1,p}(\Omega), \tag{3}$$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla g^j \, \mathrm{d}x - \int_\Omega fg^j \, \mathrm{d}x + \gamma_j = 0, \quad j = 1, \ldots, n. \tag{4}$$

Thus (3) is a variational formulation of $-\Delta_p u = f$, in Ω. Now from (4) and Divergence theorem we infer that

$$\int_\Omega (-\Delta_p u - f) g^j \, \mathrm{d}x + \sum_{i=1}^n \int_{\Gamma_i} g^j |\nabla u|^{p-2}\nabla u \cdot \mathbf{n} \, \mathrm{d}s + \gamma_j = 0, \ j = 1, \ldots, n,$$

which reduces to

$$\int_{\Gamma_j} |\nabla u|^{p-2}\nabla u \cdot \mathbf{n} \, \mathrm{d}s + \gamma_j = 0, \quad j = 1, \ldots, n.$$

It follows that (1) holds if and only if u is a critical point of J, and therefore (1) has a unique solution. ■

4. Optimization Problem

Let $\gamma_1, \ldots, \gamma_n$ are fixed real numbers and $1 < p < \infty$. Also, let f_0 is a fixed function in $L^{p'}(\Omega)$ and $\mathcal{R} := \mathcal{R}(f_0)$. It is well known that the solution u_f of problem (1) satisfies the following variational problem

$$\frac{1}{p} \int_\Omega |\nabla u_f|^p \, \mathrm{d}x - \int_\Omega f u_f \, \mathrm{d}x + \sum_{i=1}^n \gamma_i (u_f)_i = \min_{w \in W} J_f(w), \tag{5}$$

where

$$J_f(w) := \frac{1}{p} \int_\Omega |\nabla w|^p \, \mathrm{d}x - \int_\Omega fw \, \mathrm{d}x + \sum_{i=1}^n \gamma_i (w)_i.$$

By (2), when $v = u = u_f$, and (5) we deduce that

$$
\begin{aligned}
(p-1)\int_\Omega |\nabla u_f|^p \,\mathrm{d}x &= \max_{w\in W}(-pJ_f(w)) \\
&= \max_{w\in W}\left(p\int_\Omega fw \,\mathrm{d}x - \int_\Omega |\nabla w|^p \,\mathrm{d}x - p\sum_{i=1}^n \gamma_i(w)_i\right).
\end{aligned} \tag{6}
$$

We define the functional $\varphi : L^{p'}(\Omega) \to \mathbb{R}$ by

$$\varphi(f) := \int_\Omega |\nabla u_f|^p \,\mathrm{d}x.$$

Our interest is in the following optimization problems

$$\max_{f\in\mathcal{R}} \varphi(f), \tag{7}$$

and

$$\min_{f\in\mathcal{R}} \varphi(f). \tag{8}$$

We now prove some useful lemmas to be applied later.

Lemma 4.1. *The functional φ is continuous with respect to weak topology in $L^{p'}(\Omega)$.*

Proof. Let a sequence $\{f_j\}$ and f be all in $L^{p'}(\Omega)$ such that $f_j \rightharpoonup f$ in $L^{p'}(\Omega)$. To simplify notation we write u_j in place of u_{f_j}. From (6) we have

$$
\begin{aligned}
&(p-1)\varphi(f) + p\int_\Omega (f_j - f)u_f \,\mathrm{d}x \\
&= p\int_\Omega f_j u_f \,\mathrm{d}x - \int_\Omega |\nabla u_f|^p \,\mathrm{d}x - p\sum_{i=1}^n \gamma_i(u_f)_i \\
&\le (p-1)\varphi(f_j) \\
&= p\int_\Omega f u_j \,\mathrm{d}x - \int_\Omega |\nabla u_j|^p \,\mathrm{d}x - p\sum_{i=1}^n \gamma_i(u_j)_i + p\int_\Omega (f_j - f)u_j \,\mathrm{d}x \\
&\le (p-1)\varphi(f) + p\int_\Omega (f_j - f)u_j \,\mathrm{d}x.
\end{aligned} \tag{9}
$$

Since $f_j \rightharpoonup f$ we deduce that

$$\lim_{j\to\infty} \int_\Omega (f_j - f)u_f \,\mathrm{d}x = 0. \tag{10}$$

Now, we prove that

$$\lim_{j\to\infty}\int_\Omega (f_j - f)u_j \ \mathrm{d}x = 0. \tag{11}$$

From (2), when $v = u = u_j$, $f_j \rightharpoonup f$, Poincarè's inequality for $W_0^{1,p}(\Omega_0)$ and Hölder's inequality we find that

$$\begin{aligned}\|u_j\|^p &\le \left|\int_\Omega f_j u_j \ \mathrm{d}x\right| + \sum_{i=1}^n |\gamma_i (u_j)_i| \\ &\le \|f_j\|_{p'}\|u_j\|_p + C\|u_j\| \\ &\le C\|u_j\|,\end{aligned}$$

where C denotes a positive constant that can change from line to line. Note that in the above inequalities, the second inequality, we used this fact that if $i \in \{1, 2, \cdots, n\}$ then

$$|(u_j)_i|^p \mathcal{L}_2(\Omega_i) = \int_{\Omega_i} |u_j|^p \ \mathrm{d}x \le \int_{\Omega_0} |u_j|^p \ \mathrm{d}x \le C\|u_j\|^p.$$

Hence $\{u_j\} \subset W$ is a bounded sequence in $W_0^{1,p}(\Omega_0)$, thus there exists a subsequence, still denoted $\{u_j\}$, that converges weakly to $\hat{u} \in W$, because W is closed. The compact imbedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ implies that $\{u_j\}$ converges strongly to $\hat{u}$ in $L^p(\Omega)$. Thus, we derive (11). Therefore, (9), (10) and (11) complete the proof of the lemma. ■

Remark 4.1. According to the proof of the Lemma 4.1, we claim that $\hat{u}$ is equal to u_f almost every where in Ω. We know that

$$(p-1)\varphi(f_j) = p\int_\Omega f_j u_j \ \mathrm{d}x - \int_\Omega |\nabla u_j|^p \ \mathrm{d}x - p\sum_{i=1}^n \gamma_i (u_j)_i.$$

Now by the weak lower semicontinuity of the norm $\|.\|$ and (6), we derive

$$\begin{aligned}(p-1)\varphi(f) &\le p\int_\Omega f\hat{u} \ \mathrm{d}x - \int_\Omega |\nabla \hat{u}|^p \ \mathrm{d}x - p\sum_{i=1}^n \gamma_i(\hat{u})_i \\ &\le (p-1)\varphi(f).\end{aligned}$$

Therefore, the uniqueness of the maximizer of the functional $-pJ_f(.)$ implies that $\hat{u} = u_f$ almost every where in Ω.

Lemma 4.2. *The functional φ is strictly convex in $L^{p'}(\Omega)$.*

Proof. Let $0 \le t \le 1$ and $f, g \in L^{p'}(\Omega)$. For each $w \in W$ we have

$$\begin{aligned} -pJ_{tf+(1-t)g}(w) &= p\int_{\Omega}(tf+(1-t)g)w \,\mathrm{d}x - \int_{\Omega}|\nabla w|^p \,\mathrm{d}x - p\sum_{i=1}^{n}\gamma_i(w)_i \\ &= t(-pJ_f(w)) + (1-t)(-pJ_g(w)). \end{aligned}$$

Hence, from (6) we infer that

$$\varphi(tf+(1-t)g) \le t\varphi(f) + (1-t)\varphi(g);$$

thus φ is convex. Now, we show that φ is strictly convex. Suppose for some $0 < t < 1$, we have

$$\varphi(h) = t\varphi(f) + (1-t)\varphi(g),$$

where $h := tf + (1-t)g$. Thus,

$$J_h(u_h) = tJ_f(u_f) + (1-t)J_g(u_g).$$

Hence,

$$tJ_f(u_h) + (1-t)J_g(u_h) = tJ_f(u_f) + (1-t)J_g(u_g).$$

Since $0 < t < 1$, we derive $J_f(u_h) = J_f(u_f)$ and $J_g(u_h) = J_g(u_g)$. By the uniqueness of the minimizer of the functionals $J_f(.)$ and $J_g(.)$ on W, we deduce that

$$u_h = u_f = u_g, \quad \text{a.e. in } \Omega.$$

Thus, $-\Delta_p u_f = -\Delta_p u_g$ almost every where in Ω, so $f = g$ almost every where in Ω. Therefore, φ is strictly convex. ■

Lemma 4.3. *Let $f \in L^{p'}(\Omega)$. The functional φ is Gâteaux differentiable at f with derivative*

$$\varphi'(f)g = \frac{p}{p-1}\int_{\Omega} g u_f \,\mathrm{d}x,$$

for all $g \in L^{p'}(\Omega)$.

Proof. Let $\{t_j\}$ be a sequence of positive numbers that tends to zero. Let $f, g \in L^{p'}(\Omega)$ and $h_j := f + t_j(g-f)$, $j \ge 1$. So, $h_j \to f$ in $L^{p'}(\Omega)$ as $j \to 0$. From (9) we have

$$\begin{aligned} (p-1)\varphi(f) + p\int_{\Omega}(h_j - f)u_f \,\mathrm{d}x &\le (p-1)\varphi(h_j) \\ &\le (p-1)\varphi(f) + p\int_{\Omega}(h_j - f)u_j \,\mathrm{d}x, \end{aligned}$$

where $u_j := u_{h_j}$. Thus,

$$\text{(12)}\quad \frac{p}{p-1}\int_\Omega (g-f)u_f \,\mathrm{d}x \leq \frac{\varphi(f+t_j(g-f))-\varphi(f)}{t_j} \leq \frac{p}{p-1}\int_\Omega (g-f)u_j \,\mathrm{d}x.$$

As a consequence of *Remark* 4.1, $u_j \to u_f$ in $L^p(\Omega)$. This coupled with (12), implies that

$$\lim_{j\to\infty} \frac{\varphi(f+t_j(g-f))-\varphi(f)}{t_j} = \frac{p}{p-1}\int_\Omega (g-f)u_f \,\mathrm{d}x.$$

Therefore, the proof of the lemma follows. ■

Now, we are ready to prove the main results of this section.

Theorem 4.4. *The maximization problem (7) is solvable; that is, there exists $f^* \in \mathcal{R}$ such that*

$$\varphi(f^*) = \max_{f\in\mathcal{R}} \varphi(f).$$

Moreover, there exists an increasing function ξ such that $f^* = \xi(u_{f*})$ almost everywhere in Ω.

Proof. From Lemma 4.1, Lemma 4.2 and Lemma 2.4(i) we infer that there exists $f^* \in \mathcal{R}$ such that $\varphi(f) \leq \varphi(f^*)$, for all $f \in \mathcal{R}$. From Lemma 4.3, $\varphi(f)$ is Gâteaux differentiable with derivative $\frac{p}{p-1}u_f$. Since φ is strictly convex, by Lemma 2.4(ii), there is an increasing function ξ such that $f^* = \xi(u_{f^*})$. ■

Theorem 4.5. *If $f_0 > 0$ in Ω, then the minimization problem (8) has a unique solution. Moreover, if f_* be the minimizer, then $f_* = \eta(u_{f_*})$ for some decreasing function η.*

Proof. We know φ is weakly continuous in $L^{p'}(\Omega)$, Lemma 4.1, and $\overline{\mathcal{R}}$ is weakly compact, Lemma 2.1. Thus, there exists $f_* \in \overline{\mathcal{R}}$ such that

$$\varphi(f_*) = \min_{f\in\overline{\mathcal{R}}} \varphi(f).$$

Since φ is strictly convex, Lemma 4.2, we infer that f_* is unique. Now, we prove that $f_* \in \mathcal{R}$. From Lemma 2.14 of [3] we have

$$\mathcal{L}_2(\{x \in \Omega : f_*(x) > 0\} \geq \mathcal{L}_2(\{x \in \Omega : f_0(x) > 0\} = \mathcal{L}_2(\Omega),$$

so, $f_* > 0$ in Ω. This coupled with $-\Delta_p u_{f_*} = f_*$ in Ω, implies that every level set of u_{f_*} in Ω has measure zero. By applying Lemma 2.2 we derive that there exists a decreasing function η such that $\eta(u_{f_*}) \in \mathcal{R}$. Now, from Lemma 2.3(ii) we have

$$\text{(13)}\quad \int_\Omega f u_{f_*} \,\mathrm{d}x \geq \int_\Omega \eta(u_{f_*})u_{f_*} \,\mathrm{d}x, \quad \text{for all } f \in \overline{\mathcal{R}}.$$

Let $0 < t < 1$ and $f \in \overline{\mathcal{R}}$. We define $f_t := tf + (1-t)f_*$. Since $\overline{\mathcal{R}}$ is convex, Lemma 2.1(i), $f_t \in \overline{\mathcal{R}}$ for all $0 < t < 1$. From Lemma 4.3, for sufficiently small t we have

$$\varphi(f_*) \le \varphi(f_t) = \varphi(f_*) + \frac{tp}{p-1}\int_\Omega (f - f_*)u_{f_*}\,\mathrm{d}x + o(t).$$

Thus, when $t \to 0^+$ we deduce

$$\int_\Omega f u_{f_*}\,\mathrm{d}x \ge \int_\Omega f_* u_{f_*}\,\mathrm{d}x, \quad \text{for all } f \in \overline{\mathcal{R}}. \tag{14}$$

Therefore, by (13), (14) and Lemma 2.3(ii) we derive $f_* = \eta(u_{f_*})$. ∎

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