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The Minimal Dual Orlicz Surface Area

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Abstract. Petty proved that a convex body in $\mathbb{R}^n$ has the minimal surface area among its $\mathrm{SL}(n)$ images if and only if its surface area measure is isotropic. Recently, Zou and Xiong generalized this result to the Orlicz setting by introducing a new notion of minimal Orlicz surface area, and the analog of Ball's reverse isoperimetric inequality is established. In this paper, we give the dual results in Orlicz space by introducing a new notion of minimal dual Orlicz surface area. And the dual form of Ball's isoperimetric inequality is established.

1. Introduction

A classical and useful result proved by Petty [35] is the minimal surface area theorem, which states: A convex body (namely, a compact convex set with non-empty interior) in Euclidean n-space $\mathbb{R}^n$ has the minimal surface area among its $\mathrm{SL}(n)$ images if and only if its surface area measure is isotropic on the unit sphere S^{n-1}. Its importance was rediscovered in the 1990s. Clack generalized it to Minkowski space. Later, Giannopoulos and Papadimitrakis [14] used isotropic surface area measure to study the hyperplane projections of convex bodies.

During the last two decades, the Brunn-Minkowski theory [37] has been extended to the L_p-Brunn-Minkowski theory, the notions of surface area and surface area measure were extended to those of L_p-surface area and L_p-surface area measure, respectively. See the initial works of Lutwak [26, 27]. In [28], Lutwak, Yang and Zhang showed that Petty's theorem has a natural L_p-generalization: The L_p-surface area of a convex body is minimal among its $\mathrm{SL}(n)$ images if and only if its L_p-surface area measure is isotropic on S^{n-1}.

Lutwak's dual Brunn-Minkowski theory, introduced in the 1970s, helped achieving a major breakthrough in the solution of the Busemann-Petty problem in the 1990s. In contrast to the Brunn-Minkowski theory, in the dual theory, convex bodies are replaced

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by star-shaped sets, and projections onto subspaces are replaced by intersections with subspaces. The machinery of the dual theory includes dual mixed volumes and important auxiliary bodies known as intersection bodies; see [6–8, 15, 20, 21, 24, 25, 43–45]. The dual Orlicz Brunn-Minkowski theory was first proposed to study in [30]. As a more wide extension of the dual Brunn-Minkowski theory, Zhu, Zhou and Xu [47], Gardner, Hug, Weil and Ye [12, 41] established independently the dual Orlicz-Brunn-Minkowski theory for star bodies.

Recently, progress towards an Orlicz-Brunn-Minkowski theory was made by Lutwak, Yang and Zhang [29, 30] and Ludwig [23]. These theories are far more general than the L_p-Brunn-Minkowski theory, we refer the reader to [5, 11, 17, 19, 22, 23, 29–31, 34, 39, 40, 46]. In [48], Zou and Xiong generalized Petty's minimal surface area theorem to the Orlicz setting by introducing a new notion of minimal Orlicz surface area, and the analog of Ball's reverse isoperimetric inequality is established.

As a dual form of a minimum surface area of a convex body, the main goal of this paper is to seek an Orlicz extension of the minimal dual surface area. The dual surface area of a star body in $\mathbb{R}^n$ is usually defined as the integral of radial function to the $(n+1)$-th power (see [32, 41]):

$$\widetilde{S}_K = \widetilde{S}(K) = \int_{S^{n-1}} \rho_K(u)^{n+1}\,\mathrm{d}S(u),$$

where, S denotes Lebesgue measure on S^{n-1} (i.e., $(n-1)$-dimensional Hausdorff measure).

Throughout this paper, let $\phi\colon [0,\infty) \to [0,\infty)$ be convex and strictly increasing with $\phi(0) = 0$, and denote by Φ the class of those ϕ. This paper is composed of five sections. In Section 2, we propose the concept of dual Orlicz surface area: Suppose that K is a star body (about the origin) in $\mathbb{R}^n$. Its dual Orlicz surface area $\widetilde{S}_\phi(K)$ with respect to a convex function $\phi \in \Phi$, is defined by

$$\widetilde{S}_\phi(K) = \int_{S^{n-1}} \phi\left(\rho_K(u)\right) \rho_K(u)^n\,\mathrm{d}S(u).$$

In Section 3, we demonstrate that modulo orthogonal transformations, the body K has a unique $\mathrm{SL}(n)$ image with dual minimal Orlicz surface area. In view of this fact, we define the minimal dual Orlicz surface area of K with respect to ϕ by

$$\widetilde{A}_\phi(K) = \min\left\{\widetilde{S}_\phi(TK) : T \in \mathrm{SL}(n)\right\}.$$

For $\phi \in \Phi \cap C^1(0,\infty)$, namely, for smooth functions ϕ in Φ, we introduce the transformation, $\phi \mapsto \mu_\phi$, defined by $\mu_\phi(t) = t^{n+1}\phi'(t)$. Then, for each Borel set $\omega \subseteq S^{n-1}$, we write

$$\mu_\phi(K,\omega) = \int_\omega \mu_\phi\left(\rho_K(u)\right)\,\mathrm{d}S(u) = \int_\omega \phi'\left(\rho_K(u)\right)\rho_K(u)^{n+1}\,\mathrm{d}S(u).$$

In Section 4, we give a dual analog for the minimal Orlicz surface area theorem by Zou and Xiong proved in [48].

Theorem 1.1. *Suppose that K is a star body in $\mathbb{R}^n$ with the origin in its interior, and $\phi \in \Phi \cap C^1(0,\infty)$. Then $\widetilde{A}_\phi(K) = \widetilde{S}_\phi(K)$ if and only if $\mu_\phi(K,\cdot)$ is isotropic on S^{n-1}.*

In the last section, we provide bounds for the dual minimal Orlicz surface area $\widetilde{A}_\phi(K)$. When the volume of K is fixed, origin-symmetric Euclidean balls attain the minimum; the volume of dual L_∞-John ellipsoid introduced in [42] dominates it from above.

2. Preliminaries

In order to keep the paper self-contained, we collect here some basic facts from convex geometry. Good references on the theory of convex bodies are the books by Gardner [10], Gruber [16], Pisier [36], Schneider [37], and Thompson [38], etc.

As usual, $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$; $B = \{x \in \mathbb{R}^n : x \cdot x \leq 1\}$ and $S^{n-1} = \partial B$ denote the unit ball and unit sphere in $\mathbb{R}^n$, respectively. The volume of B is $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$. According to the context, one can catch clearly that the notation $|\cdot|$ has several different meanings: the absolute value, the standard Euclidean norm on $\mathbb{R}^n$, the n-dimensional volume, the absolute value of determinant of an $n \times n$ matrix, and the total mass of a finite measure. For brevity, we write $\langle x \rangle = x/|x|$, for $x \in \mathbb{R}^n \setminus \{o\}$. Let $\mathfrak{L}^n$ denote the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. The support function h_K of a compact convex set K in $\mathbb{R}^n$ is defined by

$$h_K(x) = h(K,x) = \max\{x \cdot y : y \in K\}, \quad \text{for } x \in \mathbb{R}^n. \tag{2.1}$$

Let $\mathcal{K}_o^n$ denote the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors.

A set $K \subset \mathbb{R}^n$ is said to be a star body about the origin, if the line segment from the origin to any point $x \in K$ is contained in K and K has continuous and positive radial function $\rho_K(\cdot)$. Here, the radial function of K, $\rho_K : \mathbb{R}^n \setminus \{o\} \to [0,\infty)$, is defined by

$$\rho_K(x) = \rho(K,x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{o\}. \tag{2.2}$$

Write $\mathcal{S}_o^n$ for the class of star bodies about the origin o in $\mathbb{R}^n$. $\mathcal{S}_o^n$ is often equipped with the dual Hausdorff metric $\widetilde{\delta}_H$, which is defined by

$$\widetilde{\delta}_H(K,L) = \max\left\{|\rho_K(u) - \rho_L(u)| : u \in S^{n-1}\right\} := |\rho_K(u) - \rho_L(u)|_\infty,$$

for $K, L \in \mathcal{S}_o^n$.

Star body $K \in \mathcal{S}_o^n$ can be uniquely determined by its radial function $\rho_K(\cdot)$ and vice verse. If $\lambda > 0$, we have $\rho_K(\lambda x) = \lambda^{-1}\rho_K(x)$ and $\rho_{\lambda K}(x) = \lambda \rho_K(x)$.

More generally, from the definition of the radial function it follows immediately that for $T \in \mathrm{GL}(n)$ the radial function of the image $TK = \{Ty : y \in K\}$ of $K \in \mathcal{S}_o^n$ is given by

$$\rho(TK,x) = \rho(K,T^{-1}x), \quad \text{for all } x \in \mathbb{R}^n. \tag{2.3}$$

Two star bodies $K, L \in \mathcal{S}_o^n$ are said to be dilates of each other if there is a constant $\lambda > 0$ such that $L = \lambda K$, and equivalent to $\rho_L(u) = \lambda \rho_K(u)$ for all $u \in S^{n-1}$.

For convex body $K \in \mathcal{K}_o^n$, let K^* denotes the polar of the body K. Namely,

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{for all } y \in K\}. \tag{2.4}$$

Obviously, we have $(K^*)^* = K$. From definitions (2.1), (2.2) and (2.4), we know that: If $K \in \mathcal{K}_o^n$, then the support and radial functions of K^*, the polar body of K, are defined respectively by $h_{K^*} = 1/\rho_K$ and $\rho_{K^*} = 1/h_K$. In addition, the polar body of convex body has the following property: If $K \in \mathcal{K}_o^n$, and $T \in \mathrm{GL}(n)$, then

$$(TK)^* = T^{-t} K^*.$$

Let $K \in \mathcal{S}_o^n$ with the radial function ρ_K. The dual cone-volume measure $\widetilde{V}_K$ of a star body K is a Borel measure on S^{n-1}. We define for a Borel set $\omega \subseteq S^{n-1}$ by

$$\widetilde{V}_K(\omega) = \frac{1}{n} \int_\omega \rho_K^n \, \mathrm{d}S.$$

It is convenient to use the normalized dual cone-volume measure $\widetilde{V}_K^* = \widetilde{V}_K / |K|$, of K. Observe that $\widetilde{V}_K^*$ is a probability measure on S^{n-1}. Also, $\widetilde{V}_K^*$ is $\mathrm{GL}(n)$-invariant, that is, for $T \in \mathrm{GL}(n)$ and a Borel subset $\omega \subseteq S^{n-1}$, it yields

$$\widetilde{V}_{T^{-1}K}^*(\omega) = \widetilde{V}_K^*(\langle T\omega \rangle), \tag{2.5}$$

where $\langle T\omega \rangle = \{\langle Tu \rangle : u \in \omega\}$.

Let $\phi \in \Phi$, and $K, L \in \mathcal{S}_o^n$. We define the Orlicz radial combination $K \,\widetilde{+}_\phi\, \varepsilon \circ L$ $(\varepsilon > 0)$ by

$$\rho_{K \widetilde{+}_\phi \varepsilon \circ L}^{-1}(u) = \inf \left\{ \lambda > 0 : \phi\left(\frac{1}{\lambda \rho_K(u)} \right) + \varepsilon \phi\left(\frac{1}{\lambda \rho_L(u)} \right) \leq \phi(1) \right\},$$

for all $u \in S^{n-1}$. If $\phi(t) = t^p$, $p \geq 1$, then the Orlicz radial addition $K \,\widetilde{+}_\phi\, \varepsilon \circ L$ reduces to Lutwak's radial harmonic L_p-combination $K \,\widetilde{+}_p\, \varepsilon \circ L$. Namely,

$$\rho_{K \widetilde{+}_p \varepsilon \circ L}(u)^{-p} = \rho_K(u)^{-p} + \varepsilon \rho_L(u)^{-p}.$$

According to Lemmas 3.5 and 4.1 in [47], we easily give that

$$K \,\widetilde{+}_\phi\, \varepsilon \circ L \to K, \quad \text{as } \varepsilon \to 0^+,$$

and

$$\lim_{\varepsilon \to 0^+} \frac{\rho_{K \widetilde{+}_\phi \varepsilon \circ L}(u) - \rho_K(u)}{\varepsilon} = -\frac{\rho_K(u)}{\phi_l'(1)} \phi\left(\frac{\rho_K(u)}{\rho_L(u)} \right)$$

is uniform on S^{n-1}, here ϕ_l' denotes the left-continuous derivative of ϕ at 1.

According to Theorem 4.1 and (4.7) in [47], we easily establish the following results: Let $\phi \in \Phi$, and let $K, L \in \mathcal{S}_o^n$, then we have

$$\frac{-\phi_l'(1)}{n} \lim_{\varepsilon \to 0^+} \frac{\left|K \widetilde{+}_\phi \varepsilon \circ L\right| - |K|}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \rho_K(u)^n \, \mathrm{d}S(u). \tag{2.6}$$

From (2.6), we define the dual Orlicz mixed volume $\widetilde{V}_\phi(K, L)$ of $K, L \in \mathcal{S}_o^n$ by

$$\widetilde{V}_\phi(K, L) = \int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_L}\right) \mathrm{d}\widetilde{V}_K, \quad \phi \in \Phi. \tag{2.7}$$

If $\phi(t) = t^p$, $1 \le p < +\infty$, then $\widetilde{V}_\phi(K, L)$ turns to $\widetilde{V}_{-p}(K, L)$ of the L_p-dual mixed volume of K and L. Namely, L_p-dual mixed volume of star body $K, L \in \mathcal{S}_o^n$ is expressed by (see [27])

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+p} \rho_L(u)^{-p} \, \mathrm{d}S(u). \tag{2.8}$$

From (2.8), it follows immediately that for each $K \in \mathcal{S}_o^n$ and $p \ge 1$,

$$|K| = \widetilde{V}_{-p}(K, K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n \, \mathrm{d}S(u).$$

Suppose that $K, K_i, L, L_j \in \mathcal{S}_o^n$ and $\phi, \phi_k \in \Phi$, $i, j, k \in \mathbb{N}$. If $K_i \to K$, $L_j \to L$ and $\phi_k \to \phi$, then the continuity of $\widetilde{V}_\phi(K, L)$ regarding K, L and ϕ is proved (see [49])

$$\lim_{i,j,k \to \infty} \widetilde{V}_{\phi_k}(K_i, L_j) = \widetilde{V}_\phi(K, L). \tag{2.9}$$

We easily get that if $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi$, then for $T \in \mathrm{GL}(n)$,

$$\widetilde{V}_\phi(TK, L) = |T| \, \widetilde{V}_\phi(K, T^{-1}L). \tag{2.10}$$

The L_p-Minkowski inequality for L_p-dual mixed volume was given by Lutwak [27]: If $K, L \in \mathcal{S}_o^n$ and $p \ge 1$, then

$$\widetilde{V}_{-p}(K, L) \ge |K|^{\frac{n+p}{n}} \, |L|^{\frac{-p}{n}}, \tag{2.11}$$

with equality if and only if K and L are dilates of each other.

We further establish the following dual Orlicz-Minkowski inequality by means of a similar method in [47]: Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi$. Then

$$\widetilde{V}_\phi(K, L) \ge |K| \, \phi\left(\left(\frac{|K|}{|L|}\right)^{\frac{1}{n}}\right). \tag{2.12}$$

If ϕ is strictly convex, then equality in (2.12) holds if and only if K and L are dilates of each other.

Definition 2.1. Suppose that $K \in \mathcal{S}_o^n$. Its dual Orlicz surface area $\widetilde{S}_\phi(K)$ with respect to a convex function $\phi \in \Phi$ is defined by

$$\widetilde{S}_\phi(K) = n\widetilde{V}_\phi(K, B) = n \int_{S^{n-1}} \phi(\rho_K) \,\mathrm{d}\widetilde{V}_K. \tag{2.13}$$

If $\phi(t) = t^p$, $1 \le p < +\infty$, then $\widetilde{S}_\phi(K)$ turns to $\widetilde{S}_p(K)$ of the dual L_p-surface area of K. Namely,

$$\widetilde{S}_p(K) = n\widetilde{V}_{-p}(K, B) = n \int_{S^{n-1}} \rho_K^p \,\mathrm{d}\widetilde{V}_K.$$

The following Jensen's inequality will be used in our paper.

Suppose that μ is a probability measure on a space X and $g\colon X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\phi\colon I \to \mathbb{R}$ is a convex function, then

$$\int_X \phi(g(x)) \,\mathrm{d}\mu(x) \ge \phi\left(\int_X g(x) \,\mathrm{d}\mu(x)\right). \tag{2.14}$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for μ-almost all $x \in X$ (see [18]).

Suppose that $p \neq 0$, μ is a finite Borel measure in a set X, and f is a nonnegative μ-integrable function on X. The pth mean $M_p f$ of f is defined by

$$M_p f = \left(\frac{1}{\mu(X)} \int_X f(x)^p \,\mathrm{d}\mu(x)\right)^{\frac{1}{p}},$$

$$\lim_{p\to\infty} M_p f = \max\{f(x) : x \in X\}$$

and

$$\lim_{p\to 0} M_p f = \exp\left(\frac{1}{\mu(X)} \int_X \log f(x) \,\mathrm{d}\mu(x)\right).$$

Jensen's inequality may be stated that: If $p \le q$ and $M_q f$ exists, then

$$M_p f \le M_q f, \tag{2.15}$$

with equality for $p \neq q$ if and only if f is a constant or if and only if $p = q$ (see [18]).

3. The minimal dual Orlicz surface area

In order to demonstrate the existence and uniqueness of minimal dual Orlicz surface area, we first introduce some lemmas.

Lemma 3.1. *Suppose that $K \in \mathcal{S}_o^n$, $\phi \in \Phi$ and $T \in \mathrm{GL}(n)$. Then*

$$\widetilde{S}_\phi(TK) = n|T| \int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T^{-1}B}}\right) \mathrm{d}\widetilde{V}_K.$$

Proof. Let $v = \langle T^{-1}u \rangle$. From Definition 2.1, (2.3) and (2.5), we have

$$\begin{aligned}
\widetilde{S}_\phi(TK) &= n|TK| \int_{S^{n-1}} \phi(\rho_{TK}(u)) \, \mathrm{d}\widetilde{V}^*_{TK}(u) \\
&= n|T||K| \int_{S^{n-1}} \phi(\rho_K(\langle T^{-1}u \rangle)|T \langle T^{-1}u \rangle |) \, \mathrm{d}\widetilde{V}^*_K(\langle T^{-1}u \rangle) \\
&= n|T||K| \int_{S^{n-1}} \phi(\rho_K(\langle T^{-1}u \rangle) h_{T^tB}(\langle T^{-1}u \rangle)) \, \mathrm{d}\widetilde{V}^*_K(\langle T^{-1}u \rangle) \\
&= n|T||K| \int_{S^{n-1}} \phi\left(\frac{\rho_K(\langle T^{-1}u \rangle)}{\rho_{(T^tB)^*}(\langle T^{-1}u \rangle)} \right) \mathrm{d}\widetilde{V}^*_K(\langle T^{-1}u \rangle) \\
&= n|T||K| \int_{S^{n-1}} \phi\left(\frac{\rho_K(\langle T^{-1}u \rangle)}{\rho_{T^{-1}B}(\langle T^{-1}u \rangle)} \right) \mathrm{d}\widetilde{V}^*_K(\langle T^{-1}u \rangle) \\
&= n|T||K| \int_{S^{n-1}} \phi\left(\frac{\rho_K(v)}{\rho_{T^{-1}B}(v)} \right) \mathrm{d}\widetilde{V}^*_K(v) \\
&= n|T| \int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T^{-1}B}} \right) \mathrm{d}\widetilde{V}_K,
\end{aligned}$$

as desired. $\square$

Lemma 3.2. [49] *Suppose that* $\{T_i\}_{i\in\mathbb{N}} \subset \mathrm{SL}(n)$. *Then*

$$\|T_i\| \to \infty \quad \Longleftrightarrow \quad \left\|T_i^{-1}\right\| \to \infty.$$

Thus, $\{T_i\}_{i\in\mathbb{N}}$ *is bounded from above, if and only if* $\left\{T_i^{-1}\right\}_{i\in\mathbb{N}}$ *is bounded from above.*

Denote

$$d_n(T_1, T_2) = \|T_1 - T_2\|, \quad \text{for } T_1, T_2 \in \mathfrak{L}^n.$$

Then the metric space $(\mathfrak{L}^n, d_n)$ is complete. Since $\mathfrak{L}^n$ is of finite dimension, a set in $(\mathfrak{L}^n, d_n)$ is compact if and only if it is bounded and closed.

Lemma 3.3. [49] *Suppose that* $\{T_i\}_{i\in\mathbb{N}} \subset \mathrm{SL}(n)$, *and* $T_i \to T_0 \in \mathrm{SL}(n)$ *with respect to* d_n. *Then*

$$T_iB = T_0B \quad \textit{with respect to } \widetilde{\delta}_H.$$

Inspired by the work in [4] and [49], we give the following lemma.

Lemma 3.4. *Suppose that* $K \in \mathcal{S}^n_o$ *and* $\phi \in \Phi$. *Then*

$$\lim_{\substack{T\in\mathrm{SL}(n)\\ \|T\|\to\infty}} \widetilde{S}_\phi(TK) = \infty.$$

From Lemma 3.4, Lemma 3.2, (2.10), definitions (2.13) and (2.7), we immediately obtain

Lemma 3.5. *Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi$. Then*

$$\lim_{\substack{T\in \mathrm{SL}(n)\\ \|T\|\to\infty}} \widetilde{S}_\phi(T^{-1}K) = \lim_{\substack{T\in \mathrm{SL}(n)\\ \|T\|\to\infty}} \widetilde{V}_\phi(K, TB) = \infty.$$

According to the previous lemmas in hand, we can show that the minimal dual Orlicz surface area is well-defined.

Theorem 3.6. *Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi$. Then modulo orthogonal transformations, there exists a unique solution to the minimization problem*

$$\min_{T\in \mathrm{SL}(n)} \widetilde{S}_\phi(TK).$$

Proof. Alternatively, by using dual Orlicz mixed volume, we can reformulate this theorem as follows: The unit ball B has a unique $\mathrm{SL}(n)$ image E_0 such that

$$\widetilde{V}_\phi(K, E_0) = \min\left\{\widetilde{V}_\phi(K, TB) : T \in \mathrm{SL}(n)\right\}.$$

Observe that the infimum exists, since

$$|K|\phi\left(\left(\frac{|K|}{\omega_n}\right)^{\frac{1}{n}}\right) \le \inf\left\{\widetilde{V}_\phi(K, TB) : T \in \mathrm{SL}(n)\right\} \le \widetilde{V}_\phi(K, B) < \infty,$$

where the left inequality obtained from the dual Orlicz-Minkowski inequality (2.12) and the definition (2.7).

Let

$$\mathcal{A} = \left\{T \in \mathrm{SL}(n) : \widetilde{V}_\phi(K, TB) \le \widetilde{V}_\phi(K, B)\right\}.$$

From Lemma 3.3 and (2.9), $\widetilde{V}_\phi(K, TB)$ is continuous in $T \in (\mathrm{SL}(n), d_n)$. Thus, the set $\mathcal{A}$ is closed in $(\mathrm{SL}(n), d_n)$. Meanwhile, the definition of $\mathcal{A}$ and Lemma 3.5 guarantee that $\mathcal{A}$ is bounded in $(\mathrm{SL}(n), d_n)$. Hence, $\mathcal{A}$ is compact.

Since $\widetilde{V}_\phi(K, TB)$ is continuous on $(\mathcal{A}, d_n)$, it concludes that there exists a $T_0 \in \mathcal{A}$ such that

$$\widetilde{V}_\phi(K, T_0B) = \min\left\{\widetilde{V}_\phi(K, TB) : T \in \mathcal{A}\right\} = \inf\left\{\widetilde{V}_\phi(K, TB) : T \in \mathrm{SL}(n)\right\}.$$

This demonstrates the existence of $E_0 = T_0B$.

Now, we prove the uniqueness by contradiction. Assume there are two solutions $T_1, T_2 \in \mathrm{SL}(n)$ to the considered problem, and they don't differ only by an orthogonal transformation. Let $E_1 = T_1^{-1}B$, $E_2 = T_2^{-1}B$. It easily follows that each $T \in \mathrm{SL}(n)$ can be represented in the form $T = PQ$, where P is symmetric, positive definite and Q is orthogonal. So, without loss of generality, we may assume that T_1, T_2 are symmetric and positive definite.

By the Minkowski inequality for symmetric and positive definite matrices, we have

$$\left[\det\left(\frac{T_1+T_2}{2}\right)\right]^{\frac{1}{n}} > \frac{1}{2}(\det T_1)^{\frac{1}{n}} + \frac{1}{2}(\det T_2)^{\frac{1}{n}} = 1.$$

Let

$$T_3 = \left[\det\left(\frac{T_1+T_2}{2}\right)\right]^{-\frac{1}{n}} \frac{(T_1+T_2)}{2}.$$

Then, $T_3 \in \mathrm{SL}(n)$ is symmetric.

Let $E_3 = T_3^{-1}B$. For all $u \in S^{n-1}$, we have

$$\begin{aligned} h_{E_3^*}(u) &= h_{T_3B}(u) \\ &< h_{\frac{T_1+T_2}{2}B}(u) = \left|\frac{T_1u+T_2u}{2}\right| \\ &\le \frac{|T_1u|+|T_2u|}{2} = \frac{1}{2}h_{T_1B} + \frac{1}{2}h_{T_2B}. \end{aligned}$$

Namely,

$$\frac{1}{\rho_{T_3^{-1}B}(u)} \le \frac{1}{2\rho_{T_1^{-1}B}(u)} + \frac{1}{2\rho_{T_2^{-1}B}(u)}.$$

Since ϕ is strictly increasing and convex in $[0,\infty)$, we have

$$\begin{aligned} \phi\left(\frac{\rho_K}{\rho_{T_3^{-1}B}}\right) &\le \phi\left(\frac{\rho_K}{2\rho_{T_1^{-1}B}} + \frac{\rho_K}{2\rho_{T_2^{-1}B}}\right) \\ &\le \frac{1}{2}\phi\left(\frac{\rho_K}{\rho_{T_1^{-1}B}}\right) + \frac{1}{2}\phi\left(\frac{\rho_K}{\rho_{T_2^{-1}B}}\right). \end{aligned}$$

Thus, by Lemma 3.1, we have

$$\begin{aligned} \widetilde{S}_\phi(T_3K) &= n\int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T_3^{-1}B}}\right) \mathrm{d}\widetilde{V}_K \\ &< \frac{n}{2}\int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T_1^{-1}B}}\right) \mathrm{d}\widetilde{V}_K + \frac{n}{2}\int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T_2^{-1}B}}\right) \mathrm{d}\widetilde{V}_K \\ &= \frac{1}{2}\widetilde{S}_\phi(T_1K) + \frac{1}{2}\widetilde{S}_\phi(T_2K) \\ &= \widetilde{S}_\phi(T_1K) = \widetilde{S}_\phi(T_2K). \end{aligned}$$

That is,

$$\widetilde{S}_\phi(T_3K) < \widetilde{S}_\phi(T_1K) = \widetilde{S}_\phi(T_2K).$$

However, by the previous assumption on T_1 and T_2, we have

$$\widetilde{S}_\phi(T_3K) \ge \widetilde{S}_\phi(T_1K) = \widetilde{S}_\phi(T_2K),$$

which is a contradiction. The proof is complete. $\square$

In view of Theorem 3.6, naturally, we introduce the following definition.

Definition 3.7. Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi$. The quantity

$$\widetilde{A}_\phi(K) = \min\left\{\widetilde{S}_\phi(TK) : T \in \mathrm{SL}(n)\right\}$$

is called the minimal dual Orlicz surface area of the star body K with respect to ϕ.

Obviously, $\widetilde{A}_\phi(K)$ is $\mathrm{SL}(n)$ invariant and a generalization of minimal dual surface area. If $\phi(t) = t^p$, $1 \le p < \infty$, then the notion of minimal dual Orlicz surface area reduces to that of minimal dual L_p-surface area.

4. A characterization of the minimal dual Orlicz surface area

Throughout this section, we impose a condition on $\phi \in \Phi$, that ϕ is smooth in $[0, \infty)$. Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi \cap C^1[0, \infty)$, the Borel measure $\mu_\phi(K, \cdot)$ on S^{n-1} is defined by

$$\mathrm{d}\mu_\phi(K, \cdot) = \phi'(\rho_K)\rho_K^{n+1}\,\mathrm{d}S.$$

For further discussion, we introduce the important notion of isotropy of measures. A nonnegative Borel measure μ on S^{n-1} is said to be isotropic if

$$\int_{S^{n-1}} (u \cdot v)^2\,\mathrm{d}\mu(u) = \frac{|\mu|}{n}, \quad \text{for all } v \in S^{n-1}.$$

Here, $|\mu|$ denotes the total mass of μ. The definition immediately yields

$$\int_{S^{n-1}} u_i^2\,\mathrm{d}\mu(u) = \frac{|\mu|}{n},$$

where u_i denotes the ith component of the coordinate of u. For nonzero $x \in \mathbb{R}^n \setminus \{o\}$, the notation $x \otimes x$ represents the linear operator of the rank 1 on $\mathbb{R}^n$ that takes y to $(x \cdot y)x$. It immediately gives that

$$\mathrm{tr}(x \otimes x) = |x|^2.$$

Equivalently, μ is isotropic if

$$\int_{S^{n-1}} u \otimes u\,\mathrm{d}\mu(u) = \frac{|\mu|}{n} I_n,$$

where I_n denotes the identity operator on $\mathbb{R}^n$. For more information about the isotropy, we refer to [1, 2, 4, 13, 14, 33].

The next theorem characterizes the star body with minimal Orlicz surface area.

Theorem 4.1. *Suppose that $K \in \mathcal{S}_o^n$, $\phi \in \Phi \cap C^1(0, \infty)$ and $T_0 \in \mathrm{SL}(n)$. Then the following assertions are equivalent:*

(i) $\widetilde{A}_\phi(K) = \widetilde{S}_\phi(T_0 K)$.

(ii) *The measure* $\mu_\phi(T_0K,\cdot)$ *is isotropic on* S^{n-1}.

(iii) *For all* $x \in \mathbb{R}^n$, *the transformation* T_0 *satisfies*

$$\begin{aligned}&|x|^2 \int_{S^{n-1}} |T_0 u| \rho_K(u) \phi'\left(\rho_K(u)|T_0u|\right) \mathrm{d}\widetilde{V}_K(u)\\ &= n \int_{S^{n-1}} \frac{|x\cdot T_0u|^2}{|T_0u|} \rho_K(u) \phi'\left(\rho_K(u)|T_0u|\right) \mathrm{d}\widetilde{V}_K(u).\end{aligned}$$

Proof. Firstly, we prove the equivalence of (i) and (ii).

Suppose that (i) holds. Since $\widetilde{A}_\phi(K)$ is $\mathrm{SL}(n)$ invariant, we may assume that T_0 is the $n\times n$ identity matrix I_n.

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Choose $\varepsilon_0>0$ sufficiently small so that for all $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ the matrix $I_n+\varepsilon T$ is invertible. For $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, define

$$T_\varepsilon = \frac{I_n+\varepsilon T}{|I_n+\varepsilon T|^{\frac{1}{n}}}.$$

Then $T_\varepsilon \in \mathrm{SL}(n)$. The assumption that $T_0 = I_n$ and (i) implies that for all ε,

$$\widetilde{S}_\phi(T_\varepsilon K) \ge \widetilde{S}_\phi(K).$$

According to the fact $1/\rho_{T_\varepsilon^{-1}B}(u) = h_{T_\varepsilon^t B}(u) = |T_\varepsilon u|$ for $u\in S^{n-1}$, together with the definition of $\widetilde{S}_\phi(T_\varepsilon K)$, and the equation

$$|(I_n+\varepsilon T)u| = 1+\varepsilon u\cdot Tu + O(\varepsilon^2),$$

and

$$|I_n+\varepsilon T|^{\frac{1}{n}} = 1+\varepsilon\frac{\mathrm{tr}T}{n} + O(\varepsilon^2),$$

we have

$$\begin{aligned}\widetilde{S}_\phi(T_\varepsilon K) &= n\int_{S^{n-1}} \phi\left(\rho_K(u)|T_\varepsilon u|\right) \mathrm{d}\widetilde{V}_K(u)\\ &= n\int_{S^{n-1}} \phi\left(\rho_K(u)\times \frac{1+\varepsilon u\cdot Tu+O(\varepsilon^2)}{1+\varepsilon\frac{\mathrm{tr}T}{n}+O(\varepsilon^2)}\right) \mathrm{d}\widetilde{V}_K(u).\end{aligned}$$

From the smoothness of ϕ and $|T_\varepsilon u|$ in ε, the integrand depends smoothly on ε. Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \widetilde{S}_\phi(T_\varepsilon K) = 0.$$

Calculating it directly, we have

$$\begin{aligned}
0 &= \int_{S^{n-1}} \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \phi\left(\rho_K(u) \times \frac{1+\varepsilon u\cdot Tu + O(\varepsilon^2)}{1+\varepsilon\frac{\operatorname{tr} T}{n} + O(\varepsilon^2)}\right) \mathrm{d}\widetilde{V}_K(u)\\
&= \int_{S^{n-1}} \phi'\left(\rho_K(u)\right)\left(u\cdot Tu - \frac{\operatorname{tr} T}{n}\right)\rho_K(u)\,\mathrm{d}\widetilde{V}_K(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \left(u\cdot Tu - \frac{\operatorname{tr} T}{n}\right) \mathrm{d}\mu_\phi(K,u).
\end{aligned}$$

Let $v \in S^{n-1}$ and $T = v\otimes v$. Using the facts $\operatorname{tr}(v\otimes v) = 1$ and $u\cdot(v\otimes v)u = (u\cdot v)^2$, it gives

$$\int_{S^{n-1}} (u\cdot v)^2\,\mathrm{d}\mu_\phi(K,u) = \frac{|\mu_\phi(K,u)|}{n}.$$

Thus, $\mu_\phi(K,\cdot)$ is isotropic on S^{n-1}.

Next, we show the implication "(ii) $\Rightarrow$ (i)". The proof will be completed by two steps.

Firstly, for a point $a = (a_1,\dots,a_n) \in [0,\infty)^n$, define

$$F(a) = \int_{S^{n-1}} \phi\left(\rho_K(u)\,|\operatorname{diag}(a_1,\dots,a_n)u|\right) \mathrm{d}\widetilde{V}_K(u),$$

where $\operatorname{diag}(a_1,\dots,a_n)$ denotes $n\times n$ diagonal matrix with diagonal elements $a_1,\dots,a_n$.

We aim to show that

$$F(a) \ge F(e), \quad \text{whenever } \prod_{j=1}^n a_j = 1. \tag{4.1}$$

Here, e denotes the point $(1,\dots,1)$.

From the smoothness of ϕ and $|\operatorname{diag}(a_1,\dots,a_n)u|$ in $(a_1,\dots,a_n)$, we have

$$\begin{aligned}
\frac{\partial}{\partial a_j}\bigg|_{a=e} F(a) &= \int_{S^{n-1}} \frac{\partial}{\partial a_j}\bigg|_{a=e} \phi\left(\rho_K(u)\,|\operatorname{diag}(a_1,\dots,a_n)u|\right) \mathrm{d}\widetilde{V}_K(u)\\
&= \int_{S^{n-1}} \phi'\left(\rho_K(u)\right)\rho_K(u) \frac{\partial}{\partial a_j}\bigg|_{a=e} |\operatorname{diag}(a_1,\dots,a_n)u|\ \mathrm{d}\widetilde{V}_K(u)\\
&= \int_{S^{n-1}} u_j^2 \phi'\left(\rho_K(u)\right)\rho_K(u)\,\mathrm{d}\widetilde{V}_K(u),
\end{aligned}$$

where $(u_1,\dots,u_n)$ denotes the coordinates of $u\in S^{n-1}$. From the isotropy of $\mu_\phi(K,\cdot)$, it follows that

$$\frac{\partial}{\partial a_j}\bigg|_{a=e} F(a) = \frac{|\mu_\phi(K,\cdot)|}{n}.$$

Thus,

$$\nabla F(e) = \frac{|\mu_\phi(K,\cdot)|}{n} e. \tag{4.2}$$

It can be checked that the function $F\colon [0,\infty)^n \to [0,\infty)$ is continuous and convex, and $F(\lambda a)$ is strictly increasing in $\lambda \in [0,\infty)$, for $a \in (0,\infty)^n$. Thus, $F^{-1}([0,F(e)])$ is compact, convex and of non-empty interior. Precisely, it is a convex body. Its boundary is given by the equation $F(a) = F(e)$ with $a \in (0,\infty)^n$, so (4.2) implies the vector e is an outer normal of the convex body $F^{-1}([0,F(e)])$ at the boundary point e. Consequently,

$$F^{-1}([0,F(e)]) \subset \{a \in \mathbb{R}^n : a \cdot e \le n\}\,.$$

That is to say, for all $a \in [0,\infty)^n$, if $F(a) \le F(e)$, then $a \cdot e \le n$. In contrast, for all $b = (b_1,\dots,b_n) \in (0,\infty)^n$ with $b_1 \cdots b_n = 1$, the AM-GM inequality yields that $b \cdot e \ge n$, with equality if and only if $b = e$. Hence, (4.1) is derived.

Secondly, with (4.1) in hand, we aim to show that for $T \in \mathrm{SL}(n)$, $\widetilde{S}_\phi(TK) \ge \widetilde{S}_\phi(K)$, with equality if and only if T is orthogonal.

Indeed, it is known that each $T \in \mathrm{SL}(n)$ can be represented as $T^{-1} = Q^{-1}A^{-1}P$, where P, Q are $n \times n$ orthogonal matrices, and $A = \mathrm{diag}(a_1,\dots,a_n)$ is diagonal and positive definite with $a_1 a_2 \cdots a_n = 1$. So, by Lemma 3.1, (2.5), (4.1), and Lemma 3.1 again, we have

$$\begin{aligned}
\widetilde{S}_\phi(TK) &= n\int_{S^{n-1}} \phi\left(\frac{\rho_{QK}(u)}{\rho_{A^{-1}B}(u)}\right) \mathrm{d}\widetilde{V}_{QK}(u)\\
&= n\int_{S^{n-1}} \phi\left(\rho_{QK}(u)|Au|\right) \mathrm{d}\widetilde{V}_{QK}(u)\\
&= n\int_{S^{n-1}} \phi(\rho_{QK}(u)|\,\mathrm{diag}(a_1,\dots,a_n)u|)\, \mathrm{d}\widetilde{V}_{QK}(u)\\
&\ge n\int_{S^{n-1}} \phi(\rho_{QK}(u)|\,\mathrm{diag}(1,\dots,1)u|)\, \mathrm{d}\widetilde{V}_{QK}(u)\\
&= n\int_{S^{n-1}} \phi\left(\rho_{QK}(u)\right) \mathrm{d}\widetilde{V}_{QK}(u)\\
&= \widetilde{S}_\phi(K).
\end{aligned}$$

Equality holds if and only if $(a_1,\dots,a_n) = (1,\dots,1)$, equivalently, if and only if T is orthogonal. Thus, the implication "(ii) $\Rightarrow$ (i)" is shown.

Next, we prove the equivalence of (ii) and (iii). Let $v = \left\langle T_0^{-1}u\right\rangle \in S^{n-1}$. From the definitions of $\mu_\phi(T_0K,\cdot)$ and dual cone-volume measure, (2.3) and (2.5), we have

$$\begin{aligned}
\mathrm{d}\mu_\phi(T_0K,u) &= \phi'\left(\rho_{T_0K}(u)\right)\rho_{T_0K}^{n+1}(u)\,\mathrm{d}S(u)\\
&= \phi'\left(\frac{\rho_K(\langle T_0^{-1}u\rangle)}{|T_0^{-1}u|}\right)\frac{\rho_K^{n+1}(\langle T_0^{-1}u\rangle)}{|T_0^{-1}u|}\,\mathrm{d}S(\langle T_0^{-1}u\rangle)\\
&= \phi'\left(\rho_K(\langle T_0^{-1}u\rangle)|T_0\left\langle T_0^{-1}u\right\rangle|\right)\rho_K^{n+1}(\langle T_0^{-1}u\rangle)|T_0\left\langle T_0^{-1}u\right\rangle|\,\mathrm{d}S(\langle T_0^{-1}u\rangle),
\end{aligned}$$

which immediately yields that

$$\int_{S^{n-1}} \rho_K(v)|T_0v|\phi'\left(\rho_K(v)|T_0v|\right)\,\mathrm{d}\widetilde{V}_K(v) = \frac{|\mu_\phi(T_0K,\cdot)|}{n}.$$

Meanwhile, for $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\int_{S^{n-1}} |x\cdot u|^2 \,\mathrm{d}\mu_\phi(T_0K,u) &= \int_{S^{n-1}} \frac{|x\cdot T_0\left\langle T_0^{-1}u\right\rangle|^2}{|T_0\left\langle T_0^{-1}u\right\rangle|^2}\phi'\left(\rho_K(\left\langle T_0^{-1}u\right\rangle)|T_0\left\langle T_0^{-1}u\right\rangle|\right) \\ &\quad\times \rho_K^{n+1}(\left\langle T_0^{-1}u\right\rangle)|T_0\left\langle T_0^{-1}u\right\rangle|\,\mathrm{d}S(\left\langle T_0^{-1}u\right\rangle) \\ &= n\int_{S^{n-1}} \frac{|x\cdot T_0v|^2}{|T_0v|}\phi'\left(\rho_K(v)|T_0v|\right)\rho_K(v)\,\mathrm{d}\widetilde{V}_K(v).\end{aligned}$$

With these, the equivalence of (ii) and (iii) is shown. The proof is complete. □

A direct corollary of Theorem 4.1 is:

Corollary 4.2. *A star body K in Euclidean n-space $\mathbb{R}^n$ has the minimal dual surface area $\widetilde{A}(K)$ among its $\mathrm{SL}(n)$ images if and only if its dual surface area measure $\widetilde{S}_K$ is isotropic on the unit sphere S^{n-1}.*

5. Bounds for the minimal dual Orlicz surface area

In this section, we estimate the minimal dual Orlicz surface area $\widetilde{A}_\phi(K)$. Theorem 5.1 and Theorem 5.2 give lower bounds. Theorem 5.5 and Theorem 5.7 give upper bounds.

Write

$$\widetilde{A}(K) = \min\left\{\widetilde{S}(TK) : T\in \mathrm{SL}(n)\right\}$$

for the minimal surface area of $K \in \mathcal{S}_o^n$.

The next theorem shows the relationship between $\widetilde{A}_\phi(K)$ and $\widetilde{A}(K)$.

Theorem 5.1. *Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi$. Then*

$$\widetilde{A}_\phi(K) \geq n|K|\phi\left(\frac{\widetilde{A}(K)}{n|K|}\right). \tag{5.1}$$

If K has an $\mathrm{SL}(n)$ image K' such that: (1) $\widetilde{S}_{K'}$ is isotropic; (2) $\rho_{K'}|_{\mathrm{supp}\,\widetilde{S}_{K'}}$, that is, the restriction of $\rho_{K'}$ to the support set of $\widetilde{S}_{K'}$, is constant, then equality holds in (5.1).

Conversely, if ϕ is strictly convex, then equality in (5.1) holds only if K has an $\mathrm{SL}(n)$ image K' which satisfies (1) and (2).

Proof. For $T \in \mathrm{SL}(n)$, recall that

$$\frac{\widetilde{S}_\phi(TK)}{n|K|} = \int_{S^{n-1}} \phi(\rho_{TK})\,\mathrm{d}\widetilde{V}_{TK}^*,$$

and

$$\frac{\widetilde{S}(TK)}{n|K|} = \int_{S^{n-1}} \rho_{TK}\,\mathrm{d}\widetilde{V}_{TK}^*.$$

Since ϕ is convex and $\widetilde{V}^*_{TK}$ is a probability measure, by Jensen's inequality (2.14), we have

$$\frac{\widetilde{S}_\phi(TK)}{n|K|} \geq \phi\left(\int_{S^{n-1}} \rho_{TK}\, \mathrm{d}\widetilde{V}^*_{TK}\right) = \phi\left(\frac{\widetilde{S}(TK)}{n|K|}\right),$$

which yields (5.1) by the existence of $\widetilde{A}_\phi(K)$ and $\widetilde{A}(K)$.

We proceed to prove the equality condition.

On one hand, by the condition (1), we have

$$\widetilde{A}(K) = \widetilde{S}(K').$$

On the other hand, by Theorem 4.1, the conditions (1) and (2), we know that $\mu_\phi(K', \cdot)$ is isotropic. Indeed, let's assume $\rho_{K'}|_{\operatorname{supp} \widetilde{S}_{K'}} = c > 0$ (c is a constant), then

$$\begin{aligned}
\frac{1}{\mu_\phi(K', u)} \int_{S^{n-1}} u \otimes u \, \mathrm{d}\mu_\phi(K', u) &= \frac{1}{\mu_\phi(K', u)} \int_{S^{n-1}} u \otimes u \phi'\left(\rho_{K'}(u)\right) \rho_{K'}(u)^{n+1} \, \mathrm{d}S(u) \\
&= \frac{1}{|S^{n-1}|} \int_{S^{n-1}} u \otimes u \, \mathrm{d}S(u) \\
&= I_n.
\end{aligned}$$

And then, by Theorem 4.1, it follows that

$$\widetilde{A}_\phi(K) = \widetilde{S}_\phi(K'). \tag{5.2}$$

In addition, the condition (2) can be exported that $|\widetilde{V}_{K'}| = \frac{c^n}{n}|S^{n-1}|$ and $\widetilde{S}(K') = c^{n+1}|S^{n-1}|$. By definition of $\widetilde{S}_\phi(K')$, we have

$$\begin{aligned}
\widetilde{S}_\phi(K') &= n \int_{S^{n-1}} \phi(\rho_{K'}) \, \mathrm{d}\widetilde{V}_{K'} \\
&= n \int_{S^{n-1}} \phi(c) \, \mathrm{d}\widetilde{V}_{K'} \\
&= c^n |S^{n-1}| \phi(c) \\
&= n|\widetilde{V}_{K'}| \phi\left(\frac{\widetilde{S}(K')}{n|K'|}\right) \\
&= n|K| \phi\left(\frac{\widetilde{S}(K')}{n|K|}\right).
\end{aligned} \tag{5.3}$$

Together (5.2) with (5.3), it follows that

$$\widetilde{A}_\phi(K) = \widetilde{S}_\phi(K') = n|K| \phi\left(\frac{\widetilde{S}(K')}{n|K|}\right).$$

Thus, $\widetilde{A}_\phi(K) = n|K|\phi\left(\frac{\widetilde{A}(K)}{n|K|}\right)$.

Conversely, the equality $\widetilde{A}_\phi(K) = n|K|\phi\left(\frac{\widetilde{A}(K)}{n|K|}\right)$, as well as the existence of $\widetilde{A}(K)$ and $\widetilde{A}_\phi(K)$, implies that K has two $\mathrm{SL}(n)$ images K_1 and K_2 which satisfy the following:

(3) $\widetilde{S}_\phi(K_1) = \widetilde{A}_\phi(K)$.

(4) $\widetilde{S}_\phi(K_1) = n|K|\phi\left(\frac{\widetilde{S}(K_2)}{n|K|}\right)$.

(5) For all $T \in \mathrm{SL}(n)$, $\widetilde{S}(TK_2) \geq \widetilde{S}(K_2)$, with equality if and only if T is orthogonal.

The proved inequality $\widetilde{S}_\phi(K_1) \geq n|K|\phi\left(\frac{\widetilde{S}(K_1)}{n|K|}\right)$ together with (4), yields

$$\phi\left(\frac{\widetilde{S}_{K_2}}{n|K|}\right) \geq \phi\left(\frac{\widetilde{S}_{K_1}}{n|K|}\right).$$

Since ϕ is strictly increasing, we have $\widetilde{S}_{K_2} \geq \widetilde{S}_{K_1}$. With this and (5), we conclude that K_1 differs from K_2 only by an orthogonal transformation. Thus, by minimal dual surface area theorem (Corollary 4.2), we know that $\widetilde{S}_{K_1}$ is isotropic on S^{n-1}. Moreover, by the orthogonal invariance of $\widetilde{S}$ and (4), we have

$$\widetilde{S}_\phi(K_1) = n|K|\phi\left(\frac{\widetilde{S}(K_1)}{n|K|}\right).$$

That is,

$$\int_{S^{n-1}} \phi(\rho_{K_1})\,\mathrm{d}\widetilde{V}^*_{K_1} = \phi\left(\int_{S^{n-1}} \rho_{K_1}\,\mathrm{d}\widetilde{V}^*_{K_1}\right).$$

Since $\widetilde{V}^*_{K_1}$ is a probability measure and ϕ is strictly convex, by the equality condition of Jensens inequality, it follows that $\rho_{K_1}|_{\mathrm{supp}\,\widetilde{V}^*_{K_1}}$, namely $\rho_{K_1}|_{\mathrm{supp}\,\widetilde{S}_{K_1}}$, is constant. □

Theorem 5.2. *Suppose that $K \in \mathcal{S}^n_o$ and $\phi \in \Phi$. Then*

$$\widetilde{A}_\phi(K) \geq n|K|\phi\left(\left(\frac{|K|}{|B|}\right)^{\frac{1}{n}}\right). \tag{5.4}$$

If ϕ is strictly convex, then equality in (5.4) *holds if and only if K is an origin-symmetric ellipsoid.*

Proof. Because there is a $T_0 \in \mathrm{SL}(n)$ such that $\widetilde{A}_\phi(K) = \widetilde{S}_\phi(T_0K)$, and the volume-normalized dual conical measure $\widetilde{V}^*_K$ is a probability measure on S^{n-1}, then by (2.10), Jensen's inequality (2.14), the integral formulas of L_p-dual mixed volume (2.8), dual

Minkowski inequality (2.11), and the fact that ϕ is increasing, we obtain

$$\begin{aligned}
\frac{\widetilde{A}_\phi(K)}{n|K|} = \frac{\widetilde{S}_\phi(T_0K)}{n|K|} &= \frac{\widetilde{V}_\phi(T_0K, B)}{|K|} \\
&= \int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T_0^{-1}B}}\right) \mathrm{d}\widetilde{V}_K^* \\
&\geq \phi\left(\int_{S^{n-1}} \left(\frac{\rho_K}{\rho_{T_0^{-1}B}}\right) \mathrm{d}\widetilde{V}_K^*\right) \\
&= \phi\left(\frac{\widetilde{V}_{-1}(K, T_0^{-1}B)}{|K|}\right) \\
&\geq \phi\left(\left(\frac{|K|}{|B|}\right)^{\frac{1}{n}}\right).
\end{aligned}$$

By the equality conditions of Jensen's inequality (2.14) and L_p-dual Minkowski inequality (2.11), and noting that for the linear transformation $T \in \mathrm{SL}(n)$ and an unit ball B, TB is an origin-symmetric ellipsoid, it follows that for $K \in \mathcal{S}_o^n$ equality in (5.4) holds if and only if K is an origin-symmetric ellipsoid. □

From the definition of dual Orlicz surface area, it is easily checked that for all $r > 0$,

$$\widetilde{S}_\phi(rB) = n\widetilde{V}_\phi(rB, B) = n\phi(r)|rB|.$$

We now establish the following dual Orlicz isoperimetric inequality for $\widetilde{A}_\phi(K)$. Let B_K be the origin-symmetric n-dimensional Euclidean ball with $|B_K| = |K|$. Therefore, $B_K = rB$ with $r = |K|^{1/n}|B|^{-1/n}$. Then

$$\widetilde{A}_\phi(B_K) = \widetilde{S}_\phi(B_K) = \phi(r) \cdot n|rB| = \phi\left(\left(\frac{|K|}{|B|}\right)^{\frac{1}{n}}\right) \cdot n|K|. \tag{5.5}$$

An immediate consequence of Theorem 5.2 is:

Corollary 5.3 (Dual Orlicz isoperimetric inequality)**.** *Suppose that $\phi \in \Phi$ and $K \in \mathcal{S}_o^n$, then*

$$\widetilde{A}_\phi(K) \geq \widetilde{A}_\phi(B_K). \tag{5.6}$$

If ϕ is strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid.

Proof. From Theorem 5.2 and (5.5), we have

$$\begin{aligned}
\widetilde{A}_\phi(K) &\geq n|K|\phi\left(\left(\frac{|K|}{|B|}\right)^{\frac{1}{n}}\right) \\
&= \widetilde{A}_\phi(B_K).
\end{aligned}$$

Apparently, if ϕ is strictly convex, equality holds if and only if K is an origin-symmetric ellipsoid. $\square$

The dual Orlicz isoperimetric inequality states that for all star bodies with fixed volume, the origin-symmetric Euclidean ball has the minimal dual Orlicz surface area for $\phi \in \Phi$. If $\phi(t) = t^p$ with $p \geq 1$, one can even have, by (5.6) of Corollary 5.3,

$$\frac{\widetilde{A}_p(K)}{\widetilde{A}_p(B)} \geq \left(\frac{|K|}{|B|}\right)^{\frac{n+p}{n}},$$

with equality if and only if K is an origin-symmetric ellipsoid.

In what follows, we use the dual L_∞-John ellipsoid discovered in [42] to estimate the dual minimal Orlicz surface area $\widetilde{A}_\phi(K)$. Suppose $K \in \mathcal{K}_o^n$. Recall that the dual L_∞-John ellipsoid, $\widetilde{E}_\infty$, is the unique origin-symmetric ellipsoid of minimal volume ellipsoid containing K. That is, among all origin-symmetric ellipsoids E, $\widetilde{E}_\infty$ is the unique one that solves the constrained maximization problem:

$$\max_E \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} \quad \text{subject to } \overline{\widetilde{V}}_{-\infty}(K, E) \leq 1,$$

where $\overline{\widetilde{V}}_{-\infty}(K, E) = \max\left\{\frac{\rho_K(u)}{\rho_E(u)} : u \in S^{n-1}\right\}$. Indeed, $\widetilde{E}_\infty K$ necessarily satisfies $\overline{\widetilde{V}}_{-\infty}(K, \widetilde{E}_\infty K) = 1$. Write $\overline{\widetilde{E}}_\infty K$ for $\left(|B|/|\widetilde{E}_\infty K|\right)^{1/n} \widetilde{E}_\infty K$. As it was shown in [42], $\overline{\widetilde{E}}_\infty K$ is the unique $\mathrm{SL}(n)$ image of B which satisfies

$$\overline{\widetilde{V}}_{-\infty}(K, \overline{\widetilde{E}}_\infty K) = \min\left\{\overline{\widetilde{V}}_{-\infty}(K, TB) : T \in \mathrm{SL}(n)\right\}.$$

The following lemma is needed in the proof of Theorem 5.5.

Lemma 5.4. [42] *If $K \in \mathcal{K}_0^n$, then for $T \in \mathrm{GL}(n)$, $\widetilde{E}_\infty TK = T\widetilde{E}_\infty K$.*

It is known that the classical Löowner ellipsoid (or Löwner-John ellipsoid) of convex body K is the unique ellipsoid of minimal volume ellipsoid containing K. Here we denotes the Löwner ellipsoid of K by $\widetilde{J}K$, since it can be regarded as the dual of the John ellipsoid JK (the maximal volume ellipsoid contained in K). The Löwner-John ellipsoid is extremely useful (see, for example, [2,9] for applications). In fact, if K is origin-symmetric, then $\widetilde{E}_\infty K$ is the classical Löwner ellipsoid $\widetilde{J}K$ of K.

Theorem 5.5. *Suppose that $K \in \mathcal{S}_o^n$ and $\phi \in \Phi$. Then*

$$\widetilde{A}_\phi(K) \leq n|K|\phi\left(\left(\frac{|\widetilde{E}_\infty K|}{|B|}\right)^{\frac{1}{n}}\right), \tag{5.7}$$

with equality if $K = \widetilde{E}_\infty K$ is an ellipsoid centered at the origin.

Proof. Suppose that $T \in \mathrm{SL}(n)$ and $1 \le p < \infty$. From Lemma 3.1, Jensen's inequality (2.14), and the definition of $\overline{\widetilde{V}}_\infty$, we have

$$
\begin{aligned}
\frac{\widetilde{S}_\phi(TK)}{n|K|} &= \int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T^{-1}B}}\right) \mathrm{d}\widetilde{V}_K^* \\
&\le \lim_{p\to\infty} \left(\int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_{T^{-1}B}}\right)^p \mathrm{d}\widetilde{V}_K^*\right)^{\frac{1}{p}} \\
&= \max\left\{\phi\left(\frac{\rho_K(u)}{\rho_{T^{-1}B}(u)}\right) : u \in S^{n-1}\right\} \\
&= \phi\left(\max\left\{\frac{\rho_K(u)}{\rho_{T^{-1}B}(u)} : u \in S^{n-1}\right\}\right) \\
&= \phi\left(\overline{\widetilde{V}}_\infty(K, T^{-1}B)\right).
\end{aligned} \tag{5.8}
$$

According to the condition of equality in Jensen's inequality (2.14), we see that the equality in (5.8) holds if and only if there is a constant $c > 0$ such that $\rho_K = c\rho_{T^{-1}B}$. Namely, $K = T^{-1}(cB)$ is an ellipsoid centered at the origin.

Now, from (5.7) and the definitions of $\widetilde{A}_\phi(K)$, $\overline{\widetilde{E}}_\infty K$, $\overline{\widetilde{V}}_\infty$ and $\widetilde{E}_\infty K$, it follows that

$$
\begin{aligned}
\frac{\widetilde{A}_\phi(K)}{n|K|} &\le \min\left\{\phi\left(\overline{\widetilde{V}}_\infty(K, T^{-1}B)\right) : T \in \mathrm{SL}(n)\right\} \\
&= \phi\left(\min\left\{\overline{\widetilde{V}}_\infty(K, T^{-1}B) : T \in \mathrm{SL}(n)\right\}\right) \\
&= \phi\left(\overline{\widetilde{V}}_\infty(K, \overline{\widetilde{E}}_\infty K)\right) \\
&= \phi\left(\left(\frac{|\widetilde{E}_\infty K|}{|B|}\right)^{\frac{1}{n}} \overline{\widetilde{V}}_\infty(K, \widetilde{E}_\infty K)\right) \\
&= \phi\left(\left(\frac{|\widetilde{E}_\infty K|}{|B|}\right)^{\frac{1}{n}}\right),
\end{aligned}
$$

as desired.

According to conditions of inequality (5.8) and Lemma 5.4, we see that there is a constant $c > 0$ and $T \in \mathrm{SL}(n)$ such that $K = T^{-1}(cB) = \widetilde{E}_\infty K$ is an ellipsoid centered at the origin. $\square$

A consequence of Barthe's reverse Brascamp-Lieb inequality (see [3]) is the outer volume ratio inequality which can be regarded as the dual form of Ball's volume-ratio inequality:

Lemma 5.6. [3] *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$
\frac{|K|}{|\widetilde{J}K|} \ge \frac{2^n}{n!|B|}, \tag{5.9}
$$

with equality if and only if K is a parallelotope.

If K is origin-symmetric, one precise upper bounds for $\widetilde{A}_\phi(K)$ can be obtained.

Theorem 5.7. *Suppose that $\phi \in \Phi$ and K is an origin-symmetric convex body in $\mathbb{R}^n$. Then*

$$\widetilde{A}_\phi(K) \le n|K|\phi\left(\left(\frac{n!|K|}{2^n}\right)^{\frac{1}{n}}\right). \tag{5.10}$$

Proof. Since K is an origin-symmetric convex body in $\mathbb{R}^n$, it implies that $\widetilde{E}_\infty K = \widetilde{J}K$. From Theorem 5.5 and Lemma 5.6, we have

$$\widetilde{A}_\phi(K) \le n|K|\phi\left(\left(\frac{|\widetilde{E}_\infty K|}{|B|}\right)^{\frac{1}{n}}\right) = n|K|\phi\left(\left(\frac{\widetilde{J}K}{|B|}\right)^{\frac{1}{n}}\right) \le n|K|\phi\left(\left(\frac{n!|K|}{2^n}\right)^{\frac{1}{n}}\right). \quad \square$$

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