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PERIODIC SOLUTIONS FOR A SINGULAR LIÉNARD EQUATION WITH INDEFINITE WEIGHT

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Abstract. In this paper, the existence of positive periodic solutions is studied for a singular Liénard equation where the weight function has an indefinite sign. Due to the lack of a priori estimates over the set of all possible positive periodic solutions in this equation, a new method is proposed for estimating a priori bounds of positive periodic solutions. By the use of a continuation theorem of the Mawhin coincidence degree, new conditions for existence of positive periodic solutions to the equation are obtained. The main results show that the singularity of coefficient function associated to the friction term at $x = 0$ may help the existence of periodic solutions.

1. Introduction

The purpose of this paper is to study the existence of positive T-periodic solutions for a singular Liénard equation with indefinite weight

$$x''(t) + f(x(t))x'(t) + \frac{\alpha(t)}{x^{\mu}(t)} = h(t), \tag{1.1}$$

where $f \in C((0,+\infty),\mathbb{R})$ may have a singularity at $x = 0$, $\mu \in (0,+\infty)$ is a constant, α and h are T-periodic functions with $\alpha, h \in L^1([0,T],\mathbb{R})$. Since the weight function α may change sign on $[0,T]$, the singularity of the term $\alpha(t)/x^{\mu}$

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at $x = 0$ can be regarded neither as attractive type nor as repulsive type. Just for this reason, the singularity of (1.1) is named as indefinite type.

Periodic solutions to singular differential equation has a long history. The first reference associated to it seems to be the paper of Nagumo [24] in 1943. In [26], P.J. Torres presents many periodic problems associated with singular models arising from physics, engineering, ecology and other applied sciences. In the past years, the periodic problem for second order differential equations with singularities was extensively studied, both, the case of Liénard type [9], [18], [29], and the case without friction term [5], [10], [13], [14], [18]–[20], [25], [29]. We notice that the weight function $\alpha(t)$ associated to the singular restoring force term $\alpha(t)/x^{\mu}$ in the equation considered in [5], [9], [10], [13], [14], [18]–[20], [25], [28], [29] does not change sign on $[0,T]$ and the coefficient function $f(x)$ in the friction term $f(x)x'$ is required to be continuous at $x = 0$ [9], [18], [28], [29]. In [7], [11], [12], [22], the authors considered periodic problem for the equation as (1.1) in the case where the function $f(x)$ has a singularity at $x = 0$ and the weight function $\alpha(t)$ has a definite sign. However, although it is quite relevant for applications, the study of periodic problem with singularity and indefinite weight has not been sufficiently developed yet. We only find a few articles [2]–[4] and [15] considering the problem of periodic solutions to the singular like

$$x''(t) = \frac{\alpha(t)}{x^{\mu}}, \tag{1.2}$$

where the sign of weight function $\alpha(t)$ can change on $[0,T]$. In [4], $\alpha(t)$ is required to be piecewise-constant with two pieces, and in [15], $\alpha(t)$ is required to have a finite number of sign-changes, i.e. there are pairwise disjoint intervals $[a_k, b_k]$ $(k = 1, \ldots, n)$ such that

$$\begin{aligned}
\alpha(t) &\geq 0 \quad \text{for a.e. } t \in \bigcup_{k=1}^{n} [a_k, b_k], \\
\alpha(t) &\leq 0 \quad \text{for a.e. } t \in [0,T] \setminus \bigcup_{k=1}^{n} [a_k, b_k].
\end{aligned}$$

Some relations between the order μ of the singularity of restoring force $\alpha(t)/x^{\mu}$ and the order of the zeros of $\alpha(t)$ are needed [3], [15]. For other recent developments on the study of this topic, we refer the reader to [1], [6], [16], [17], [21]. As far as we know, there are no results on the existence of periodic solutions to the singular Liénard equation of the form (1.1).

Motivated by this, the aim of this paper is to study the periodic problem for (1.1). The proof of main results rely on a continuation theorem of the coincidence degree theory established by Mawhin. Since the weight function $\alpha(t)$ has indefinite sign, generally, there is lack of any a priori estimate over the set of all possible positive T-periodic solutions to equation (1.1) with a parameter λ

$$x''(t) + \lambda f(x(t))x'(t) + \frac{\lambda\alpha(t)}{x^{\mu}(t)} = \lambda h(t), \quad \lambda \in (0,1). \tag{1.3}$$

Such a priori estimate is crucial for us to apply some continuation theorem of coincidence degree theory [23], [8]. To overcome these difficulties, we propose a new method for estimating *a priori bounds* of all possible positive T-periodic solutions to (1.3), where a priori estimates are treated on a suitable open subset V of C^1_T rather than on whole space C^1_T (see Theorem 3.1), and so the requirements on a priori estimates are weakened. We allow $f(x)$ to have a singularity at $x = 0$, and the relation between the order μ of the singularity of $\alpha(t)/x^{\mu}$ and the order of singularity of $f(x)$ at $x = 0$ is investigated. Such a relation plays an important role in estimating *a priori bounds* of positive T-periodic solutions from below. Moreover, in (1.1), the weak singularity condition $\mu \in (0,1)$ is allowed, which is essentially different from the strong singularity condition $\mu \in [1,+\infty)$ needed in [15] and [27].

2. Preliminaries

Throughout this paper, let $C_T = \{x \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t) \text{for all } t \in \mathbb{R}\}$ with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$, and $C^1_T = \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\}$ with the norm $||x||_{C^1_T} = \max\{||x||_\infty, ||x'||_\infty\}$. For any T-periodic function $y(t)$ with $y \in L^1([0,T],\mathbb{R})$, let $y_+(t) = \max\{y(t), 0\}$, $y_-(t) = -\min\{y(t), 0\}$, and $\overline{y} = (1/T)\int_0^T y(s)\,ds$. Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in \mathbb{R}$, $\overline{y} = \overline{y_+} - \overline{y_-}$.

Lemma 2.1 ([11]). *Let $u \in [0,\omega] \to \mathbb{R}$ be an arbitrary absolutely continuous function with $u(0) = u(\omega)$. Then the inequality*

$$\Big(\max_{t\in[0,\omega]} u(t) - \min_{t\in[0,\omega]} u(t)\Big)^2 \le \frac{\omega}{4}\int_0^{\omega} |u'(s)|^2 d\,s$$

holds.

Now, we introduce a continuation theorem of the coincidence degree established by Mawhin, which is the theoretic basis of this paper.

Let X and Y be Banach spaces, and let $L\colon D(L) \subset X \to Y$ be a Fredholm operator with index zero, where $D(L)$ denotes the domain of the operator L. Then, $\operatorname{Im} L$ is a closed subset of Y and $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$. This implies that there are two continuous projectors $P\colon X \to \ker L$ and $Q\colon Y \to Y$ satisfying $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Then we have

$$X = \ker L \oplus \ker P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Since $\ker L \cap (D(L) \cap \ker P) = \{0\}$, the restriction $L_P := L|_{D(L)\cap\ker P} \to \operatorname{Im} L$ is invertible. Denote by K_P the inverse of L_P. Suppose that $\Omega \subset X$ is an open bounded subset, a continuous operator $N\colon \overline{\Omega} \to Y$ is said to be L-compact on $\overline{\Omega}$, provided that $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact and $QN\colon \overline{\Omega} \to Y$ is bounded.

Lemma 2.2 ([8]). *Let X and Y be two real Banach spaces. Suppose that $L\colon D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N\colon \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied*:

(a) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;*
(b) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \ker L'$;*
(c) *The Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where $J\colon \operatorname{Im} Q \to \ker L$ is an isomorphism.*

Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega}$.

In order to study the existence of positive periodic solutions to equation (1.1), we list the following assumptions:

(H1) $\overline{h} > 0$, $\overline{\alpha} > 0$, $\alpha(t) \leq 0, t \in J$; $\alpha(t) > 0$, $t \in [0,T] \setminus J$, where J is a closed subset of $[0,T]$.
(H2) $\overline{h} < 0$, $\overline{\alpha} < 0$, $\alpha(t) \geq 0$, $t \in I$; $\alpha(t) < 0$, $t \in [0,T] \setminus I$, where I is a closed subset of $[0,T]$.

Now, we embed equation (1.1) into the following equations family with a parameter $\lambda \in (0,1)$

$$x''(t) + \lambda f(x(t))x'(t) + \lambda \frac{\alpha(t)}{x^\mu(t)} = \lambda h(t), \quad \lambda \in (0,1). \tag{2.1}$$

3. Main results

In this section, let

$$F(x) = \int_1^x f(s)\,ds, \quad x \in (0,+\infty), \tag{3.1}$$

where $f(x)$ is the coefficient function of the friction term $f(x)\,x'$ in (1.1).

Theorem 3.1. *Suppose that assumption* (H1) *holds. If there are two constants*

$$\varepsilon \in \left(0, \left(\frac{\overline{\alpha}}{\overline{h}}\right)^{1/\mu}\right) \tag{3.2}$$

$$\gamma_1 \in (\varepsilon, A(\varepsilon)) \cap \left(0, \left(\frac{\overline{\alpha}}{\overline{h}}\right)^{1/\mu}\right), \tag{3.3}$$

such that

$$\inf_{x\in(0,\gamma_1)} \left(F(x) - \frac{T\,\overline{\alpha}_-}{x^\mu}\right) > \max_{x\in[A(\varepsilon),M_0]} F(x) + T\overline{h}_+ \tag{3.4}$$

or

$$\sup_{x\in(0,\gamma_1)}\left(F(x)+\frac{T\,\overline{\alpha}_-}{x^\mu}\right)<\min_{x\in[A(\varepsilon),M_0]}F(x)-T\overline{h}_+, \tag{3.5}$$

where $F(x)$ is determined by (3.1), $A(\varepsilon)=\varepsilon(\overline{\alpha}_+/(\overline{\alpha}_-+\varepsilon^\mu\overline{h}))^{1/\mu}$,

$$M_0:=\left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu}+\frac{\sqrt{T}}{2}A_0,$$

$$A_0=\frac{1}{2}\left(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu}+T\overline{h}_+\right)\sqrt{T}+\left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/(2\mu)}\left(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu}+T\overline{h}_+\right)^{1/2},$$

then equation (1.1) *has at least one positive T-periodic solution.*

PROOF. Let us define $V=\{x\in C^1_T: x(t)>0,\ t\in[0,T];\ x(t)>\varepsilon,\ t\in J\}$, where $J\subset[0,T]$ is determined in assumption (H1). Clearly, $V\subset C^1_T$ is an open set. Suppose that $u\in\overline{V}$ and u is a positive T-periodic solution to equation (2.1). Then

$$u''(t)+\lambda f(u(t))u'(t)+\frac{\lambda\alpha(t)}{u^\mu(t)}=\lambda h(t),\quad \lambda\in(0,1). \tag{3.6}$$

Integrating it over $[0,T]$, we obtain

$$\int_0^T\frac{\alpha(s)}{u^\mu(s)}\,ds=T\overline{h}, \tag{3.7}$$

i.e.

$$\int_J\frac{\alpha(s)}{u^\mu(s)}\,ds+\int_{[0,T]\setminus J}\frac{\alpha(s)}{u^\mu(s)}\,ds=T\overline{h}.$$

It follows from (H1) that

$$-\int_J\frac{\alpha_-(s)}{u^\mu(s)}\,ds+\int_0^T\frac{\alpha_+(s)}{u^\mu(s)}\,ds=T\overline{h},$$

which together with the definition of the set V yields

$$\begin{aligned}\int_0^T\frac{\alpha_+(s)}{u^\mu(s)}\,ds&=\int_J\frac{\alpha_-(s)}{u^\mu(s)}\,ds+T\overline{h}\\&\le\int_J\frac{\alpha_-(s)}{\varepsilon^\mu}\,ds+T\overline{h}=\int_0^T\frac{\alpha_-(s)}{\varepsilon^\mu}\,ds+T\overline{h}=\frac{T}{\varepsilon^\mu}\,\overline{\alpha}_-+T\overline{h}.\end{aligned} \tag{3.8}$$

Thus, there is a point $\xi\in[0,T]$ such that

$$\frac{\overline{\alpha}_+}{u^\mu(\xi)}\le\frac{\overline{\alpha}_-}{\varepsilon^\mu}+\overline{h},\quad\text{i.e.}\quad u(\xi)\ge\varepsilon\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_-+\varepsilon^\mu\overline{h}}\right)^{1/\mu}=A(\varepsilon).$$

The assumption $\varepsilon\in\big(0,(\overline{\alpha}/\overline{h})^{1/\mu}\big)$ in (3.2) implies that

$$u(\xi)\ge A(\varepsilon)>\varepsilon. \tag{3.9}$$

Furthermore, the inequality

$$\int_0^T\frac{\alpha_+(s)}{u^\mu(s)}\,ds\ge T\overline{h},$$

which can be obtained from (3.7), implies that there is a constant $\eta > 0$ such that

$$u(\eta) \le \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu}. \tag{3.10}$$

On the other hand, multiplying (3.6) with $u(t)$, and integrating it on $[0,T]$, we have

$$\int_0^T |u'(t)|^2\,dt = \lambda\int_0^T \frac{\alpha(t)u(t)}{u^\mu(t)}\,dt - \lambda\int_0^T u(t)h(t)\,dt,$$

i.e.

$$\begin{aligned}\int_0^T |u'(t)|^2\,dt &= \lambda\int_J \frac{\alpha(t)u(t)}{u^\mu(t)}\,dt + \lambda\int_{[0,T]\setminus J} \frac{\alpha(t)u(t)}{u^\mu(t)}\,dt - \lambda\int_0^T u(t)h(t)\,dt\\ &\le \int_{[0,T]\setminus J} \frac{\alpha(t)u(t)}{u^\mu(t)}\,dt + \int_0^T u(t)h_-(t)\,dt\\ &\le \|u\|_\infty\bigg(\int_0^T \frac{\alpha_+(t)}{u^\mu(t)}\,dt + \int_0^T h_-(t)\,dt\bigg),\end{aligned}$$

which together with (3.8) yields

$$\int_0^T |u'(t)|^2\,dt \le \|u\|_\infty\bigg(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h} + T\overline{h}_-\bigg) = \|u\|_\infty\bigg(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg). \tag{3.11}$$

Moreover, by using Lemma 2.1 and (3.10), we get

$$\|u\|_\infty \le \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu} + \frac{\sqrt{T}}{2}\bigg(\int_0^T |u'(s)|^2\,ds\bigg)^{1/2}. \tag{3.12}$$

Substituting it into (3.11), we have

$$\begin{aligned}\int_0^T &|u'(t)|^2\,dt\\ &\le \bigg[\left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu} + \frac{\sqrt{T}}{2}\bigg(\int_0^T |u'(s)|^2\,ds\bigg)^{1/2}\bigg]\bigg(\frac{T\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg)\\ &= \frac{\sqrt{T}}{2}\bigg(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg)\bigg(\int_0^T |u'(s)|^2\,ds\bigg)^{1/2} + \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu}\bigg(\frac{T\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg).\end{aligned}$$

If we set

$$X = \bigg(\int_0^T |u'(s)|^2\,ds\bigg)^{1/2},$$

$$A = \frac{\sqrt{T}}{2}\bigg(\frac{T\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg) \quad\text{and}\quad B = \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu}\bigg(\frac{T\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg),$$

then the above inequality can be written as $X^2 \le AX + B$, which results in $X \le (A + \sqrt{A^2+4B})/2 \le A + B^{1/2}$, i.e.

$$\bigg(\int_0^T |u'(s)|^2 ds\bigg)^{1/2} \le \frac{1}{2}\bigg(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg)\sqrt{T} + \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/(2\mu)}\bigg(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\bigg)^{1/2}.$$

Thus, it follows from (3.12) that

$$||u||_\infty \le \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/\mu} + \frac{\sqrt{T}}{2}A_0 := M_0, \tag{3.13}$$

where

$$A_0 = \frac{1}{2}\left(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\right)\sqrt{T} + \left(\frac{\overline{\alpha}_+}{\overline{h}}\right)^{1/(2\mu)}\left(\frac{T\,\overline{\alpha}_-}{\varepsilon^\mu} + T\overline{h}_+\right)^{1/2}.$$

Furthermore, we can conclude that there exist points $t_1, t_2 \in \mathbb{R}$ such that

$$u(t_1) = \max_{t\in[0,T]} u(t), \qquad u(t_2) = \min_{t\in[0,T]} u(t),$$

$$0 < t_2 - t_1 < T. \tag{3.14}$$

In fact, if $0 < t_1 - t_2 < T$, (3.14) follows directly from the case that t_2 is replaced by $t_2 + T$. From (3.9) and (3.13), we have $A(\varepsilon) \le u(t_1) \le M_0$, and then

$$A_2 := \min_{x\in[A(\varepsilon),M_0]} F(x) \le F(u(t_1)) \le \max_{x\in[A(\varepsilon),M_0]} F(x) := A_1. \tag{3.15}$$

Integrating (3.6) over the interval $[t_1, t_2]$, we obtain

$$F(u(t_2)) = F(u(t_1)) - \int_{t_1}^{t_2} \frac{\alpha(s)}{u^\mu(s)}\,ds + \int_{t_1}^{t_2} h(s)\,ds. \tag{3.16}$$

In virtue of (3.15), we get

$$\begin{aligned} F(u(t_2)) &\le A_1 + \int_{t_1}^{t_2} \frac{\alpha_-(s)}{u^\mu(s)}\,ds + \int_{t_1}^{t_2} h_+(s)\,ds \\ &\le A_1 + \int_0^T \frac{\alpha_-(s)}{u^\mu(s)}\,ds + \int_0^T h_+(s)\,ds \le A_1 + \frac{T\,\overline{\alpha}_-}{u^\mu(t_2)} + T\overline{h}_+. \end{aligned}$$

i.e.

$$F(u(t_2)) - \frac{T\,\overline{\alpha}_-}{u^\mu(t_2)} \le A_1 + T\overline{h}_+.$$

By using condition (3.4), we have

$$\min_{t\in[0,T]} u(t) = u(t_2) > \gamma_1. \tag{3.17}$$

In virtue of (3.15) again, (3.16) gives us that

$$\begin{aligned} F(u(t_2)) &\ge A_2 - \int_{t_1}^{t_2} \frac{\alpha_+(s)}{u^\mu(s)}\,ds - \int_{t_1}^{t_2} h_-(s)\,ds \\ &\ge A_2 - \int_0^T \frac{\alpha_+(s)}{u^\mu(s)}\,ds - \int_0^T h_-(s)\,ds \\ &= A_2 - \int_0^T \frac{\alpha(s)}{u^\mu(s)}\,ds - \int_0^T \frac{\alpha_-(s)}{u^\mu(s)}\,ds - T\overline{h}_-. \end{aligned}$$

It follows from (3.7) that

$$F(u(t_2)) \ge A_2 - T\overline{h} - T\overline{h}_- - \int_0^T \frac{\alpha_-(s)}{u^\mu(s)}\,ds \ge A_2 - T\overline{h}_+ - \frac{T\,\overline{\alpha}_-}{u^\mu(t_2)},$$

i.e.

$$F(u(t_2)) + \frac{T\,\overline{\alpha}_-}{u^\mu(t_2)} \geq A_2 - T\overline{h}_+,$$

which together with (3.5) also yields (3.17). Hence, (3.17) is obtained under either condition (3.4) or condition (3.5).

Next, if u attains its maximum over $[0,T]$ at $t_1 \in [0,T]$, then $u'(t_1)$=0 and we see from (3.6) that

$$u'(t) = \lambda \int_{t_1}^{t} \left[-f(u(s))u'(s) - \frac{\alpha(s)}{u^\mu(s)} + h(s) \right] ds, \quad \text{for all } t \in [t_1, t_1+T].$$

Then, we have

$$\begin{aligned} |u'(t)| &\leq \lambda |F(u(t)) - F(u(t_1))| + \lambda \int_{t_1}^{t_1+T} \frac{|\alpha(s)|}{u^\mu(s)}\, ds + \lambda \int_{t_1}^{t_1+T} |h(s)|\, ds \\ &\leq 2 \max_{\gamma_1 \leq u \leq M_0} |F(u)| + \int_0^T \frac{|\alpha(s)|}{u^\mu(s)}\, ds + \int_0^T |h(s)|\, ds. \end{aligned}$$

It follows from (3.17) that

$$|u'(t)| \leq 2 \max_{\gamma_1 \leq u \leq M_0} |F(u)| + \frac{T|\overline{\alpha}|}{\gamma_1^\mu} + T|\overline{h}| := M_1, \quad \text{for all } t \in [0,T].$$

and then

$$\max_{t\in[0,T]} |u'(t)| \leq M_1. \tag{3.18}$$

Let us define $X = C_T^1$ and $Y = L^1([0,T],\mathbb{R})$. Define $L\colon D(L) \subset X \to Y$ by $Lx = x''$, where $D(L) = \{x \in X : x'' \in L^1([0,T],\mathbb{R})\}$. $\ker L = \mathbb{R}$, $\operatorname{Im} L = \big\{y \in Y : \int_0^T y(t)\, dt = 0\big\}$. It is easy to see that L is a Fredholm operator with index zero. Define

$$V_1 = \{x \in C_T^1 : \gamma_1 < x(t) < M_0 + 1 := m_1,\ t \in [0,T];\ \|x'\|_\infty < M_1 + 1 := m_2\},$$

$\Omega = V \cap V_1$ and $N\colon \overline{\Omega} \to Y$ by

$$(Nx)(t) = -f(x(t))x'(t) - \frac{\alpha(t)}{x^\mu(t)} + h(t), \quad t \in [0,T].$$

It is easy to see from (3.3) that $(\partial V \cap \overline{V_1}) = \emptyset$, which together with the fact $\partial\Omega \subset (\partial V \cap \overline{V_1}) \cup (\overline{V} \cap \partial V_1)$ gives $\partial\Omega \subset (\overline{V} \cap \partial V_1)$. So we can prove that condition (a) of Lemma 2.2 is satisfied. In fact, if condition (a) of Lemma 2.2 does not hold, then there are $\lambda_0 \in (0,1)$ and $x_0 \in \partial\Omega$ such that $Lx_0 = \lambda_0 N x_0$. It follows from the fact $\partial\Omega \subset (\overline{V} \cap \partial V_1)$ that

$$x_0 \in (\overline{V} \cap \partial V_1) \tag{3.19}$$

and $Lx_0 = \lambda_0 N x_0$. However, from (3.13), (3.17) and (3.18), we see that, if $x_0 \in \overline{V}$ such that $Lx_0 = \lambda_0 N x_0$, then

$$\gamma_1 < x_0(t) < m_1, \quad t \in [0,T]; \quad \|x'\|_\infty < m_2.$$

According to the definition of V_1, we see that $x_0 \notin \partial V_1$. This contradicts to the above conclusion (3.19).

If $x \in (\overline{V} \cap \partial V_1) \cap \ker L$, then $x(t) \equiv \gamma_1$ or $x(t) \equiv m_1$. By this, we have

$$QN\gamma_1 = \frac{1}{T}\int_0^T \bigg(-\frac{\alpha(t)}{\gamma_1^\mu} + h(t)\bigg)\,dt = -\frac{\overline{\alpha}}{\gamma_1^\mu} + \overline{h}$$

and

$$QNm_1 = \frac{1}{T}\int_0^T \bigg(-\frac{\alpha(t)}{m_1^\mu} + h(t)\bigg)\,dt = -\frac{\overline{\alpha}}{m_1^\mu} + \overline{h}.$$

It follows from assumption of $\gamma_1 \in (0, (\overline{\alpha}/\overline{h})^{1/\mu})$ in (3.3) and the definition of m_1 that

$$QN(\gamma_1) < 0 \quad \text{and} \quad QN(m_1) > 0,$$

which gives that

$$QNx \neq 0 \quad \text{for all } x \in \partial\Omega \cap \ker L \tag{3.20}$$

and

$$\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0. \tag{3.21}$$

(3.20) and (3.21) imply that condition (b) and condition (c) of Lemma 2.2 are satisfied. Thus, by using Lemma 2.2, we see that (1.1) has at least one positive T-periodic solution. □

Corollary 3.2. *Assume that $\alpha(t) \geq 0$ for almost every $t \in [0,T]$ with $\overline{\alpha} > 0$, and*

$$\lim_{x\to 0^+} |F(x)| = +\infty. \tag{3.22}$$

Then, equation (1.1) *has a positive T-periodic solution if and only if $\overline{h} > 0$.*

Proof. Let $u(t)$ be a positive T-periodic solution to (1.1), then

$$u''(t) + f(u(t))u'(t) + \frac{\alpha(t)}{u^\mu(t)} = h(t). \tag{3.23}$$

The necessity follows by integrating equation (3.23) over $[0,T]$ and using the condition of $\alpha(t) \geq 0$ for almost every $t \in [0,T]$ with $\overline{\alpha} > 0$. Below, we will prove the sufficiency. Suppose that $\overline{h} > 0$. From the condition $\alpha(t) \geq 0$ for almost every $t \in [0,T]$ with $\overline{\alpha} > 0$, we see that $\overline{\alpha}_- = 0$, $\overline{\alpha}_+ = \overline{\alpha}$, and assumption (H1) holds, where $J = \{t \in [0,T] : \alpha(t) = 0\}$. Thus, the constants of $A(\varepsilon)$ and M_0 in Theorem 3.1 are replaced by

$$A(\varepsilon) = \bigg(\frac{\overline{\alpha}}{\overline{\overline{h}}}\bigg)^{1/\mu} \quad \text{and} \quad M_0 = \bigg(\frac{\overline{\alpha}}{\overline{\overline{h}}}\bigg)^{1/\mu} + \frac{\sqrt{T}}{2}\bigg[\frac{T^{3/2}\overline{h}_+}{2} + \bigg(\frac{\overline{\alpha}}{\overline{\overline{h}}}\bigg)^{1/(2\mu)}\big(T\overline{h}_+\big)^{1/2}\bigg],$$

which are all independent of ε. In addition, assumption (3.22) implies that there is a constant

$$\gamma_1 \in \left(0, \left(\frac{\overline{\alpha}}{\overline{h}}\right)^{1/\mu}\right)$$

such that

$$\begin{aligned}\text{(3.24)}\qquad \inf_{x\in(0,\gamma_1)}\left(F(x)-\frac{T\,\overline{\alpha}_-}{x^\mu}\right) &= \inf_{x\in(0,\gamma_1)} F(x)\\ &> \max_{x\in\left[(\overline{\alpha}/\overline{h})^{1/\mu},M_0\right]} F(x)+T\overline{h}_+ = \max_{x\in[A(\varepsilon),M_0]} F(x)+T\overline{h}_+\end{aligned}$$

or

$$\text{(3.25)}\qquad \sup_{x\in(0,\gamma_1)}\left(F(x)+\frac{T\overline{\alpha}_-}{x^\mu}\right) = \sup_{x\in(0,\gamma_1)} F(x) < \min_{x\in[A(\varepsilon),M_0]} F(x)-T\overline{h}_+.$$

Take $\varepsilon=\gamma_1/2$, then conditions of (3.2) and (3.3) are satisfied. Furthermore, (3.24) (or (3.25)) verifies condition (3.4) (or (3.5)). Thus, by using Theorem 3.1, we see that there is a positive T-periodic solution to (1.1). □

Corollary 3.3. *Assume that* (H1) *holds with* $\overline{\alpha}_->0$, *and the function*

$$G(x) := F(x)-\frac{T\,\overline{\alpha}_-}{x^\mu}$$

is decreasing in $(0,+\infty)$ *with*

$$\text{(3.26)}\qquad \lim_{x\to0^+} G(x)=+\infty,$$

If there is a constant $\sigma_0\in(0,1)$ *such that*

$$\text{(3.27)}\qquad \lim_{x\to0^+}\frac{f(x)+\dfrac{\mu T\,\overline{\alpha}_-}{x^{\mu+1}}}{f(\sigma_0A(x))}\left(\frac{\overline{\alpha}_-}{\overline{\alpha}_+}\right)^{1/\mu} > \sigma_0,$$

then equation (1.1) *has a positive* T*-periodic solution, where*

$$A(x)=x\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_-+x^\mu\overline{h}}\right)^{1/\mu}.$$

Proof. From (3.26), one can easily find that $F(x)$ is also decreasing in $(0,+\infty)$, and

$$\lim_{x\to0^+} F(x)=+\infty,$$

which together with

$$A'(x)=\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_-+x^\mu\overline{h}}\right)^{1/\mu}\frac{\overline{\alpha}_-}{\overline{\alpha}_-+x^\mu\overline{h}}>0,\quad x\in(0,+\infty)$$

yields

$$\lim_{x\to0^+} F(\sigma_0A(x))=+\infty.$$

By using condition (3.27), we have

$$\lim_{x\to 0^+} \frac{G(x)}{F(\sigma_0 A(x)) + T\overline{h}_+} > 1.$$

Thus, there is a constant $\delta \in \big(0, (\overline{\alpha}/\overline{h})^{1/\mu}\big)$ such that

$$G(x) > F(\sigma_0 A(x)) + T\overline{h}_+ \quad \text{for all } x \in (0, \delta]. \tag{3.28}$$

By using the monotonicity of $G(x)$, $F(x)$ and $A(x)$, we have

$$\inf_{x\in(0,\delta)} G(x) = G(\delta) > F(\sigma_0 A(\delta)) + T\overline{h}_+ = \sup_{x\in[\sigma_0 A(\delta),+\infty)} \big(F(x) + T\overline{h}_+\big). \tag{3.29}$$

Since

$$x < A(x) < x\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_-}\right)^{1/\mu}, \quad x \in (0, \delta], \tag{3.30}$$

it follows that for the above constant $\delta \in \big(0, (\overline{\alpha}/\overline{h})^{1/\mu}\big)$, there exists an $\varepsilon \in (0, \delta)$, which is very close to the number δ such that

$$A(\varepsilon) > \max\{\delta, \sigma_0 A(\delta)\}. \tag{3.31}$$

This implies that there are two constants ε and δ with

$$\varepsilon \in \left(0, \left(\frac{\overline{\alpha}}{\overline{\overline{h}}}\right)^{1/\mu}\right) \tag{3.32}$$

and

$$\delta \in (\varepsilon, A(\varepsilon)) \cap \left(0, \left(\frac{\overline{\alpha}}{\overline{\overline{h}}}\right)^{1/\mu}\right). \tag{3.33}$$

Furthermore, (3.31) and (3.29) gives

$$\inf_{x\in(0,\delta)} G(x) > \sup_{x\in[A(\varepsilon),+\infty)} \big(F(x) + T\overline{h}_+\big). \tag{3.34}$$

Clearly, (3.32) implies that condition (3.2) holds, (3.33) implies that condition (3.3) holds, and (3.34) implies that condition (3.4) holds. Thus, the conclusion follows from Theorem 3.1 directly. □

Analogously to the proof of Corollary 3.3, we can obtain the following result.

COROLLARY 3.4. *Assume that* (H1) *holds with* $\overline{\alpha_-} > 0$, *and the function*

$$F(x) + \frac{T\overline{\alpha}_-}{x^\mu}$$

is increasing in $(0, +\infty)$ *with*

$$\lim_{x\to 0^+} \left(F(x) + \frac{T\,\overline{\alpha}_-}{x^\mu}\right) = -\infty.$$

If there is a constant $\sigma \in (0,1)$ such that

$$\lim_{x\to 0^+} \frac{f(x) - \dfrac{\mu T\,\overline{\alpha}_-}{x^{\mu+1}}}{f(\sigma A(x))}\left(\frac{\overline{\alpha}_-}{\overline{\alpha}_+}\right)^{1/\mu} > \sigma,$$

then equation (1.1) *has a positive T-periodic solution.*

Theorem 3.5. *Suppose that assumption* (H2)*holds. If there are two constants*

$$\varepsilon_0 \in \left(0, \left(\frac{\overline{\alpha}}{\overline{\overline{h}}}\right)^{1/\mu}\right) \quad \text{and} \quad \gamma_2 \in (\varepsilon_0, B(\varepsilon_0)) \cap \left(0, \left(\frac{\overline{\alpha}}{\overline{\overline{h}}}\right)^{1/\mu}\right) \tag{3.35}$$

such that

$$\inf_{x\in(0,\gamma_2)}\left(F(x) - \frac{T\overline{\alpha}_+}{x^\mu}\right) > \max_{x\in[B(\varepsilon_0),M_2]} F(x) + T\overline{h}_- \tag{3.36}$$

or

$$\sup_{x\in(0,\gamma_2)}\left(F(x) + \frac{T\overline{\alpha}_+}{x^\mu}\right) < \min_{x\in[B(\varepsilon_0),M_2]} F(x) - T\overline{h}_-, \tag{3.37}$$

where

$$B(\varepsilon_0) = \varepsilon_0\left(\frac{\overline{\alpha}_-}{\overline{\alpha}_+ - \varepsilon_0^\mu \overline{h}}\right)^{1/\mu}, \qquad M_2 := \left(\frac{\overline{\alpha}_-}{|\overline{h}|}\right)^{1/\mu} + \frac{\sqrt{T}}{2}B_0,$$

$$B_0 = \frac{\left(\dfrac{T\,\overline{\alpha}_+}{\varepsilon_0^\mu} + T\overline{h}_-\right)\sqrt{T}}{2} + \left(\frac{\overline{\alpha}_-}{|\overline{h}|}\right)^{1/(2\mu)}\left(\frac{T\,\overline{\alpha}_+}{\varepsilon_0^\mu} + T\overline{h}_-\right)^{1/2},$$

then equation (1.1) *has at least one positive T-periodic solution.*

Since the argument works almost exactly as the proof of Theorem 3.1, we omit it here.

By applying Theorem 3.5, and using the arguments which are similar to the ones in the proofs of Corollaries 3.3 and 3.4, we obtain the following two corollaries.

Corollary 3.6. *Assume that* (H2) *holds with $\overline{\alpha}_+ > 0$, and the function*

$$G_1(x) := F(x) - \frac{T\overline{\alpha}_+}{x^\mu}$$

is decreasing in $(0,+\infty)$ with $\lim_{x\to 0^+} G_1(x) = +\infty$. If there is a constant $\sigma_1 \in (0,1)$ such that

$$\lim_{x\to 0^+} \frac{f(x) + \dfrac{\mu T\,\overline{\alpha}_+}{x^{\mu+1}}}{f(\sigma_1 B(x))}\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_-}\right)^{1/\mu} > \sigma_1,$$

then equation (1.1) *has a positive T-periodic solution, where*

$$B(x) = x\left(\frac{\overline{\alpha}_-}{\overline{\alpha}_+ - x^\mu \overline{h}}\right)^{1/\mu}$$

is determined in Theorem 3.5.

COROLLARY 3.7. *Assume that* (H2) *holds with* $\overline{\alpha_+} > 0$*, and the function*

$$F(x) + \frac{T\,\overline{\alpha}_+}{x^\mu}$$

is increasing in $(0, +\infty)$ *with*

$$\lim_{x\to 0^+}\left(F(x) + \frac{T\,\overline{\alpha}_+}{x^\mu}\right) = -\infty.$$

If there is a constant $\sigma_2 \in (0,1)$ *such that*

$$\lim_{x\to 0^+}\frac{f(x) - \dfrac{\mu T\,\overline{\alpha}_+}{x^{\mu+1}}}{f(\sigma_2 B(x))}\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_-}\right)^{1/\mu} > \sigma_2,$$

then equation (1.1) *has a positive* T*-periodic solution.*

REMARK 3.8. From conditions of (3.4), (3.5) in Theorem 3.1, as well as (3.26) in Corollary 3.3, one can find that the function $F(x)$ has a singularity at $x = 0$, and the order of singularity of function $F(x)$ at $x = 0$ is required to be no less than μ which is the order of singularity of restoring force $\alpha(t)/x^\mu$ at $x = 0$. This relation is crucial for us to estimate a priori bounds of periodic solutions from below. In this sense, the singularity associated to $f(x)$ at $x = 0$ can help periodic solutions to exist.

EXAMPLE 3.9. Consider the following equation

$$x''(t) - \frac{30\,x'(t)}{x^2(t)} + \frac{\alpha(t)}{x^{1/2}} = h(t), \tag{3.38}$$

where $\alpha, h\colon R \to R$ are $2\pi-$periodic functions with

$$\alpha(t) = \begin{cases} 10\,\pi\sin t & \text{for } t\in[0,\pi],\\ \pi\sin t & \text{for } t\in[\pi,2\pi],\end{cases}$$

and

$$h(t) = \begin{cases} 2.4\,\pi\sin t & \text{for } t\in[0,\pi/2],\\ 0.4\,\pi\cos t & \text{for } t\in[\pi/2,2\pi].\end{cases}$$

We can chose $J := [\pi, 2\pi]$ such that (H1) holds, and by simple calculating, we have $\overline{\alpha_+} = 10$, $\overline{\alpha_-} = 1$, $\overline{h_+} = 1.4$ and $\overline{h} = 1$. Corresponding to (1.1), we have $T = 2\pi$, $f(x) = -30/x^2$, $\mu = 1/2$ and then

$$F(x) = \int_1^x f(s)\,ds = \frac{30}{x} - 30.$$

Since

$$A(\varepsilon) = \varepsilon\left(\frac{\overline{\alpha}_+}{\overline{\alpha}_- + \varepsilon^\mu\overline{h}}\right)^{1/\mu} = \frac{10^2\varepsilon}{(1+\varepsilon^\mu)^2}$$

and $\big(0,(\overline{\alpha}\overline{h})^{1/\mu}\big)=(0,81)$, the constant ε can be chosen as $\varepsilon=0.01$ such that $\varepsilon\in\big(0,(\overline{\alpha}/\overline{h})^{1}/\mu\big)$ and $\varepsilon<A(\varepsilon)=100/121$. Furthermore,

$$A_1=\max_{x\in[A(\varepsilon),M_0]}F(x)=F(A(\varepsilon))=6.3.$$

Now, we set $\gamma_1=0.02$, then $\gamma_1\in(0,81)=\big(0,(\overline{\alpha}/\overline{h})^{1/\mu}\big)$, $\gamma_1\in(\varepsilon,A(\varepsilon))$ and

$$\begin{aligned}\inf_{x\in(0,\gamma_1)}\bigg(F(x)-\frac{T\overline{\alpha_-}}{x^\mu}\bigg)&=\inf_{x\in(0,0.02)}\bigg(\frac{30}{x}-\frac{2\pi}{x^{1/2}}-30\bigg)\\&=1470-\frac{20\pi}{\sqrt{2}}>1400,\\A_1+T\overline{h}_+&=6.3+2.8\pi.\end{aligned}$$

Thus,

$$\inf_{x\in(0,0.02)}\bigg(F(x)-\frac{T\overline{\alpha_-}}{x^\mu}\bigg)>A_1+T\overline{h_+},$$

which implies that assumption (3.4) holds. Thus, by using Theorem 3.1, we have that equation (3.38) has at least one positive 2π-periodic solution.

Example 3.10. Consider the following equation

$$x''(t)-\bigg(\frac{1}{x^\eta(t)}+\frac{1}{x^{\mu+1}(t)}\bigg)x'(t)+\frac{\alpha(t)}{x^\mu}=h(t), \tag{3.39}$$

where $\alpha,h\colon\mathbb{R}\to\mathbb{R}$ are T-periodic functions and in $L^1([0,T],\mathbb{R})$ with $\overline{\alpha}>0$ and $\overline{h}>0$, η and μ are positive constants with $\eta>\mu+1$. If $0\le\overline{\alpha_-}\le1/(\mu T)$, then

$$F(x)-\frac{T\,\overline{\alpha}_-}{x^\mu}=\frac{1}{(\eta-1)\,x^{\eta-1}}+\bigg(\frac{1}{\mu}-T\,\overline{\alpha}_-\bigg)\frac{1}{x^\mu}-\frac{1}{\eta-1}-\frac{1}{\mu}.$$

This gives that $F(x)-T\,\overline{\alpha}_-/x^\mu$ is decreasing in $(0,+\infty)$ and

$$\lim_{x\to0^+}\bigg(F(x)-\frac{T\,\overline{\alpha}_-}{x^\mu}\bigg)=+\infty.$$

Besides, we can chose

$$\sigma_0\in\bigg(\bigg(\frac{\overline{\alpha}_-}{\overline{\alpha}_+}\bigg)^{1/\mu},1\bigg)$$

such that

$$\lim_{x\to0^+}\frac{f(x)+\frac{\mu T\,\overline{\alpha}_-}{x^{\mu+1}}}{f(\sigma_0A(x))}\bigg(\frac{\overline{\alpha}_-}{\overline{\alpha}_+}\bigg)^{1/\mu}=\sigma_0^\eta\bigg(\frac{\overline{\alpha}_+}{\overline{\alpha}_-}\bigg)^{(\eta-1)/\mu}>\sigma_0.$$

Thus, by using Corollary 3.3, we see that (3.39) has a positive T-periodic solution.

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