Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 1, 2019, 81–108
DOI: 10.12775/TMNA.2019.026

CLASSIFICATION OF RADIAL SOLUTIONS TO HÉNON TYPE EQUATION ON THE HYPERBOLIC SPACE

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Abstract. We devote this paper to classifying radial solutions of a weighted semilinear elliptic equation on the hyperbolic space. More precisely, for a weighted Lane–Emden equation on the hyperbolic space, we shall study the sign and asymptotic behavior of the radial solutions. We shall also show the existence of fast-decay sign-changing solutions to the Lane–Emden equation on the hyperbolic space.

1. Introduction

In this paper, we shall investigate the structure of radial solutions to the following weighted semilinear elliptic equation:

$$-\Delta_g u = (\sinh r)^\alpha |u|^{p-1} u \quad \text{in } \mathbb{H}^N, \tag{H}$$

where $N \geq 2$, $p > 1$, and $\alpha > -2$. Here, $\mathbb{H}^N$ denotes the N-dimensional hyperbolic space in terms of the spherical coordinates, $r > 0$ represents the geodesic distance on $\mathbb{H}^N$, and Δ_g denotes the Laplace–Beltrami operator on $\mathbb{H}^N$.

The structure of radial solutions to semilinear elliptic equations has attracted a great interest. In particular, the following Hénon type equation has been well

2010 *Mathematics Subject Classification.* Primary: 58J05; Secondary: 35B05,58K55.
Key words and phrases. Semilinear elliptic equation; decay rate; sign-changing solutions.
The author was supported by Research Fellow of Japan Society for the Promotion of Science (No. 16J01320).

considered:

$$-\Delta u = K|u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where K denotes a given function in $\mathbb{R}^N$. The structure of radial solutions to (1.1) was firstly descibed by W.-M. Ni [16] in 1982. Under assumptions on the decay rate of K, he proved the existence and non-existence of positive radial solutions to (1.1) for the case of $p = (N+2)/(N-2)$. Thereafter, for each $p > 1$, in [10], [12], [17], the existence of positive solutions to (1.1) was shown when K decays faster than or equal to $|x|^{-2}$ at ∞. Moreover, under various assumptions on K and p, there is a large number of results on the sign and asymptotic behavior of radial solutions to (1.1) (see [5], [11], [13], [15], [17], [20]–[23] and references therein). Here, we focus on the known result on the positivity of radial solutions to the following Hénon equation:

$$-\Delta v = |x|^\alpha |v|^{p-1}v \quad \text{in } \mathbb{R}^N, \tag{E}$$

where $N \geq 3$, $p > 1$, and $\alpha > -2$. For each $\beta > 0$, we denote by $v_\beta = v_\beta(r)$ the radial solution of (E) satisfying $v_\beta(0) = \beta$. Moreover, let $p_s(N, \alpha)$ be the Sobolev exponent given by

$$p_s(N, \alpha) = \frac{N + 2 + 2\alpha}{N - 2}.$$

Then radial solutions of the equation (E) satisfy the following conditions ([17], [20]):

- If $p < p_s(N, \alpha)$, then v_β has infinitely many zeros in $(0, \infty)$;
- If $p \geq p_s(N, \alpha)$, then v_β is positive in $(0, \infty)$.

Therefore, we observe that $p_s(N, \alpha)$ is critical with respect to the existence of positive radial solutions of (E).

On the other hand, from 2000s, the literature of semilinear elliptic equations on $\mathbb{H}^N$ has increased and the following Lane-Emden equation is now well-investigated ([1]–[4], [10], [14], [18], [19]):

$$-\Delta_g u = |u|^{p-1}u \quad \text{in } \mathbb{H}^N, \tag{L}$$

where $N \geq 2$ and $p > 1$. In order to introduce the known results on (L), we denote by u_β the radial solution of (L) with $u_\beta(0) = \beta$. The radial solution u_β to (L) satisfies the following ([4]):

- If $p < p_s(N, 0)$, then u_β has finitely many zeros in $(0, \infty)$;
- If $p \geq p_s(N, 0)$, then u_β is positive in $(0, \infty)$.

Here, the case where u_β has finitely many zeros in $(0, \infty)$ includes the case of $u_\beta > 0$ in $(0, \infty)$. Indeed, for $p < p_s(N, 0)$ and sufficiently small $\beta > 0$, u_β is positive in $(0, \infty)$. Thus $p_s(N, 0)$ is not critical on the existence of positive radial solutions to (L).

Comparing the structure of radial solutions of (L) with that of (E), we observe that (L) admits no critical exponent on the existence of positive radial solutions and no solutions with infinitely many zeros for any $p > 1$. Then we can say that the structure of solutions of (L) is not as "rich" as that of (E). Recalling that the Sobolev exponent p_s depends on the weight $|x|^\alpha$ of (E), we are interested in the following problem:

Problem 1.1. Is the structure of solutions of (L) with suitable weight as "rich" as that of (E)?

Following the motivation, we consider the weighted equation (H). Here we mention the weight in (H). From the analogue of (E), one of natural choice of weight is power of geodesic distance on $\mathbb{H}^N$. However, we infer from [9] that such weighted equation admits no critical exponent on the existence of positive radial solutions. On the other hand, positive radial solutions to (E) can be classified into two types with respect to the decay rate. Indeed, decay rate of the radial solution to (E) is equal to or slower than that of the fundamental solution of Laplace equation ([20]). Here, the fundamental solution depends on the elliptic operator $-\Delta$. Thus, regarding the classification of radial solutions, we infer that it is suitable to choose the related function to the metric of $\mathbb{H}^N$ as the weight. Now, the weight function $\sinh r$ in (H) appears in the metric of $\mathbb{H}^N$.

In order to state the main theorem of this paper, we prepare notations. We denote by u_β the radial solution of (H) with $u_\beta(0) = \beta$. Then, inspired by the classification of radial solutions in [20], we define several classes of $\{u_\beta\}_{\beta>0}$:

Definition 1.2.

(a) We say that u_β is Type O, if u_β has infinitely many zeros in $(0, \infty)$.
(b) We say that u_β is Type R, if u_β has finitely many zeros in $(0, \infty)$ and satisfies $(\sinh r)^{N-1}|u_\beta(r)| \to \gamma$ as $r \to \infty$ for some $\gamma > 0$.
(c) We say that u_β is Type S, if u_β has finitely many zeros in $(0, \infty)$ and satisfies $(\sinh r)^{N-1}|u_\beta(r)| \to \infty$ as $r \to \infty$.

Here, the case where u_β has finitely many zeros in $(0, \infty)$ includes the case of $u_\beta > 0$ in $(0, \infty)$. We define the exponent $p_b(N, \alpha)$ as

$$p_b(N, \alpha) = \frac{N - 1 + 2\alpha}{N - 1}.$$

The main result of this paper is stated as follows:

Theorem 1.3. *Let $N \geq 3$, $p > 1$, and $\alpha > 0$. Then the following hold*:
(a) *Let $p < p_b(N, \alpha)$. Then u_β is Type O for any $\beta > 0$*;
(b) *Let $p_b(N, \alpha) < p < p_s(N, \alpha)$. Then there exists $\beta_H = \beta_H(N, p, \alpha) > 0$ such that the following hold*:
   (i) *If $\beta < \beta_H$, then u_β is positive in $(0, \infty)$ and is Type* S;

(ii) *If* $\beta = \beta_H$*, then* u_β *is positive in* $(0, \infty)$ *and is Type* R;

(iii) *If* $\beta > \beta_H$*, then* u_β *is sign-changing and has finitely many zeros in* $(0, \infty)$.

(c) *Let* $p > p_s(N, \alpha)$*. Then* u_β *is positive in* $(0, \infty)$ *and is Type* S *for any* $\beta > 0$.

Remark 1.4. (a) The assumption $\alpha > 0$ is sharp. Indeed, when $\alpha = 0$, although Theorem 1.3 holds, the assertion (a) does not occur for $p_b(N, 0) = 1$.

(b) When $p \in (p_b(N, \alpha), p_s(N, \alpha))$, the threshold $\beta = \beta_H$ arises from the variational view. Indeed, u_{β_H} coincides with the positive solution which is obtained by the variational methods in Appendix of [6], while u_β does not belong to the energy space for $\beta < \beta_H$.

(c) The assertion (c) has already been proved in Theorems 3.1–3.2 of [6].

(d) For $p \in (p_b(N, \alpha), p_s(N, \alpha))$ and $\beta > \beta_H$, it is an outstanding problem whether u_β is Type S or Type R, and how many zeros u_β has in $(0, \infty)$. Moreover, for the cases of $p = p_b(N, \alpha)$ and $p = p_s(N, \alpha)$, the structure of radial solutions is an open problem. In addition, when $\alpha < 0$, the structure of radial solutions is not clarified for any $p > 1$.

We observe from Theorem 1.3 that there exist two critical exponents with respect to the sign of solutions. Indeed, $p_b(N, \alpha)$ is critical on the existence of positive radial solutions, while $p_s(N, \alpha)$ is critical with respect to the existence of sign-changing radial solutions. Theorem 1.3 also implies the existence of solutions of Type O. Thus, we can say that the structure of radial solutions of (H) is as "rich" as that of (E).

We shall obtain a further result on radial solutions to (H). Regarding the existence of radial solutions of Type R, we obtain the following result:

Theorem 1.5. *Let* $N \geq 3$*,* $\alpha > -2$*, and* $p \in (\max\{1, p_b(N, \alpha)\}, p_s(N, \alpha))$*. Then there exist a strictly increasing divergent positive sequence* $\{\beta_k\}$ *and a positive sequence* $\{\gamma_k\}$*,* $k = 0, 1, 2, \dots$ *such that* u_{β_k} *has just* k *zeros in* $(0, +\infty)$ *and satisfies* $(\sinh r)^{N-1} u_{\beta_k}(r) \to (-1)^k \gamma_k$ *as* $r \to \infty$.

Remark that $\beta_0 = \beta_H$ if $\alpha > 0$, where β_H is defined in Theorem 1.3. We observe from Theorem 1.5 that there exist radial solutions of Type R for the case of $\alpha > 0$, $p_b(N, \alpha) < p < p_s(N, \alpha)$ and $\beta > \beta_H$ in Theorem 1.3. Moreover, Theorem 1.5 also implies that the equation (L) has sign-changing radial solutions of Type R when $p \in (1, p_s(N, 0))$. Here, for the equation (L), the existence of positive radial solutions of Type R has been already proved for $p \in (1, p_s(N, 0))$ ([4], [14]).

The paper is organized as follows: To begin with, in Section 2, we prepare notations and auxiliary lemmas. In Section 3, we shall prove Theorem 1.3 (a).

Moreover, we devote Section 4 to showing Theorem 1.5. Finally, we verify Theorem 1.3 (b) in Section 5. Here, Theorem 1.3 (c) was proved in [6].

2. Auxiliary lemmas

We devote this section to preparing notations and lemmas necessary for the classification of radial solutions to (H). To begin with, we describe the N-dimensional hyperbolic space as $\mathbb{H}^N$. Let $\mathbb{H}^N$ be a manifold admitting a pole o and whose metric g is denoted, in the polar coordinates around o, by

$$ds^2 = dr^2 + (\sinh r)^2\, d\Theta^2, \quad r > 0,\ \Theta \in \mathbb{S}^{N-1},$$

where $d\Theta^2$ denotes the canonical metric on the unit sphere $\mathbb{S}^{N-1}$, r is the geodesic distance between o and a point (r, Θ). Now, we shall define radial solutions to (H). For each $\beta > 0$, we consider the following problem:

$$\text{(Hr)} \qquad \begin{cases} u''(r) + \dfrac{N-1}{\tanh r} u'(r) + (\sinh r)^\alpha |u(r)|^{p-1} u(r) = 0 & \text{in } (0, +\infty), \\ u(0) = \beta. \end{cases}$$

Then, the solution of (Hr) satisfies the following properties:

Lemma 2.1. *Let $N \geq 2$, $p > 1$, and $\alpha > -2$. Then the problem* (Hr) *admits a unique global solution $u_\beta \in C([0,\infty)) \cap C^2((0,\infty))$ and the following hold*:

(a) $\lim_{r\to 0} (\sinh r)^{-\alpha/2} u'_\beta(r) = 0$;
(b) $|u_\beta(r)| < \beta$ *in* $(0, +\infty)$;
(c) *If $u_\beta > 0$ in (R, ∞) for some $R > 0$, then* $\lim_{r\to\infty} (\sinh r)^{N-1} u'_\beta(r) < 0$;
(d) *If $u_\beta > 0$ in $[0, R)$ for some $R > 0$, then $u'_\beta < 0$ in $(0, R)$, i.e. u_β is strictly monotone decreasing in $(0, R)$.*

Proof. To begin with, we consider the following integral equation in $[0, \infty)$:

$$u(r) = \beta - \int_0^r \frac{1}{(\sinh t)^{N-1}} \int_0^t (\sinh s)^{N-1+\alpha} |u(s)|^{p-1} u(s)\, ds\, dt. \tag{2.1}$$

By a standard theory, (2.1) has a unique global solution u_β and then we can verify that u_β is the unique solution of (Hr) (e.g. see [8]). Now we shall prove the assertion (a). Differentiating (2.1) with respect to r, we see that

$$u'_\beta(r) = -\frac{1}{(\sinh r)^{N-1}} \int_0^r (\sinh s)^{N-1+\alpha} |u_\beta(s)|^{p-1} u_\beta(s)\, ds. \tag{2.2}$$

Since $u_\beta(0) = \beta > 0$, we may choose sufficiently small $\xi > 0$ such that $u_\beta(r) > 0$ in $[0, \xi)$. Then u_β is strictly monotone decreasing in $[0, \xi)$ and using (2.2) again, we obtain

$$\frac{|u'_\beta(r)|}{(\sinh r)^{\alpha/2}} \leq \frac{\beta^p}{(\sinh r)^{N-1+\alpha/2}} \int_0^r (\sinh s)^{N+\alpha} \frac{ds}{\tanh s} = \frac{\beta^p}{N+\alpha} (\sinh r)^{1+\alpha/2}.$$

By $1+\alpha/2>0$, the assertion (a) holds. Next, multiplying the equation in (Hr) by $u'_\beta/(\sinh r)^\alpha$, we observe that the following identity holds:

$$F'(r) = -\left(N-1+\frac{\alpha}{2}\right)\frac{(u'_\beta(r))^2}{(\sinh r)^\alpha \tanh r}, \tag{2.3}$$

where

$$F(r) = \frac{(u'_\beta(r))^2}{2(\sinh r)^\alpha} + \frac{|u_\beta(r)|^{p+1}}{p+1}.$$

Hence, since $N-1+\alpha/2>0$, it follows that F is strictly monotone decreasing in $[0,\infty)$. Thus, combining assertion (a) with the monotonicity of F, we see that $F(r)\le F(0)$ in $(0,+\infty)$. Then assertion (b) holds. We shall show assertion (c). From (2.2), we deduce that

$$-\{(\sinh r)^{N-1}u'_\beta\}' = (\sinh r)^{N-1+\alpha}|u_\beta(r)|^{p-1}u_\beta(r). \tag{2.4}$$

Since $u_\beta>0$ in (R,∞), we see that $(\sinh r)^{N-1}u'_\beta$ is strictly decreasing in (R,∞). Then we claim that $\lim_{r\to\infty}(\sinh r)^{N-1}u'_\beta(r)<0$. Suppose not, there exists $\gamma\ge 0$ such that $(\sinh r)^{N-1}u'_\beta(r)\to\gamma$ as $r\to\infty$. It follows from the monotonicity of $(\sinh r)^{N-1}u'_\beta$ that $u'_\beta>0$ in (R,∞) and then $u_\beta(r)>u_\beta(R)$ in (R,∞). Thus, integrating (2.4) over (R,r), we obtain as $r\to\infty$,

$$(\sinh r)^{N-1}u'_\beta(r) = (\sinh R)^{N-1}u'_\beta(R) - \int_R^r (\sinh s)^{N-1+\alpha}u_\beta^p(s)\,ds \to -\infty.$$

This is a contradiction and assertion (c) holds. Finally, we shall show assertion (d). Since $u_\beta>0$ in $[0,R)$, from (2.4) it follows that $(\sinh r)^{N-1}u'_\beta$ is strictly decreasing in $(0,R)$. Moreover, in view of assertion (a), we observe that $(\sinh r)^{N-1}u'_\beta\to 0$ as $r\to 0$. Hence, it holds that $(\sinh r)^{N-1}u'_\beta<0$ in $(0,R)$ and we complete the proof of the assertion (d). □

In the following, we denote by $u_\beta=u_\beta(r)$ the solution of (Hr). Then u_β also satisfies the following:

Lemma 2.2. *Let $N\ge 2$, $p>1$, and $\alpha>-2$. Then the following hold*:

(a) $\lim_{\beta\downarrow 0}\beta^{-1}u_\beta(r)=1$ *uniformly in* $r\in[0,\operatorname{arc\,sinh}1]$;

(b) $\lim_{\beta\downarrow 0}\beta^{-1}u'_\beta(r)\sinh r=0$ *uniformly in* $r\in[0,\operatorname{arc\,sinh}1]$.

Proof. Setting $U_\beta(r):=\beta^{-1}u_\beta(r)$, we infer from (2.2) that

$$U'_\beta(r) = -\frac{\beta^{p-1}}{(\sinh r)^{N-1}}\int_0^r (\sinh s)^{N-1+\alpha}|U_\beta(s)|^{p-1}U_\beta(s)\,ds. \tag{2.5}$$

Since Lemma 2.1 (b) implies that $|U_\beta(r)|\le 1$ in $r\in[0,+\infty)$, it holds that

$$|U'_\beta(r)\sinh r| \le \frac{\beta^{p-1}}{(\sinh r)^{N-2}}\int_0^r \frac{(\sinh s)^{N+\alpha}}{\tanh s}\,ds = \frac{\beta^{p-1}}{N+\alpha}(\sinh r)^{\alpha+2}$$

in $r \in [0, \operatorname{arc\,sinh} 1]$. Then

$$|U'_\beta(r)\sinh r| \le \frac{\beta^{p-1}}{N+\alpha} \quad \text{in } r \in [0, \operatorname{arc\,sinh} 1]$$

and the assertion (b) holds. Moreover, integrating (2.5) over $[0, r]$, we obtain

$$|U_\beta(r) - 1| \le \frac{\beta^{p-1}}{N+\alpha}\int_0^r \frac{(\sinh t)^{\alpha+2}}{\tanh t}\,dt \le \frac{\beta^{p-1}}{(N+\alpha)(\alpha+2)}$$

in $r \in [0, \operatorname{arc\,sinh} 1]$. Thus we derive the assertion (a). □

In this paper, applying Definition 1.2 introduced in Section 1, we shall classify $\{u_\beta\}_{\beta>0}$. To this aim, we state the following lemma:

Lemma 2.3. *Let $N \ge 2$, $p > 1$, and $\alpha > -2$. Then u_β is classified into one of three types in Definition* 1.2, *i.e. Type* O, *Type* R, *and Type* S.

Proof. We only have to show that if the number of zeros of u_β is finite on $(0, \infty)$, then $(\sinh r)^{N-1}|u_\beta(r)| \to 0$ does not occur as $r \to \infty$. We may assume that there exists $R > 0$ such that $u_\beta > 0$ in (R, ∞). Suppose not, $(\sinh r)^{N-1}u_\beta(r) \to 0$ as $r \to \infty$. Then, we deduce from l'Hospital's rule that $(\sinh r)^{N-1}u'_\beta(r) \to 0$ as $r \to \infty$. However, Lemma 2.1 (c) implies that $\lim_{r\to\infty}(\sinh r)^{N-1}u'_\beta(r) < 0$. This is a contradiction. □

Now, we introduce the following function which plays an important rule to show the sign of radial solutions:

$$\Psi(r) = \frac{\varphi(r)(u'_\beta(r))^2}{2(\sinh r)^\alpha} + \frac{\varphi(r)|u_\beta(r)|^{p+1}}{p+1} + \frac{(\sinh r)^{N-1}u_\beta(r)u'_\beta(r)}{p+1}, \tag{2.6}$$

where $\varphi(r) = \int_0^r (\sinh s)^{N-1+\alpha}\,ds$. Here, making use of the Pohozaev type identity, we construct the function Ψ. Indeed, setting

$$h(r) = \frac{1}{2} + \frac{1}{p+1} - \frac{(N-1+\alpha/2)\varphi(r)}{(\sinh r)^{N-1+\alpha}\tanh r}, \tag{2.7}$$

we infer that Ψ satisfies the following:

Lemma 2.4. *Let $N \ge 2$, $p > 1$, and $\alpha > -2$. Then, the following identity holds*:

$$\Psi'(r) = (\sinh r)^{N-1}h(r)|u'_\beta(r)|^2 \quad \textit{in } (0, \infty). \tag{2.8}$$

Proof. Multiplying the equation in (Hr) by $\varphi(r)u'_\beta(r)/(\sinh r)^\alpha$, we derive

$$\begin{aligned}\left(\frac{\varphi(r)(u'_\beta(r))^2}{2(\sinh r)^\alpha} + \frac{\varphi(r)|u_\beta(r)|^{p+1}}{p+1}\right)' - \frac{(\sinh r)^{N-1+\alpha}|u(r)|^{p+1}}{p+1} \\ = \frac{(\sinh r)^{N-1}(u'_\beta(r))^2}{2} - \left(N-1+\frac{\alpha}{2}\right)\frac{\varphi(r)(u'_\beta(r))^2}{(\sinh r)^\alpha(\tanh r)}.\end{aligned} \tag{2.9}$$

Moreover, multiplying the equation in (2.4) by $u_\beta(r)$, we have

$$\big((\sinh r)^{N-1}u_\beta u_\beta'\big)' = (\sinh r)^{N-1}(u_\beta')^2 - (\sinh r)^{N-1+\alpha}|u_\beta|^{p+1}. \tag{2.10}$$

Then, multiplying (2.10) by $(p+1)^{-1}$ and adding it to (2.9), we obtain (2.8).□

Moreover, Ψ satisfies the following properties:

LEMMA 2.5. *Let $N\geq 2$, $p>1$, and $\alpha>-2$. Then, it holds that* $\lim_{r\to 0}\Psi(r)=0$.

PROOF. We have

$$\varphi(r) \leq \int_0^r (\sinh s)^{N+\alpha}\frac{ds}{\tanh s} = \frac{1}{N+\alpha}(\sinh r)^{N+\alpha}. \tag{2.11}$$

Then it follows from (2.11) and Lemma 2.1 (a) that $\Psi(r)\to 0$ as $r\to 0$. □

LEMMA 2.6. *Let $N\geq 3$, $\alpha>-2$, and $1<p\leq p_b(N,\alpha)$. Then, it holds that $\Psi>0$ and $\Psi'>0$ in $(0,\infty)$.*

PROOF. Since

$$\frac{\varphi(r)}{\tanh r} < \int_0^r (\sinh s)^{N-1+\alpha}\frac{ds}{\tanh s} = \frac{1}{N-1+\alpha}(\sinh r)^{N-1+\alpha},$$

the following estimate holds:

$$h(r) > \frac{1}{2} + \frac{1}{p+1} - \frac{N-1+\alpha/2}{N-1+\alpha} \geq 0,$$

where the last inequality is equivalent to $p\leq p_b(N,\alpha)$. Then it follows from Lemmas 2.4–2.5 that $\Psi'>0$ and $\Psi>0$ in $(0,\infty)$. □

LEMMA 2.7. *Let $N\geq 3$, $\alpha>-2$, and $p_b(N,\alpha)<p<p_s(N,\alpha)$. Then, there exists $R=R(N,p,\alpha)>0$ such that Ψ is strictly monotone increasing in $(0,R)$ and is strictly monotone decreasing in $(R,+\infty)$.*

PROOF. In order to investigate the sign of Ψ, we shall study the sign of h, where h was defined in (2.7). To this aim, we set two functions $g_1, g_2\colon [0,\infty)\to\mathbb{R}$ as $g_1(r) = h(r)(\sinh r)^{N-1+\alpha}\tanh r$ and $g_2(r) = g_1'(r)(\sinh r)^{-(N-1+\alpha)}$. It is easy to verify that g_2 is strictly monotone decreasing in $(0,\infty)$ and

$$g_2(0) = -\frac{N-2}{2} + \frac{N+\alpha}{p+1} > 0, \qquad \lim_{r\to\infty} g_2(r) = -\frac{N-1}{2} + \frac{N-1+\alpha}{p+1} < 0.$$

Here, we used the assumption $p_b(N,\alpha)<p<p_s(N,\alpha)$. Therefore we find a unique constant $\widetilde{R}=\widetilde{R}(N,p,\alpha)>0$ such that $g_2(\widetilde{R})=0$. Then g_1 is monotone increasing in $(0,\widetilde{R})$ and monotone decreasing in $(\widetilde{R},\infty)$. Now, we have $g_1(0)=0$ and it follows from l'Hospital's rule that $\lim_{r\to\infty} g_1(r) < 0$. Thus, there exists a unique $R=R(N,p,\alpha)>0$ such that $g_1(R)=0$. Applying the definition of g_1 and (2.8), we complete the proof. □

3. Proof of Theorem 1.3 (a)

In this section, we shall investigate the solution u_β to (Hr) for $p < p_b(N,\alpha)$ and prove Theorem 1.3 (a). To this aim, we prepare the following lemma:

Lemma 3.1. *Let $N \geq 2$, $p > 1$, and $\alpha > 0$. If there exists $R > 0$ such that $u_\beta > 0$ in (R,∞), then there exist $C = C(N,p,\alpha) > 0$ and $\widetilde{R} \geq R$ such that*

$$u_\beta(r)(\sinh r)^{\alpha/(p-1)} \leq C \quad \text{in } (\widetilde{R},\infty). \tag{3.1}$$

Proof. Using Lemma 2.1 (c), we find $R_0 > R$ such that $u_\beta' < 0$ in (R_0,∞). Thus, integrating (2.4) over (R_0,r), we infer that

$$\frac{u_\beta'(r)}{u_\beta^p(r)} \leq -\frac{1}{(\sinh r)^{N-1}} \int_{R_0}^r (\sinh s)^{N-1+\alpha}\,ds \quad \text{in} \quad (R_0,\infty). \tag{3.2}$$

Since

$$\lim_{r\to\infty} (\sinh r)^{-(N-1+\alpha)} \int_{R_0}^r (\sinh s)^{N-1+\alpha}\,ds = (N-1+\alpha)^{-1},$$

there exists $R_1 \geq R_0$ such that

$$\int_{R_0}^r (\sinh s)^{N-1+\alpha}\,ds > C(\sinh r)^{N-1+\alpha} \quad \text{in } (R_1,\infty).$$

Hence, from (3.2) it follows that $(u_\beta(r))^{-p}u_\beta'(r) < -C(\sinh r)^\alpha$ in (R_1,∞). Integrating this inequality over (R_1,r), we obtain

$$u_\beta^{1-p}(r) > C\int_{R_1}^r (\sinh s)^\alpha\,ds \quad \text{in } (R_1,\infty).$$

Similarly, since there exists $\widetilde{R} \geq R_1$ such that

$$\int_{R_1}^r (\sinh s)^\alpha\,ds > C(\sinh r)^\alpha \quad \text{in } (\widetilde{R},\infty),$$

it holds that $u_\beta^{1-p}(r) > C(\sinh r)^\alpha$ in $(\widetilde{R},\infty)$. □

Now, we prove Theorem 1.3 (a).

Proof of Theorem 1.3 (a). We prove the assertion by contradiction. Assume that u_β has finitely many zeros in $(0,\infty)$. Then we may assume that there exists $R > 0$ such that $u_\beta > 0$ in (R,∞). Thus, by Lemma 3.1, we find $C > 0$ and $\widetilde{R} \geq R$ such that (3.1) holds. Since

$$p < p_b(N,\alpha) \;\Leftrightarrow\; \frac{\alpha}{p-1} > \frac{N-1+\alpha}{p+1} \;\Leftrightarrow\; \frac{N-1+\alpha}{p+1} > \frac{N-1}{2},$$

we observe from (3.1) that $(\sinh r)^{(N-1+\alpha)/(p+1)}u_\beta \to 0$ and $(\sinh r)^{(N-1)/2}u_\beta \to 0$ as $r \to \infty$. Moreover, using l'Hospital's rule, we obtain $(\sinh r)^{(N-1)/2}u_\beta' \to 0$ as $r \to \infty$. Then we derive $\Psi(r) \to 0$ as $r \to \infty$, where Ψ is defined in (2.6). This is a contradiction to Lemma 2.6 and we complete the proof. □

4. Proof of Theorem 1.5

In this section, we show Theorem 1.5, i.e. the existence of radial solutions of Type R. Now, we shall prepare some notations and lemmas. For $\beta > 0$, let $u_\beta = u_\beta(r)$ be the solution of (Hr). Then we introduce the Prüfer transformation

$$\begin{cases} u_\beta(r) = \rho_\beta(r)\cos\theta_\beta(r), \\ -(\sinh r)^{N-1}u'_\beta(r) = \rho_\beta(r)\sin\theta_\beta(r), \end{cases} \tag{4.1}$$

where $\rho_\beta = \rho_\beta(r) = \left\{u_\beta^2 + ((\sinh r)^{N-1}u'_\beta)^2\right\}^{1/2} > 0$ and $\theta_\beta = \theta_\beta(r)$. Moreover, $\rho_\beta(r)$ and $\theta_\beta(r)$ are continuous functions on r satisfying $\rho_\beta(0) = \beta$ and $\theta_\beta(0) = 0$. Now, we study properties of ρ_β and θ_β.

Lemma 4.1. *Let $N \geq 3$, $p > 1$ and $\alpha > -2$. It holds that $\theta'_\beta(r) \geq 0$ in $r \in (0, +\infty)$. In particular, $\theta'_\beta > 0$ if $\theta_\beta = (j + 1/2)\pi$ for any $j \in \mathbb{N} \cup \{0\}$.*

Proof. Differentiating two equalities in (4.1) with respect to r, we obtain

$$\rho'_\beta \cos\theta_\beta - \rho_\beta\theta'_\beta \sin\theta_\beta = -(\sinh r)^{-(N-1)}\rho_\beta \sin\theta_\beta, \tag{4.2}$$

$$\rho'_\beta \sin\theta_\beta + \rho_\beta\theta'_\beta \cos\theta_\beta = (\sinh r)^{N-1+\alpha}|u_\beta|^{p-1}\rho_\beta \cos\theta_\beta. \tag{4.3}$$

Then, multiplying (4.2) by $-\sin\theta_\beta$ and (4.3) by $\cos\theta_\beta$, and adding them, we see that $\theta'_\beta = (\sinh r)^{-(N-1)}(\sin\theta_\beta)^2 + (\sinh r)^{N-1+\alpha}|u_\beta|^{p-1}(\cos\theta_\beta)^2 \geq 0$. Furthermore, when $\theta_\beta = (j + 1/2)\pi$, it holds that $\theta'_\beta > 0$ in $r \in (0, +\infty)$. $\square$

Lemma 4.2. *Let $N \geq 3$, $p > 1$, and $\alpha > -2$. It holds that*

$$\lim_{\beta \downarrow 0} \rho_\beta(\operatorname{arc\,sinh} 1) = \lim_{\beta \downarrow 0} \theta_\beta(\operatorname{arc\,sinh} 1) = 0.$$

Proof. Applying Lemma 2.2, we obtain $\rho_\beta(\operatorname{arc\,sinh} 1) \to 0$ as $\beta \downarrow 0$. Similarly, it follows from Lemma 2.2 that $\cos\theta_\beta(r) \to 1$ and $\sin\theta_\beta(r) \to 0$ as $\beta \downarrow 0$ uniformly in $r \in [0, \operatorname{arc\,sinh} 1]$. Therefore we see that for some integer $j \geq 0$, $\theta_\beta(r) \to 2j\pi$ as $\beta \downarrow 0$ uniformly in $r \in [0, \operatorname{arc\,sinh} 1]$. Then recalling that $\theta_\beta(0) = 0$, we derive $j = 0$. $\square$

Lemma 4.3. *Let $N \geq 3$, $\alpha > -2$, and $1 < p < p_s(N, \alpha)$. It holds that*

$$\lim_{\beta \to +\infty} \theta_\beta(\operatorname{arc\,sinh} 1) = +\infty.$$

Proof. Let $\beta > 1$ and set $v_\beta(r) = \beta^{-1}u_\beta(\beta^{-(p-1)/(\alpha+2)}r)$. Since u_β satisfies (2.1), v_β satisfies the following integral equation:

$$v_\beta = 1 - \int_0^r \left(\frac{\beta^{-(p-1)/(\alpha+2)}t}{t\sinh(\beta^{-(p-1)/(\alpha+2)}t)}\right)^{N-1} \cdot \int_0^t \left(\frac{s\sinh(\beta^{-(p-1)/(\alpha+2)}s)}{\beta^{-(p-1)/(\alpha+2)}s}\right)^{N-1+\alpha} |v_\beta|^{p-1}v_\beta \, ds\, dt. \tag{4.4}$$

Making use of the Ascoli–Arzelà theorem, we shall prove that $\{v_\beta\}_{\beta>1}$ is a convergent sequence in C^0 sense. We observe from Lemma 2.1 (b) that for any $\beta > 0$,

$$|v_\beta(r)| \le \beta^{-1}\beta = 1 \quad \text{in } r \in [0,\infty). \tag{4.5}$$

Fix $R > 0$. Setting $f(z) = z/\sinh z$ in $z \in (0,\infty)$, we have $f(z) \to 1$ as $z \to 0$. Then, for $\beta > 0$ large enough and any $r_1, r_2 \in [0,R]$ with $r_1 < r_2$,

$$\begin{aligned}
&|v_\beta(r_2) - v_\beta(r_1)| \\
&\quad \le \int_{r_1}^{r_2} \frac{f(\beta^{-(p-1)/(\alpha+2)}t)^{N-1}}{t^{N-1}} \int_0^t \frac{s^{N-1+\alpha}}{(f(\beta^{-(p-1)/(\alpha+2)}s))^{N-1+\alpha}}\,ds\,dt \\
&\quad \le C\int_{r_1}^{r_2} \frac{1}{t^{N-1}} \int_0^t s^{N-1+\alpha}\,ds\,dt \le C\big(r_2^{\alpha+2} - r_1^{\alpha+2}\big).
\end{aligned}$$

We consider the case of $\alpha \ge -1$. It follows from the mean value theorem that

$$|v_\beta(r_2) - v_\beta(r_1)| \le CR^{\alpha+1}(r_2 - r_1). \tag{4.6}$$

If $\alpha \in (-2,-1)$, then we can verify that $(z^{\alpha+2}-1)/(z-1)^{\alpha+2} < 1$ in $z \in (1,\infty)$. Therefore, taking $z = r_2/r_1$, we see that

$$|v_\beta(r_2) - v_\beta(r_1)| \le C(r_2 - r_1)^{\alpha+2}. \tag{4.7}$$

Applying (4.5)–(4.7) and the Ascoli–Arzelà theorem, we find $\widehat{v} \in C([0,R])$ such that $v_\beta \to \widehat{v}$ as $\beta \to \infty$ in $C([0,R])$. Recalling that v_β is the solution of the integral equation (4.4), we infer that $\widehat{v}$ is the solution of

$$\widehat{v}(s) = 1 - \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1+\alpha}\,|\widehat{v}(s)|^{p-1}\widehat{v}(s)\,ds\,dt.$$

Therefore $\widehat{v} \in C^2((0,R])$ and $\widehat{v}$ satisfies

$$\begin{cases} \widehat{v}'' + \dfrac{N-1}{r}\widehat{v}' + r^\alpha|\widehat{v}|^{p-1}\,\widehat{v} = 0, \\ \widehat{v}(0) = 1, \end{cases}$$

i.e. $\widehat{v}$ is a radial solution of (E) on $[0,R]$. Since $R \in (0,+\infty)$ is arbitrary, $\widehat{v}$ is a radial solution of (E) on $[0,+\infty)$. Here, the number of zeros of u_β in $[0,\operatorname{arc\,sinh} 1]$ is equal to that of v_β in $[0,\beta^{(p-1)/(\alpha+2)}\operatorname{arc\,sinh} 1]$. Therefore, since $\widehat{v}$ has infinitely many zeros in $[0,+\infty)$ (e.g. see Theorem 2. (c) in [20]), the number of zeros of u_β in $[0,\operatorname{arc\,sinh} 1]$ diverges to $+\infty$ as $\beta \to +\infty$. Then, we complete the proof. □

Next, for $\gamma > 0$, we consider the problem

$$\begin{cases} \widetilde{u}''(r) + \dfrac{N-1}{\tanh r}\,\widetilde{u}'(r) + (\sinh r)^\alpha|\widetilde{u}(r)|^{p-1}\widetilde{u}(r) = 0 \quad \text{in } (0,+\infty), \\ \lim\limits_{r\to+\infty}(\sinh r)^{N-1}\widetilde{u}(r) = \gamma. \end{cases} \tag{4.8}$$

Then the solution of (4.8) satisfies the following properties:

LEMMA 4.4. *Let $N \geq 3$, $\alpha > -2$, and $p > \max\{1, p_b(N,\alpha)\}$. Then the problem* (4.8) *admits a unique global solution $\widetilde{u}_\gamma = \widetilde{u}_\gamma(r)$ and the following hold*:

(a) $(\sinh r)^{N-1}|\widetilde{u}_\gamma(r)| \leq \gamma$ *in* $r \in (0,\infty)$;

(b) $\displaystyle\lim_{\gamma\to 0} \gamma^{-1}\,\widetilde{u}_\gamma = \int_r^{+\infty} \frac{N-1}{(\sinh t)^{(N-1)}}\,dt$ *uniformly in* $r \in [\operatorname{arc}\sinh 1, +\infty)$;

(c) $\displaystyle\lim_{\gamma\to 0} \gamma^{-1}(\sinh r)^{N-1}\,\widetilde{u}'_\gamma = -(N-1)$ *uniformly in* $r \in [\operatorname{arc}\sinh 1, +\infty)$.

PROOF. To begin with, under the transformation

$$\sinh s = \frac{1}{\sinh r}, \qquad w(s) = (\sinh r)^{N-1}\widetilde{u}(r),$$

problem (4.8) is rewritten as follows:

$$\begin{cases} w'' + \dfrac{Nw'}{\tanh s} + (N-1)w + (\sinh s)^\delta |w|^{p-1}w = 0 \quad \text{in } (0,+\infty), \\ w(0) = \gamma, \end{cases} \tag{4.9}$$

where $\delta = (N-1)p - (N+1+\alpha)$. Here $p > p_b(N,\alpha)$ yields $\delta > -2$. Then we consider the following integral equation:

$$w = \gamma - \int_0^s \frac{1}{(\sinh t)^N}\int_0^t (\sinh z)^N((N-1)w + (\sinh z)^\delta |w|^{p-1}w)\,dz\,dt. \tag{4.10}$$

By a standard theory, (4.10) has a unique global solution w_γ and then we can verify that w_γ is the unique solution of (4.9) (e.g. see [8]). Thus (4.8) also has the unique global solution $\widetilde{u}_\gamma$. Now, we shall prove the assertion (a). Multiplying the equation in (4.8) by $(\sinh r)^{2(N-1)}\widetilde{u}'_\gamma(r)$ and setting

$$G(r) = \frac{(\widetilde{u}'_\gamma(r))^2}{2}(\sinh r)^{2(N-1)} + \frac{|\widetilde{u}_\gamma(r)|^{p+1}}{p+1}(\sinh r)^{2(N-1)+\alpha},$$

we can verify that $G' \geq 0$ in $r \in (0,\infty)$. Then it follows from l'Hospital's rule and $p > p_b(N,\alpha)$ that $G(r) \leq \lim_{r\to\infty} G(r) = ((N-1)\gamma)^2/2$ in $r \in (0,\infty)$. Therefore, we obtain $|\widetilde{u}'_\gamma(r)| \leq (N-1)\gamma(\sinh r)^{-(N-1)}$ in $r \in (0,\infty)$ and

$$|\widetilde{u}_\gamma(r)| \leq \int_r^\infty |\widetilde{u}'_\gamma(z)|\,dz \leq \int_r^\infty \frac{(N-1)\gamma}{(\sinh z)^{N-1}\tanh z}\,dz = \frac{\gamma}{(\sinh r)^{N-1}}$$

in $r \in (0,\infty)$. We derive the assertion (a). We shall prove the assertions (b) and (c). Letting $\widetilde{U}_\gamma(r) := \gamma^{-1}\widetilde{u}_\gamma(r)$, we have

$$\begin{cases} \big((\sinh r)^{N-1}\widetilde{U}'_\gamma(r)\big)' + \gamma^{p-1}(\sinh r)^{N-1+\alpha}|\widetilde{U}_\gamma(r)|^{p-1}\widetilde{U}_\gamma(r) = 0, \\ \displaystyle\lim_{r\to+\infty}(\sinh r)^{N-1}\widetilde{U}_\gamma(r) = 1. \end{cases} \tag{4.11}$$

Using l'Hospital's rule, we obtain $\lim_{r\to+\infty}(\sinh r)^{N-1}\widetilde{U}'_\gamma(r) = -(N-1)$. Hence, integrating the equation in (4.11) over $[r,+\infty)$, we derive

$$N - 1 + (\sinh r)^{N-1}\widetilde{U}'_\gamma(r) = \int_r^{+\infty} \gamma^{p-1}(\sinh t)^{N-1+\alpha}|\widetilde{U}_\gamma(t)|^{p-1}\widetilde{U}_\gamma(t)\,dt. \tag{4.12}$$

Since the assertion (a) implies that $\left|\widetilde{U}_\gamma\right| \le (\sinh r)^{-(N-1)}$ in $r \in (0,\infty)$, we have

$$\left|N-1+(\sinh r)^{N-1}\widetilde{U}_\gamma'(r)\right| \le \int_r^{+\infty} \gamma^{p-1}(\sinh t)^{\alpha-(N-1)(p-1)}\,dt \le \gamma^{p-1}C$$

in $r \in [\operatorname{arc\,sinh} 1, +\infty)$, where $\alpha-(N-1)(p-1) < 0$. Thus, we derive the assertion (c). Multiplying (4.12) by $(\sinh r)^{-(N-1)}$ and integrating over $[r,+\infty)$, we infer from the assertion (a) that

$$\left|\widetilde{U}_\gamma(r) - \int_r^{+\infty} \frac{N-1}{(\sinh t)^{N-1}}\,dt\right| \le \gamma^{p-1}\int_r^{+\infty}\int_s^{+\infty}(\sinh t)^{\alpha-(N-1)(p-1)}\,dt\,ds \le \gamma^{p-1}C$$

in $r \in [\operatorname{arc\,sinh} 1, +\infty)$. Then the assertion (b) holds. $\square$

In the following, for $\gamma > 0$, we denote by $\widetilde{u}_\gamma = \widetilde{u}_\gamma(r)$ the solution of the problem (4.8). Now again, we use Prüfer transformation

$$\begin{cases} \widetilde{u}_\gamma(r) = \widetilde{\rho}_\gamma(r)\cos\widetilde{\theta}_\gamma(r), \\ -(\sinh r)^{N-1}\,\widetilde{u}_\gamma'(r) = \widetilde{\rho}_\gamma(r)\sin\widetilde{\theta}_\gamma(r), \end{cases}$$

where $\widetilde{\rho}_\gamma = \left\{\widetilde{u}_\gamma^2 + ((\sinh r)^{N-1}\widetilde{u}_\gamma')^2\right\}^{1/2} > 0$. Here, $\widetilde{\rho}_\gamma$ and $\widetilde{\theta}_\gamma$ are continuous functions on r and satisfy the following asymptotic behavior as $r \to +\infty$:

Lemma 4.5. *Let N, α, and p satisfy the assumption in Lemma 4.4. Then it holds that* $\lim_{r\to+\infty}\widetilde{\rho}_\gamma(r) = (N-1)\gamma$ *and* $\lim_{r\to+\infty}\widetilde{\theta}_\gamma(r) = \pi/2$.

Proof. Set $w_\gamma(s) = (\sinh r)^{N-1}\widetilde{u}_\gamma(r)$, where $\sinh s = (\sinh r)^{-1}$. To begin with, we shall prove that $w_\gamma'(s)\sinh s \to 0$ as $s \downarrow 0$. Since $\widetilde{u}_\gamma$ is the solution of (4.8), w_γ is the solution of (4.10). Then, differentiating (4.10) with respect to s, we have

$$w_\gamma' = -\frac{1}{(\sinh s)^N}\int_0^s \left\{(N-1)(\sinh t)^N w_\gamma + (\sinh t)^{N+\delta}|w_\gamma|^{p-1}w_\gamma\right\}dt. \tag{4.13}$$

Since $w_\gamma(0) = \gamma > 0$, we may choose sufficiently small $\xi > 0$ such that $w_\gamma' < 0$ in $(0,\xi)$. Hence, we observe from (4.13) that

$$|w_\gamma'(s)| \le C\int_0^s \frac{(\sinh t)^{N+1} + (\sinh t)^{N+1+\delta}}{(\sinh s)^N \tanh t}\,dt = C\left\{\sinh s + (\sinh s)^{1+\delta}\right\}$$

in $s \in (0,\xi)$. Thus it holds that $w_\gamma'(s)\sinh s \to 0$ as $s \to 0$ and then

$$\lim_{r\to+\infty}(\sinh r)^{N-1}\,\widetilde{u}_\gamma'(r) = \lim_{s\downarrow 0} -(w_\gamma'(s)\sinh s + (N-1)w_\gamma(s)\cosh s) = -(N-1)\gamma.$$

Therefore, we have $\widetilde{\rho}_\gamma(r) \to (N-1)\gamma$ as $r \to +\infty$. Since $\cos\widetilde{\theta}_\gamma(r) \to 0$ and $\sin\widetilde{\theta}_\gamma(r) \to 1$ as $r \to +\infty$, we also obtain $\lim_{r\to+\infty}\widetilde{\theta}_\gamma(r) = \pi/2$. $\square$

Then, we shall state the properties of $(\widetilde{\rho}_\gamma, \widetilde{\theta}_\gamma)$.

Lemma 4.6. *Let N, α, and p satisfy the assumption in Lemma* 4.4. *Then it holds that $\widetilde{\theta}'_\gamma(r) \geq 0$ in $(0, +\infty)$. In particular, $\widetilde{\theta}'_\gamma > 0$ if $\widetilde{\theta}_\gamma = (-j + 1/2)\pi$ for any $j \in \mathbb{N}$.*

Making use of the same method as in the proof of Lemma 4.1, we can prove Lemma 4.6. Hence we omit the proof of Lemma 4.6.

Lemma 4.7. *Let N, α, and p satisfy the assumption in Lemma* 4.4. *Then it holds that $\lim_{\gamma\downarrow 0} \widetilde{\rho}_\gamma(\operatorname{arc\,sinh} 1) = 0$. Moreover, there exists $\Theta \in (0, \pi/2)$ such that $\lim_{\gamma\downarrow 0} \widetilde{\theta}_\gamma(\operatorname{arc\,sinh} 1) = \Theta$.*

Proof. Making use of Lemma 4.4 (b)–(c), we have $\lim_{\gamma\downarrow 0} \widetilde{\rho}_\gamma(\operatorname{arc\,sinh} 1) = 0$. Moreover, there exist $\Theta = \Theta(r) \in (0, \pi/2)$ and $j \in \mathbb{Z}$ such that $\widetilde{\theta}_\gamma(r) \to \Theta - 2j\pi$ as $\gamma \downarrow 0$ uniformly in $r \in [\operatorname{arc\,sinh} 1, +\infty)$. Recalling that $\widetilde{\theta}_\gamma(r) \to \pi/2$ as $r \to +\infty$, we see that $j = 0$. $\square$

Lemma 4.8. *Let N, α, and p satisfy the assumption in Lemma* 4.4. *Then it holds that*

$$\lim_{\gamma\to+\infty} \widetilde{\theta}_\gamma(\operatorname{arc\,sinh} 1) = -\infty.$$

Proof. Let $\gamma > 1$ and set $\widetilde{v}_\gamma(s) = \gamma^{-1} w_\gamma(\gamma^{-(p-1)/(\delta+2)} s)$. Since $w_\gamma(s)$ satisfies (4.10), $\widetilde{v}_\gamma(s)$ satisfies the following integral equation:

$$\begin{aligned}(4.14)\qquad \widetilde{v}_\gamma = 1 &- \int_0^s \left(\frac{\gamma^{-(p-1)/(\delta+2)} t}{t \sinh(\gamma^{-(p-1)/(\delta+2)} t)} \right)^N \\ &\cdot \int_0^t \left(\frac{z \sinh(\gamma^{-(p-1)/(\delta+2)} z)}{\gamma^{-(p-1)/(\delta+2)} z} \right)^{N+\delta} |\widetilde{v}_\gamma|^{p-1} \widetilde{v}_\gamma \, dz \, dt \\ &- \int_0^s \left(\frac{\gamma^{-(p-1)/(\delta+2)} t}{t \sinh(\gamma^{-(p-1)/(\delta+2)} t)} \right)^N \\ &\cdot \int_0^t \left(\frac{z \sinh(\gamma^{-(p-1)/(\delta+2)} z)}{\gamma^{-(p-1)/(\delta+2)} z} \right)^N (N-1)\gamma^{-2(p-1)/(\delta+2)} \widetilde{v}_\gamma \, dz \, dt.\end{aligned}$$

We shall check that for any fixed $S > 0$, $\{\widetilde{v}_\gamma\}_{\gamma>0}$ is a convergent sequence in C^0 sense by making use of the Ascoli–Arzelà theorem. Now, Lemma 4.4 (a) implies that

$$(4.15)\qquad |\widetilde{v}_\gamma(s)| \leq \gamma^{-1}\gamma = 1 \quad \text{in } s \in [0, \infty).$$

Fix $S > 0$. Applying the same method to obtain (4.6)–(4.7), we observe from (4.14) that for any $s_1, s_2 \in [0, S]$ satisfying $s_1 < s_2$ and sufficiently large $\gamma > 0$,

$$(4.16)\qquad |\widetilde{v}_\gamma(s_2) - \widetilde{v}_\gamma(s_1)| \leq \begin{cases} C(S + S^{\delta+1})(s_2 - s_1) & \text{for } \delta \geq -1, \\ CS(s_2 - s_1) + C(s_2 - s_1)^{\delta+2} & \text{for } \delta \in (-2, -1). \end{cases}$$

Applying (4.15)–(4.16) and the Ascoli–Arzelà theorem, we find $\widetilde{v} \in C([0,S])$ such that $\widetilde{v}_\gamma \to \widetilde{v}$ as $\gamma \to \infty$ in $C([0,S])$. Since $\widetilde{v}_\gamma$ is the solution of the integral equation (4.14), we verify that $\widetilde{v}$ is the solution of

$$\widetilde{v}(s) = 1 - \int_0^s \frac{1}{t^N} \int_0^t z^{N+\delta} |\widetilde{v}|^{p-1} \widetilde{v}\, dz\, dt.$$

Thus $\widetilde{v} \in C^2((0,S])$ and $\widetilde{v}$ satisfies

$$\begin{cases} \widetilde{v}''(s) + \dfrac{N}{s}\, \widetilde{v}'(s) + s^\delta |\widetilde{v}(s)|^{p-1}\, \widetilde{v}(s) = 0, \\ \widetilde{v}(0) = 1. \end{cases} \tag{4.17}$$

Moreover, we observe that $\widetilde{v}$ is the solution of (4.17) on $[0,+\infty)$, for $S \in (0,+\infty)$ is arbitrary. Now, the number of zeros of $\widetilde{u}_\gamma$ in $[\operatorname{arc\,sinh} 1, +\infty)$ is equal to that of $\widetilde{v}_\gamma$ in $[0, \gamma^{(p-1)/(\delta+2)} \operatorname{arc\,sinh} 1]$. Furthermore, since $p > p_b(N,\alpha)$ is equivalent to $p < p_s(N+1,\delta)$, $\widetilde{v}$ has infinitely many zeros in $[0,+\infty)$ (e.g. see Theorem 2 (c) in [20]). Thus, the number of zeros of $\widetilde{u}_\gamma$ in $[\operatorname{arc\,sinh} 1, +\infty)$ diverges to $+\infty$ as $\gamma \to +\infty$. $\square$

Then, making use of the method applied in [22], we shall prove Theorem 1.5.

Proof of Theorem 1.5. We denote Γ and $\widetilde{\Gamma}_k$ by

$$\begin{aligned} \Gamma &= \{(\rho_\beta(\operatorname{arc\,sinh} 1), \theta_\beta(\operatorname{arc\,sinh} 1); \beta \in (0,+\infty))\}, \\ \widetilde{\Gamma}_k &= \{(\widetilde{\rho}_\gamma(\operatorname{arc\,sinh} 1), \widetilde{\theta}_\gamma(\operatorname{arc\,sinh} 1) + k\pi; \gamma \in (0,+\infty))\}, \end{aligned}$$

where $k \in \mathbb{N} \cup \{0\}$. It follows from the uniqueness of solutions to the initial value problems that Γ and $\widetilde{\Gamma}_k$ are continuous curves and do not intersect itself. Similarly, $\widetilde{\Gamma}_j$ and $\widetilde{\Gamma}_k$ do not intersect if $j \neq k$. Using Lemmas 4.2–4.3 and 4.7–4.8, we observe that Γ intersects $\widetilde{\Gamma}_k$ for every k. Then we define β_k by the smallest β such that Γ intersects $\widetilde{\Gamma}_k$. Since $\widetilde{\Gamma}_j$ and $\widetilde{\Gamma}_k$ do not intersect, β_k increases strictly on k. Moreover, there exists γ_k such that

$$\begin{cases} \theta_{\beta_k}(\operatorname{arc\,sinh} 1) = \widetilde{\theta}_{\gamma_k}(\operatorname{arc\,sinh} 1) + k\pi, \\ \rho_{\beta_k}(\operatorname{arc\,sinh} 1) = \widetilde{\rho}_{\gamma_k}(\operatorname{arc\,sinh} 1). \end{cases}$$

Therefore, we obtain

$$\begin{cases} u_{\beta_k}(\operatorname{arc\,sinh} 1) = (-1)^k\, \widetilde{u}_{\gamma_k}(\operatorname{arc\,sinh} 1), \\ u'_{\beta_k}(\operatorname{arc\,sinh} 1) = (-1)^k\, \widetilde{u}'_{\gamma_k}(\operatorname{arc\,sinh} 1). \end{cases}$$

Then $u_{\beta_k} = (-1)^k\, \widetilde{u}_{\gamma_k}$ in $(0,+\infty)$, i.e. $(\sinh r)^{N-1} u_{\beta_k}(r) \to (-1)^k \gamma_k$ as $r \to \infty$. Furthermore, Lemma 4.5 implies that $\theta_{\beta_k}(r) = \widetilde{\theta}_{\gamma_k}(r) + k\pi \to \pi/2 + k\pi$ as $r \to \infty$. Hence, u_{β_k} has exactly k zeros in $(0,\infty)$. $\square$

5. Proof of Theorem 1.3 (b)

5.1. Existence of threshold. In this subsection, we shall prove the following theorem:

THEOREM 5.1. *Let $N \geq 3$, $\alpha > 0$, and $p_b(N,\alpha) < p < p_s(N,\alpha)$. Then there exists a unique $\beta_H > 0$ such that*

(a) *If $\beta < \beta_H$, then $u_\beta > 0$ in $[0,\infty)$ and $u_\beta \notin H^1(\mathbb{H}^N)$.*
(b) *$u_{\beta_H} > 0$ in $[0,\infty)$ and $u_{\beta_H} \in H^1(\mathbb{H}^N)$.*

Now, we prepare notations and lemmas. To begin with, we shall study decay rates of radial solutions.

LEMMA 5.2. *Let $N \geq 2$, $\alpha > 0$ and $p_b(N,\alpha) < p \leq p_s(N,\alpha)$. Suppose that $u_\beta \in H^1(\mathbb{H}^N)$ and $u_\beta > 0$ in $[0,\infty)$. Then it holds that*

$$\lim_{r\to\infty} \frac{\log u_\beta(r)}{r} = \lim_{r\to\infty} \frac{\log |u_\beta'(r)|}{r} = \lim_{r\to\infty} \frac{u_\beta'(r)}{u_\beta(r)} = -(N-1). \tag{5.1}$$

PROOF. $u_\beta \in H^1(\mathbb{H}^N)$ implies that $(\sinh r)^{(N-1)/2} u_\beta(r) \to 0$ as $r \to \infty$. Thus, since $p > p_b(N,\alpha)$ is equivalent to $\alpha/(p-1) < (N-1)/2$, it holds that

$$\lim_{r\to\infty} (\sinh r)^{\alpha/(p-1)} u_\beta(r) = 0. \tag{5.2}$$

Then, for $\varepsilon > 0$ small enough, there exists $r_\varepsilon > 0$ such that $(\sinh r)^\alpha u_\beta^p(r) \leq \varepsilon u_\beta(r)$ and $(\tanh r)^{-1} \leq 1 + \varepsilon$ in $[r_\varepsilon, \infty)$. Using Lemma 2.1 (d), we obtain the following estimates in $[r_\varepsilon, \infty)$:

$$u_\beta'' + (N-1)(1+\varepsilon)u_\beta' \leq u_\beta'' + \frac{N-1}{\tanh r} u_\beta' + (\sinh r)^\alpha u_\beta^p = 0,$$
$$u_\beta'' + (N-1)u_\beta' + \varepsilon u_\beta \geq u_\beta'' + \frac{N-1}{\tanh r} u_\beta' + (\sinh r)^\alpha u_\beta^p = 0.$$

Then, applying the same method as in the proof of Lemma 3.4 of [14], we complete the proof. □

Next, we shall show the uniqueness of a positive radial solution in $H^1(\mathbb{H}^N)$. To the aim, we define notations. We set

$$\widehat{u}_\beta(r) := (\sinh r)^q u_\beta(r), \quad q = \frac{\alpha + 2(N-1)}{p+3}, \quad \widehat{q} = q(p-1) - \alpha. \tag{5.3}$$

Then, by direct calculations, $\widehat{u}_\beta$ satisfies

$$(\sinh r)^{\widehat{q}}\, \widehat{u}_\beta''(r) + \frac{1}{2}\big((\sinh r)^{\widehat{q}}\big)' \widehat{u}_\beta'(r) + G(r)\widehat{u}_\beta(r) + \widehat{u}_\beta^p(r) = 0, \tag{5.4}$$

where

$$G(r) = A(\sinh r)^{\widehat{q}} + B(\sinh r)^{\widehat{q}-2}, \quad A = q^2 - q(N-1), \quad B = A + q.$$

Moreover, we set the following function:

$$E_\beta(r) := \frac{1}{2}(\sinh r)^{\widehat{q}}\big(\widehat{u}'_\beta(r)\big)^2 + \frac{\widehat{u}_\beta^{p+1}(r)}{p+1} + \frac{1}{2}\,G(r)\,\widehat{u}_\beta^2(r). \tag{5.5}$$

Then, multiplying (5.4) by $\widehat{u}'_\beta(r)$, we have

$$\frac{d}{dr}E_\beta(r) = \frac{1}{2}\,G'(r)\,\widehat{u}_\beta^2(r). \tag{5.6}$$

Now, we shall state properties of E_β and G.

Lemma 5.3. *Let N, α, and p satisfy the assumption in Theorem* 5.1. *Then the following hold*:

(a) *There exists $R > 0$ such that $G' > 0$ in $(0, R)$, $G'(R) = 0$, and $G' < 0$ in (R, ∞)*;
(b) *$E_\beta(r) \to 0$ as $r \to 0$*;
(c) *If $u_\beta \in H^1(\mathbb{H}^N)$ and $u_\beta > 0$ in $[0, \infty)$, then $E_\beta(r) \to 0$ as $r \to \infty$.*

Proof. First we prove the assertion (a). Since

$$G'(r) = (A\widehat{q}(\sinh r)^2 + B(\widehat{q} - 2))(\sinh r)^{\widehat{q}-2}(\tanh r)^{-1},$$

it is sufficient to investigate the sign of constants $A\widehat{q}$ and $B(\widehat{q} - 2)$. We can verify that $p > p_b(N, \alpha)$ is equivalent to $\widehat{q} > 0$ and $p < p_s(N, \alpha)$ is equivalent to $\widehat{q} < 2$. Moreover, it follows from $p > p_b(N, \alpha)$ that $A < B < 0$. Hence, we see that $A\widehat{q} < 0$ and $B(\widehat{q} - 2) > 0$. Therefore, using the definition of G' and the monotonicity of $(\sinh r)^2$, we obtain the assertion (i).

We prove the assertion (b). $E_\beta(r)$ can be expressed as $E_\beta(r) = E^1_\beta(r)E^2_\beta(r)$, where

$$E^1_\beta(r) = \frac{(\sinh r)^{(p+1)q-\alpha-2}u_\beta^2}{2},$$
$$E^2_\beta(r) = \left\{\left(q\cosh r + \frac{u'_\beta \sinh r}{u_\beta}\right)^2 + \frac{2(\sinh r)^{\alpha+2}\,u_\beta^{p-1}}{p+1} + A(\sinh r)^2 + B\right\}.$$

Since $(p + 1)q - \alpha - 2 > 0$, we see that $E^1_\beta(r) \to 0$ as $r \to 0$. Moreover, Lemma 2.1 (a) implies that $|E^2_\beta(0)| < \infty$. Thus the assertion (b) holds. We shall prove the assertion (c). Since $((p+1)q-\alpha)/2 < N-1$ and $u_\beta \in H^1(\mathbb{H}^N)$, it holds that $(\sinh r)^2E^1_\beta(r) \to 0$ as $r \to \infty$. Furthermore, we observe from (5.1)–(5.2) that $\lim_{r\to\infty} |(\sinh r)^{-2}E^2_\beta(r)| < \infty$ and the assertion (c) holds. □

Making use of Lemma 5.3, we prepare further lemmas for the proof of Theorem 5.1.

Lemma 5.4. *Let N, α, and p satisfy the assumption in Theorem* 5.1. *Let $\beta > 0$. Moreover, assume that there exists $r_\beta > 0$ such that $u_\beta > 0$ in $[0, r_\beta)$ and $u_\beta(r_\beta) = 0$. Then the following hold*:

(a) *Let* $\widetilde{\beta} < \beta$. *Then there exists* $r_1 \in (0, r_\beta)$ *such that* $u_\beta(r_1) = u_{\widetilde{\beta}}(r_1)$ *and* $u_{\widetilde{\beta}} > 0$ *in* $[0, r_1]$.

(b) *In addition, suppose that* $u_{\widetilde{\beta}} > 0$ *in* $[0, r_\beta)$ *and* $u_{\widetilde{\beta}}(r_\beta) = 0$. *Then there exists* $r_2 \in (r_1, r_\beta)$ *such that* $u_\beta(r_2) = u_{\widetilde{\beta}}(r_2)$.

Proof. Since Lemma 2.4 (b) in [7] yields the existence of r_1, we complete the proof of the assertion (a). In the following, we shall prove the assertion (b). Suppose not, $u_\beta < u_{\widetilde{\beta}}$ in (r_1, r_β). We set $\gamma(r) = u_{\widetilde{\beta}}(r)/u_\beta(r)$. Then, it follows from l'Hospital's rule that

$$\gamma(r_\beta) = \lim_{r \to r_\beta} \frac{u_{\widetilde{\beta}}(r)}{u_\beta(r)} = \lim_{r \to r_\beta} \frac{u'_{\widetilde{\beta}}(r)}{u'_\beta(r)} = \frac{u'_{\widetilde{\beta}}(r_\beta)}{u'_\beta(r_\beta)}. \tag{5.7}$$

Now we claim that $u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}} > 0$ in $(0, r_\beta)$. The equation in (Hr) implies that u_β and $u_{\widetilde{\beta}}$ satisfy

$$\big((\sinh r)^{N-1}(u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}})\big)' = (\sinh r)^{N-1+\alpha} u_\beta u_{\widetilde{\beta}} \big(u_\beta^{p-1} - u_{\widetilde{\beta}}^{p-1}\big). \tag{5.8}$$

We see that $(\sinh r)^{N-1}(u_\beta(r)u'_{\widetilde{\beta}}(r) - u'_\beta(r)u_{\widetilde{\beta}}(r)) \to 0$ as $r \to 0$ by making use of Lemma 2.1 (a). Since $u_\beta > u_{\widetilde{\beta}}$ in $[0, r_1)$, (5.8) implies that $u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}} > 0$ in $(0, r_1)$. We shall also verify that $u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}} > 0$ in (r_1, r_β). Suppose not, there exists $r_3 \in (r_1, r_\beta)$ such that $u_\beta(r_3)u'_{\widetilde{\beta}}(r_3) - u'_\beta(r_3)u_{\widetilde{\beta}}(r_3) = 0$. Then, we observe from (5.8) that $u_\beta(r_\beta)u'_{\widetilde{\beta}}(r_\beta) - u'_\beta(r_\beta)u_{\widetilde{\beta}}(r_\beta) < 0$, for $u_\beta < u_{\widetilde{\beta}}$ in (r_1, r_β). This is a contradiction to $u_\beta(r_\beta) = u_{\widetilde{\beta}}(r_\beta) = 0$. Therefore, we derive

$$u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}} > 0 \quad \text{in } (0, r_\beta). \tag{5.9}$$

Then it holds that $\gamma' = (u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}})/u_\beta^2 > 0$ in $(0, r_\beta)$. Namely, γ is strictly monotone increasing in $[0, r_\beta]$. On the other hand, we deduce from (5.3)–(5.6) and Lemma 5.3 (b) that

$$\begin{aligned} \frac{1}{2}(\sinh r_\beta)^{\widehat{q}+2q}\big((u'_{\widetilde{\beta}}(r_\beta))^2 - \gamma^2(r)(u'_\beta(r_\beta))^2\big) &= E_{\widetilde{\beta}}(r_\beta) - \gamma^2(r)E_\beta(r_\beta) \\ &= E_{\widetilde{\beta}}(\varepsilon) - \gamma^2(r)E_\beta(\varepsilon) + \frac{1}{2}\int_\varepsilon^{r_\beta} G'(\tau)\big((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(r)(\widehat{u}_\beta(\tau))^2\big)\,d\tau \\ &\to \frac{1}{2}\int_0^{r_\beta} G'(\tau)\big((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(r)(\widehat{u}_\beta(\tau))^2\big)\,d\tau \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{5.10}$$

for $r, \varepsilon \in (0, r_\beta)$. Applying Lemma 5.3 (a), we find $R > 0$ such that $G' > 0$ in $(0, R)$, $G'(R) = 0$, and $G' < 0$ in (R, ∞). Then, we shall show a contradiction for the following two cases: (1) $R \ge r_\beta$; (2) $R < r_\beta$. For the case of (1), it holds that $G' > 0$ on $(0, r_\beta)$. Moreover, we infer from the monotonicity of γ that $\gamma < \gamma(r_\beta)$, i.e. $\widehat{u}_{\widetilde{\beta}} < \gamma(r_\beta)\widehat{u}_\beta$ in $(0, r_\beta)$. Hence, (5.7) and (5.10) for the case of $r = r_\beta$ imply that

$$0 = \frac{1}{2}\int_0^{r_\beta} G'(\tau)\big((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(r_\beta)(\widehat{u}_\beta(\tau))^2\big)\,d\tau < 0.$$

This is a contradiction. Next we consider the case of (2). Then, $G' > 0$ in $(0, R)$ and $G' < 0$ in (R, r_β). Since we see that $\widehat{u}_{\widetilde{\beta}} < \gamma(R)\widehat{u}_\beta$ in $(0, R)$ and $\widehat{u}_{\widetilde{\beta}} > \gamma(R)\widehat{u}_\beta$ in (R, r_β), it holds that $G'((\widehat{u}_{\widetilde{\beta}})^2 - \gamma^2(R)(\widehat{u}_\beta)^2) < 0$ in $(0, r_\beta)$. Therefore, we deduce from (5.7) and (5.10) that

$$\begin{aligned}
0 &= (\sinh r_\beta)^{\widehat{q}+2q}\big((u'_{\widetilde{\beta}}(r_\beta))^2 - \gamma^2(r_\beta)(u'_\beta(r_\beta))^2\big)\\
&< (\sinh r_\beta)^{\widehat{q}+2q}\big((u'_{\widetilde{\beta}}(r_\beta))^2 - \gamma^2(R)(u'_\beta(r_\beta))^2\big)\\
&= \int_0^{r_\beta} G'(\tau)\big((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(R)(\widehat{u}_\beta(\tau))^2\big)\,d\tau < 0.
\end{aligned}$$

This is a contradiction and we complete the proof of the assertion (b). □

Lemma 5.5. *Let N, α, and p satisfy the assumption in Theorem 5.1. Suppose that $\beta, \widetilde{\beta} > 0$ with $\beta \neq \widetilde{\beta}$. If there exist $r_1, r_2 > 0$ such that*

$$r_1 < r_2, \quad u_\beta(r_1) = u_{\widetilde{\beta}}(r_1), \quad u_\beta(r_2) = u_{\widetilde{\beta}}(r_2), \quad u_\beta > u_{\widetilde{\beta}} > 0 \quad \text{in } (r_1, r_2),$$

then there exists $C = C(N, p, \alpha) > 0$ such that

$$\int_{r_1}^{r_2} u_\beta^{p+1}(s)(\sinh s)^{N-1+\alpha}\,ds \geq C.$$

Proof. We set $v := u_\beta - u_{\widetilde{\beta}}$ and define the function $\chi\colon \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ by $\chi = 1$ in $[r_1, r_2]$ and $\chi = 0$ in $[0, r_1) \cup (r_2, +\infty)$. It follows from $p \in (p_b(N, \alpha), p_s(N, \alpha))$ and Lemma A.2 in [6] that $\|\omega^{\alpha/(p+1)} v\chi\|_{L^{p+1}(\mathbb{B}^N)} \leq C\|\nabla_{\mathbb{B}^N}(v\chi)\|_{L^2(\mathbb{B}^N)}$. Here, $\mathbb{B}^N$ denotes the ball model of the hyperbolic space, $\nabla_{\mathbb{B}^N}$ is the gradient operator on $\mathbb{B}^N$ and $\omega(r) = \sinh r$ (see Appendix in [6]). Then, it holds that

$$\begin{aligned}
\bigg(\int_{r_1}^{r_2} v^{p+1}(s)(\sinh s)^{N-1+\alpha}\,ds\bigg)^{2/(p+1)} &= \|\omega^{\alpha/(p+1)} v\chi\|^2_{L^{p+1}(\mathbb{B}^N)}\\
\leq C\|\nabla_{\mathbb{B}^N}(v\chi)\|^2_{L^2(\mathbb{B}^N)} &= C\int_{r_1}^{r_2} \big(u_\beta^p(s) - u_{\widetilde{\beta}}^p(s)\big)v(s)(\sinh s)^{N-1+\alpha}\,ds,
\end{aligned}$$

where integrating by parts, we derive the last equality. We observe from the mean value theorem and Hölder's inequality that

$$\begin{aligned}
&\bigg(\int_{r_1}^{r_2} v^{p+1}(s)(\sinh s)^{N-1+\alpha}\,ds\bigg)^{2/(p+1)}\\
&\qquad\leq C\int_{r_1}^{r_2} u_\beta^{p-1}(s)v^2(s)(\sinh s)^{N-1+\alpha}\,ds\\
&\qquad\leq C\bigg(\int_{r_1}^{r_2} u_\beta^{p+1}(s)(\sinh s)^{N-1+\alpha}\,ds\bigg)^{(p-1)/(p+1)}\\
&\qquad\qquad\cdot\bigg(\int_{r_1}^{r_2} v^{p+1}(s)(\sinh s)^{N-1+\alpha}\,ds\bigg)^{2/(p+1)}.
\end{aligned}$$

Thus we obtain the required inequality immediately. □

Lemma 5.6. *Let N, α, and p satisfy the assumption in Theorem* 5.1. *Suppose that $\beta, \widetilde{\beta} > 0$ with $\beta \neq \widetilde{\beta}$. If $u_\beta \in H^1(\mathbb{H}^N)$ and there exists $r_1 > 0$ such that*

$$u_\beta(r_1) = u_{\widetilde{\beta}}(r_1), \quad u_\beta > u_{\widetilde{\beta}} > 0 \quad \textit{in } (r_1, \infty),$$

then there exists $C = C(N, p, \alpha) > 0$ such that

$$\int_{r_1}^{\infty} u_\beta^{p+1}(s)(\sinh s)^{N-1+\alpha}\, ds \geq C.$$

Proof. We can verify that $u_{\widetilde{\beta}} \in H^1(\mathbb{H}^N)$. Indeed, since $u_\beta > u_{\widetilde{\beta}} > 0$ in (r_1, ∞), we have $u_{\widetilde{\beta}} \in L^2(\mathbb{H}^N)$. Moreover, it follows from Lemma 2.1 (c) and Lemma A.2 in [6] that

$$\begin{aligned}\int_0^r (\sinh s)^{N-1}(u_{\widetilde{\beta}}')^2\, ds &\leq \int_0^r (u_{\widetilde{\beta}}')^2(\sinh s)^{N-1}\, ds - (\sinh r)^{N-1} u_{\widetilde{\beta}} u_{\widetilde{\beta}}' \\ &= -\int_0^r \big[(\sinh s)^{N-1} u_{\widetilde{\beta}}'\big]' u_{\widetilde{\beta}}\, ds = \int_0^r (\sinh s)^{N-1+\alpha} u_{\widetilde{\beta}}^{p+1}\, ds \\ &\leq \int_0^r (\sinh s)^{N-1+\alpha} u_\beta^{p+1}\, ds \leq C\bigg(\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_\beta|_{\mathbb{B}^N}^2\, dV_{\mathbb{B}^N}\bigg)^{(p+1)/2},\end{aligned}$$

for $r > 0$ large enough. Therefore, it holds that $u_{\widetilde{\beta}} \in H^1(\mathbb{H}^N)$. Setting $v := u_\beta - u_{\widetilde{\beta}}$ and defining the function $\chi\colon \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ by $\chi = 1$ in $[r_1, \infty)$ and $\chi = 0$ in $[0, r_1)$, we apply the same method as in the proof of Lemma 5.5. Then we derive the required estimate. □

Lemma 5.7. *Let N, α, and p satisfy the assumption in Theorem* 5.1. *Suppose that $R > 0$. Then, the following problem admits at most one solution*:

$$\begin{cases} u''(r) + \dfrac{N-1}{\tanh r}\, u'(r) + (\sinh r)^\alpha u^p(r) = 0 & \textit{in } [0, R), \\ u'(0) = 0, \quad u(R) = 0, \quad u > 0 & \textit{in } [0, R). \end{cases} \tag{5.11}$$

Proof. Suppose not, u_{β_0} and u_{β_1} are the solutions of (5.11), where $u_{\beta_0}(0) = \beta_0$, $u_{\beta_1}(0) = \beta_1$, and $\beta_0 > \beta_1$. Lemma 5.4 (b) implies that u_{β_0} and u_{β_1} intersect each other more than twice in $[0, R)$. We define the following set:

$$\begin{aligned} B := \{\beta \in (0, \beta_0) : &\text{there exist } r_{\beta,1} > 0 \text{ and } r_{\beta,2} > 0 \text{ such that} \\ &r_{\beta,1} < r_{\beta,2} < R, \\ &u_\beta < u_{\beta_0} \text{ in } [0, r_{\beta,1}),\ u_\beta(r_{\beta,1}) = u_{\beta_0}(r_{\beta,1}), \\ &u_\beta > u_{\beta_0} \text{ in } (r_{\beta,1}, r_{\beta,2}),\ u_\beta(r_{\beta,2}) = u_{\beta_0}(r_{\beta,2})\}. \end{aligned}$$

Since $\beta_1 \in B$, we see that $B \neq \emptyset$. Moreover, it follows from continuous dependence of u_β on β that B is open. Now, we claim that $\inf B = 0$. To the aim, suppose not, $\inf B > 0$. Then we see that $r_{\beta,2} \to R$ as $\beta \downarrow \inf B$, i.e. $u_{\inf B}(R) = 0$. It follows from Lemma 5.4 (b) that $u_{\inf B}$ and u_{β_0} intersect more than twice in $[0,R)$. This is a contradiction and we obtain $\inf B = 0$. Then, we derive $r_{\beta,1}, r_{\beta,2} \to R$ as $\beta \to 0$ and

$$\int_{r_{\beta,1}}^{r_{\beta,2}} (\sinh s)^{N-1+\alpha}\, u_{\beta_0}^{p+1}\, ds \to 0 \quad \text{as } \beta \to 0.$$

This is a contradiction to Lemma 5.5. □

Lemma 5.8. *Let N, α, and p satisfy the assumption in Theorem* 5.1. *Let $\beta_0 > 0$ and assume that $u_{\beta_0} > 0$ in $[0,\infty)$. Then the following hold*:

(a) *Suppose that $\beta_1 \in (0,\beta_0)$. Then $u_{\beta_1} > 0$ in $[0,\infty)$.*
(b) *In addition, assume that $u_{\beta_0} \in H^1(\mathbb{H}^N)$. Then $u_{\beta_0} - u_{\beta_1}$ has exactly one zero in $[0,\infty)$.*

Proof. To begin with, we shall prove the assertion (a). Suppose not, u_{β_1} has a zero in $[0,\infty)$. Defining the set B as $B := \{\beta \in (0,\beta_0) : u_\beta \text{ has a zero in } [0,\infty)\}$, we deduce from $\beta_1 \in B$ and continuous dependence of u_β on β that $B \neq \emptyset$ and B is open. Moreover, let $(\beta_2,\beta_3) \subset B$ be the largest open interval containing β_1 and r_β be the first zero of u_β in $[0,\infty)$. Then it holds that $r_\beta \to \infty$ as $\beta \downarrow \beta_2$ and $r_\beta \to \infty$ as $\beta \uparrow \beta_3$. Since r_β is continuous on β, for sufficiently large $R > 0$, there exist $\beta_R, \widetilde{\beta}_R > 0$ such that $r_{\beta_1} < r_{\beta_R} = r_{\widetilde{\beta}_R} = R$ and $\beta_2 < \beta_R < \beta_1 < \widetilde{\beta}_R < \beta_3$. This is a contradiction to Lemma 5.7. Therefore, the assertion (a) holds.

In the following, we shall show the assertion (b). Now, we claim that u_{β_0} and u_{β_1} intersect in $[0,\infty)$. Suppose not, $u_{\beta_0} > u_{\beta_1}$ in $[0,\infty)$. Then it follows from $u_{\beta_0} \in H^1(\mathbb{H}^N)$ that $(\sinh r)^{(N-1)/2} u_{\beta_1}(r) \to 0$ and $(\sinh r)^{(N-1)/2} u'_{\beta_0}(r) \to 0$ as $r \to \infty$. Since u_{β_0} and u_{β_1} satisfy (5.8), we infer from Lemma 2.1 (a) and (d) that

$$\begin{aligned}\int_0^r (\sinh s)^{N-1+\alpha} u_{\beta_0} u_{\beta_1} \big(u_{\beta_0}^{p-1} - u_{\beta_1}^{p-1}\big)\, ds& \\ = (\sinh r)^{N-1} (u_{\beta_0} u'_{\beta_1} - u'_{\beta_0} u_{\beta_1}) &\le -(\sinh r)^{N-1} u'_{\beta_0} u_{\beta_1} \to 0\end{aligned}$$

as $r \to \infty$. This is a contradiction to $u_{\beta_0} > u_{\beta_1}$ in $[0,\infty)$. Thus u_{β_0} and u_{β_1} intersect in $[0,\infty)$. Then, suppose not, u_{β_0} and u_{β_1} intersect more than twice in $[0,\infty)$. We denote by $\widehat{B}$ the following set:

$$\begin{aligned}\widehat{B} := \{\beta \in (0,\beta_0) : {}&\text{there exist } r_{\beta,1} > 0 \text{ and } r_{\beta,2} > 0 \text{ such that} \\ & r_{\beta,1} < r_{\beta,2}, \\ & u_\beta < u_{\beta_0} \text{ in } [0, r_{\beta,1}),\ u_\beta(r_{\beta,1}) = u_{\beta_0}(r_{\beta,1}), \\ & u_\beta > u_{\beta_0} \text{ in } (r_{\beta,1}, r_{\beta,2}),\ u_\beta(r_{\beta,2}) = u_{\beta_0}(r_{\beta,2})\}.\end{aligned}$$

It follows from $\beta_1 \in \widehat{B}$ and continuous dependence of u_β on β that $\widehat{B} \neq \emptyset$ and $\widehat{B}$ is open. Then we see that $r_{\beta,2} \to \infty$ and

$$\int_{r_{\beta,2}}^{\infty} u_{\beta_0}^{p+1}(s)(\sinh s)^{N-1+\alpha}\, ds \to 0 \quad \text{as } \beta \downarrow \inf \widehat{B}.$$

This is a contradiction to Lemma 5.5 or Lemma 5.6. Hence, we observe that u_{β_0} and u_{β_1} intersect only once in $[0,\infty)$. □

Now, we shall prove Theorem 5.1.

Proof of Theorem 5.1. To begin with, we shall claim that there exists a unique $\beta_H > 0$ such that $u_{\beta_H} > 0$ in $[0,\infty)$ and $u_{\beta_H} \in H^1(\mathbb{H}^N)$. Suppose not, there exist $\beta > 0$ and $\widetilde{\beta} > 0$ such that $\beta > \widetilde{\beta}$, $u_\beta, u_{\widetilde{\beta}} \in H^1(\mathbb{H}^N)$, and $u_\beta, u_{\widetilde{\beta}} > 0$ in $[0,\infty)$. We infer from Lemma 5.8 (b) that $u_\beta - u_{\widetilde{\beta}}$ has exactly one zero in $[0,\infty)$. Then, applying the same method to obtain (5.9), we derive $u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}} > 0$ in $(0,\infty)$. Here, we use the fact that $(\sinh r)^{N-1}(u_\beta u'_{\widetilde{\beta}} - u'_\beta u_{\widetilde{\beta}}) \to 0$ as $r \to \infty$, for $u_\beta, u_{\widetilde{\beta}} \in H^1(\mathbb{H}^N)$. Hence, defining the function γ as $\gamma = u_{\widetilde{\beta}}/u_\beta$, we infer that $\gamma' > 0$ in $(0,\infty)$. We observe from (5.6) that

$$\begin{aligned} E_{\widetilde{\beta}}(\widehat{r}) - \gamma^2(r)E_\beta(\widehat{r}) &= E_{\widetilde{\beta}}(\varepsilon) - \gamma^2(r)E_\beta(\varepsilon) \\ &\quad + \frac{1}{2}\int_\varepsilon^{\widehat{r}} G'(\tau)((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(r)(\widehat{u}_\beta(\tau))^2)\, d\tau. \end{aligned}$$

for $r, \widehat{r}, \varepsilon \in (0,\infty)$. Using Lemma 5.3 (b)–(c), we see that as $\varepsilon \to 0$ and $\widehat{r} \to \infty$,

$$0 = \frac{1}{2}\int_0^\infty G'(\tau)\big((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(r)(\widehat{u}_\beta(\tau))^2\big)\, d\tau. \tag{5.12}$$

Moreover, from Lemma 5.3 (a), there exists a unique $R > 0$ such that $G' > 0$ in $(0,R)$, $G'(R) = 0$, and $G' < 0$ in (R,∞). Therefore, we observe from the monotonicity of γ that

$$G'(\tau)\big((\widehat{u}_{\widetilde{\beta}}(\tau))^2 - \gamma^2(R)(\widehat{u}_\beta(\tau))^2\big) < 0 \quad \text{in } \tau \in (0,\infty).$$

This is a contradiction to (5.12) for the case of $r = R$. Thus, the assertion (b) holds. Furthermore, applying Lemma 5.8 (a), we obtain the assertion (a). □

5.2. Sign of radial solutions. In this subsection, we shall investigate the sign of u_β for $p \in (p_b(N,\alpha), p_s(N,\alpha))$ and $\beta > \beta_H$. Here, $\beta_H > 0$ is defined in Theorem 5.1. Moreover, we assume that $N \geq 3$, $\alpha > 0$, and $p_b(N,\alpha) < p < p_s(N,\alpha)$ throughout in this subsection. Now, we prepare lemmas.

Lemma 5.9. *For sufficiently large $\beta > \beta_H$, there exists $r_\beta > 0$ such that the following hold*:

(a) $u_\beta(r_\beta) = 0$ *and* $u_\beta > 0$ *in* $[0, r_\beta)$;
(b) $r_\beta \to 0$ *as* $\beta \to \infty$;
(c) $u_\beta - u_{\beta_H}$ *has exactly one zero in* $[0, r_\beta]$.

Proof. Set $v_\beta(r) = \beta^{-1} u_\beta(\beta^{-(p-1)/(\alpha+2)} r)$. Making use of the same method as in the proof of Lemma 4.3, we see that for $R > 0$, there exists $\widetilde{v} \in C([0,R])$ such that $v_\beta \to \widetilde{v}$ as $\beta \to \infty$ in $C([0,R])$. Since $R > 0$ is arbitrary, we infer that $\widetilde{v}$ is the solution of

$$\begin{cases} \widetilde{v}'' + \dfrac{N-1}{r}\,\widetilde{v}' + r^\alpha |\,\widetilde{v}|^{p-1}\widetilde{v} = 0 & \text{in } [0,\infty), \\ \widetilde{v}(0) = 1. \end{cases}$$

Then $\widetilde{v}$ has infinitely many zeros in $[0,+\infty)$ (e.g. see Theorem 2 (c) in [20]). Here, we denote by r_β and $\widetilde{r}$ the first zero point of u_β and $\widetilde{v}$ in $[0,+\infty)$, respectively. Thus, since the first zero point of v_β is $\beta^{(p-1)/(\alpha+2)} r_\beta$, we deduce from the convergence of v_β in $C([0,\infty))$ that $\beta^{(p-1)/(\alpha+2)} r_\beta \to \widetilde{r}$ as $\beta \to \infty$. Hence assertions (a) and (b) hold true.

Now, we shall prove assertion (c). Since $u_{\beta_H} > 0$ in $[0,\infty)$, u_β and u_{β_H} intersect at least once in $[0,r_\beta]$. Then, suppose not, u_β and u_{β_H} intersect more than twice in $[0,r_\beta]$. Namely, there exist $r_{\beta,1}, r_{\beta,2} > 0$ such that $r_{\beta,1} < r_{\beta,2} < r_\beta$, $u_\beta > u_{\beta_H}$ in $[0, r_{\beta,1})$, $u_\beta(r_{\beta,1}) = u_{\beta_H}(r_{\beta,1})$, $u_\beta < u_{\beta_H}$ in $(r_{\beta,1}, r_{\beta,2})$, and $u_\beta(r_{\beta,2}) = u_{\beta_H}(r_{\beta,2})$. Applying the same method as in the proof of the assertion (b), we see that $r_{\beta,1} \to 0$ and $r_{\beta,2} \to 0$ as $\beta \to \infty$. Therefore, we have

$$\int_{r_{\beta,1}}^{r_{\beta,2}} (\sinh s)^{N-1+\alpha} u_{\beta_H}^{p+1}(s)\,ds \to 0.$$

This is a contradiction to Lemma 5.5. Thus the assertion (c) holds. □

Lemma 5.10. *There exists $\beta_* \geq \beta_H$ such that the following hold*:

(a) *If $\beta > \beta_*$, then u_β is sign-changing in* $[0,\infty)$;
(b) *If $\beta \leq \beta_*$, then $u_\beta > 0$ in* $[0,\infty)$.

Proof. We define the set B as $B := \{\beta \in (0,\infty) : u_\beta \text{ has a zero in } [0,\infty)\}$. Lemma 5.9 implies that $B \neq \emptyset$. Moreover, we deduce from continuous dependence of u_β on β that B is open. Then, we shall claim that B is connected. Suppose not, there exists the largest connected open interval $(\beta_2, \beta_3) \subset B$, where $\beta_2 < \beta_3$. Setting r_β as the first zero of u_β in $[0,\infty)$, we have $r_\beta \to \infty$ as $\beta \to \beta_2$ and $r_\beta \to \infty$ as $\beta \to \beta_3$. Then, for sufficiently large $R > 0$, there exist $\beta_R, \widehat{\beta}_R > 0$ such that $r_{\beta_R} = r_{\widehat{\beta}_R} = R$ and $\beta_2 < \beta_R < \widehat{\beta}_R < \beta_3$. This is a contradiction to Lemma 5.7. Hence, B is connected. Making use of Theorem 5.1 and Lemma 5.9, we complete the proof. □

Concerning $\beta_* \geq \beta_H$ defined in Lemma 5.10, we obtain the following results:

Lemma 5.11. *Let $r_\beta > 0$ be the first zero of u_β in $[0,\infty)$. If $\beta > \beta_*$, then $u_\beta - u_{\beta_H}$ has exactly one zero in $[0, r_\beta]$.*

Proof. Lemma 5.10 (a) implies that $r_\beta < \infty$ for $\beta > \beta_*$. Since $\beta_* \geq \beta_H$ and $u_{\beta_H} > 0$ in $[0,\infty)$, $u_\beta - u_{\beta_H}$ has at least one zero in $[0, r_\beta]$ for $\beta > \beta_*$. Moreover, it follows from Lemma 5.9 (c) that for sufficiently large $\beta > \beta_H$, $u_\beta - u_{\beta_H}$ has exactly one zero in $[0, r_\beta]$. Then, suppose not, there exists $\beta_1 > \beta_*$ such that $u_{\beta_1} - u_{\beta_H}$ has more than two zeros in $[0, r_\beta)$. Thus, setting

$$\begin{aligned} B = \{\beta \in (\beta_*, \infty) : \ & \text{there exist } r_{\beta,1} > 0 \text{ and } r_{\beta,2} > 0 \text{ such that} \\ & r_{\beta,1} < r_{\beta,2} < r_\beta, \\ & u_\beta > u_{\beta_H} \text{ in } [0, r_{\beta,1}),\ u_\beta(r_{\beta,1}) = u_{\beta_H}(r_{\beta,1}), \\ & u_\beta < u_{\beta_H} \text{ in } (r_{\beta,1}, r_{\beta,2}),\ u_\beta(r_{\beta,2}) = u_{\beta_H}(r_{\beta,2})\}, \end{aligned}$$

we infer that $B \neq \emptyset$ and $\sup B < \infty$. Since $u_{\beta_H} > 0$ in $[0,\infty)$ and $r_{\beta,2} \to \infty$ as $\beta \uparrow \sup B$, we also derive $r_\beta \to \infty$ as $\beta \uparrow \sup B$. This is a contradiction to Lemma 5.10 (a). $\square$

Then we obtain the following result:

Theorem 5.12. *If $\beta > \beta_H$, then u_β is sign-changing in* $[0,\infty)$.

Proof. We shall prove that $\beta_* = \beta_H$ by contradiction. Suppose not, $\beta_* > \beta_H$. Now, let $\beta_1 < \beta_H$ and Lemma 5.8 (b) implies that $u_{\beta_1} - u_{\beta_H}$ has exactly one zero in $[0,\infty)$. Thus, applying the same argument as in Lemma 7.3 of [4], we see that $u_{\beta_*} - u_{\beta_H}$ has at least one zero in $[0,\infty)$. Then we deduce from Lemma 5.11 that $u_{\beta_*} - u_{\beta_H}$ has exactly one zero in $[0,\infty)$. On the other hand, applying Theorem 5.1 and $u_{\beta_H} \in H^1(\mathbb{H}^N)$, we see that the decay of u_{β_*} is slower than that of u_{β_H} as $r \to \infty$. Therefore, there exits $R > 0$ such that $u_{\beta_*} > u_{\beta_H}$ in (R,∞). Since $u_{\beta_*} - u_{\beta_H}$ has exactly one zero in $[0,\infty)$, this is a contradiction. Hence, we derive $\beta_* = \beta_H$. Making use of Lemma 5.10, we complete the proof. $\square$

5.3. Finiteness of number of zeros and asymptotic behavior. In this subsection, we shall complete the proof of Theorem 1.3 (b). To begin with, we investigate the finiteness of number of zeros.

Theorem 5.13. *Let $N \geq 3$, $\alpha > 0$, and $p_b(N,\alpha) < p < p_s(N,\alpha)$. Then u_β has finitely many zeros in* $[0,\infty)$.

Proof. Suppose not, there exists $\beta > 0$ such that u_β has infinitely many zeros in $[0,\infty)$. We denote by r_k the k-th zero of u_β. Then, one of the following two cases must occur: (1) There exists $R > 0$ such that $r_k \to R$ as $k \to \infty$; (2) $r_k \to \infty$ as $k \to \infty$.

Firstly, we consider the case (1). Let $\widehat{r}_k$ be the first local maximum point of $|u_\beta|$ in (r_k, r_{k+1}). It follows from (2.3) and the monotonicity of F that

$$\begin{aligned}|F(R)-F(r_k)| > |F(\widehat{r}_k)-F(r_k)| &= \left|\int_{r_k}^{\widehat{r}_k} F'(s)\,ds\right| > C\int_{r_k}^{\widehat{r}_k}(u_\beta'(s))^2\,ds\\ &= -C\int_{r_k}^{\widehat{r}_k} u_\beta''(s)u_\beta(s)\,ds > C\int_{r_k}^{\widehat{r}_k} u_\beta'(s)u_\beta(s)\,ds = Cu_\beta^2(\widehat{r}_k).\end{aligned}$$

Therefore, $u_\beta(\widehat{r}_k)\to 0$ as $k\to\infty$. Recalling that $u_\beta'(\widehat{r}_k)=0$ for $k\in\mathbb{N}$, we see that this is a contradiction.

Secondly, we consider the case (2). Then (2.8) implies that

$$\int_{r_k}^{r}(\sinh s)^{N-1}h(s)|u_\beta'(s)|^2\,ds = \Psi(r)-\Psi(r_k). \tag{5.13}$$

Here, we observe from the proof of Lemma 2.7 that there exists $\widehat{R}>0$ such that h is bounded, $h<0$, and Ψ is strictly monotone decreasing in $(\widehat{R},\infty)$. Since $\Psi(r_k)\ge 0$ for any $k>0$, there exists $\gamma\ge 0$ such that

$$\lim_{r\to\infty}\Psi(r)=\gamma, \quad 0\le\gamma<\Psi<\infty \quad \text{in } (\widehat{R},\infty). \tag{5.14}$$

Applying (5.13), we see that for sufficiently large $k>0$,

$$0<-\int_{r_k}^{\infty}(\sinh s)^{N-1}h(s)|u_\beta'(s)|^2\,ds = \Psi(r_k)-\gamma\le\Psi(r_k)<\infty.$$

Thus, it holds that $(\sinh r)^{(N-1)/2}u_\beta'(r)\to 0$ as $r\to\infty$. Since $p>p_b(N,\alpha)$ is equivalent to $(N-1)/2>\alpha/(p-1)$, we have $(\sinh r)^{\alpha/(p-1)}u_\beta'(r)\to 0$ as $r\to\infty$. Then, setting $v_\beta(r)=(\sinh r)^{\alpha/(p-1)}u_\beta(r)$ and denoting by m_k the maximum point of $|v_\beta|$ in (r_k, r_{k+1}), we see that $u_\beta'(m_k)/u_\beta(m_k) = -\alpha(p-1)^{-1}(\tanh m_k)^{-1}$ for $v_\beta'(m_k)=0$. Therefore, we infer that

$$\begin{aligned}\frac{\Psi(m_k)}{(\sinh m_k)^{N-1}(u_\beta'(m_k))^2} &= \frac{\varphi(m_k)}{2(\sinh m_k)^{N-1+\alpha}} - \frac{(p-1)\tanh m_k}{(p+1)\alpha}\\ &\quad + \left(\frac{(p-1)\tanh m_k}{\alpha}\right)^{p+1}\frac{\varphi(m_k)}{(\sinh m_k)^{N-1+\alpha}}\frac{(\sinh m_k)^\alpha|u_\beta'(m_k)|^{p-1}}{p+1}\\ &\to \frac{1}{2(N-1+\alpha)} - \frac{p-1}{(p+1)\alpha}\end{aligned}$$

as $k\to\infty$. If $p>p_b(N,\alpha)$, then $(2(N-1+\alpha))^{-1}-(p-1)((p+1)\alpha)^{-1}<0$. Hence, for sufficiently large $k\in\mathbb{N}$, we obtain $\Psi(m_k)<0$. However, this is a contradiction to (5.14). □

In the following, we shall study the asymptotic behavior of $u_\beta\notin H^1(\mathbb{H}^N)$. To this aim, we prepare lemmas.

LEMMA 5.14. *Let $N \geq 3$, $p > 1$, and $\alpha > 0$. If $u_\beta \notin H^1(\mathbb{H}^N)$ and there exists $R > 0$ such that $u_\beta > 0$ in $[R, \infty)$, then there exist no triplets $(C_0, \widehat{R}, \varepsilon)$ with $C_0 > 0$, $\widehat{R} > 0$, and $\varepsilon \in (0, (N-1)/2)$ such that the following estimate holds*:

$$u_\beta(r) \leq C_0 (\sinh r)^{-(N-1-\varepsilon)} \quad \textit{in } [\widehat{R}, \infty).$$

PROOF. Suppose not, there exist a triplet $(C_0, \widehat{R}, \varepsilon)$ with $C_0 > 0$, $\widehat{R} > 0$, and $\varepsilon \in (0, (N-1)/2)$ such that $u_\beta(r) \leq C_0 (\sinh r)^{-(N-1-\varepsilon)}$ in $[\widehat{R}, \infty)$. Then, since

$$\begin{aligned} \int_{\widehat{R}}^{r} (\sinh s)^{N-1} u_\beta^2(s)\, ds &\leq C_0^2 \int_{\widehat{R}}^{r} (\sinh s)^{-(N-1-2\varepsilon)} \frac{ds}{\tanh s} \\ &\leq \frac{C_0^2}{N-1-2\varepsilon} (\sinh \widehat{R})^{-(N-1-2\varepsilon)} \end{aligned} \tag{5.15}$$

in $[\widehat{R}, \infty)$, it holds that $u_\beta \in L^2(\mathbb{H}^N)$. On the other hand, we choose $\delta > 0$ satisfying $N-1-2(\varepsilon+\delta) > 0$. Then we obtain $(\sinh r)^{N-1-(\varepsilon+\delta)} u_\beta(r) \to 0$ as $r \to \infty$. Using l'Hospital's rule, we also have $(\sinh r)^{N-1-(\varepsilon+\delta)} u_\beta'(r) \to 0$ as $r \to \infty$. Thus there exist $C_1 > 0$ and $R_1 > 0$ such that $|u_\beta'(r)| \leq C_1 (\sinh r)^{-(N-1-(\varepsilon+\delta))}$ in $[R_1, \infty)$. Therefore, applying the similar method to derive (5.15), we see that

$$\int_{R_1}^{r} (\sinh s)^{N-1} |u_\beta'(s)|^2\, ds \leq \frac{C_1^2 (\sinh R_1)^{-(N-1-2(\varepsilon+\delta))}}{N-1-2(\varepsilon+\delta)} \tag{5.16}$$

in $r \in (R_1, \infty)$. Hence, it follows from (5.15)–(5.16) that $u_\beta \in H^1(\mathbb{H}^N)$. This is a contradiction. $\square$

LEMMA 5.15. *Let $N \geq 3$, $\alpha > 0$, and $p > p_b(N, \alpha)$. If $u_\beta \notin H^1(\mathbb{H}^N)$ and there exists $R > 0$ such that $u_\beta > 0$ in $[R, \infty)$, then the following does not hold*:

$$u_\beta(r) = o\big((\sinh r)^{-\alpha/(p-1)}\big) \quad \textit{as } r \to +\infty.$$

PROOF. We claim that there exist no triplets (C, r_0, δ) with $C > 0$, $r_0 \geq 0$, and $\delta > 0$ such that $u_\beta(r) \leq C (\sinh r)^{-\alpha/(p-1)-\delta}$ in $[r_0, \infty)$. Applying Lemma 5.14 and the same argument as in Lemma 3.6 of [6], we can verify the claim. Furthermore, making use of the nonexistence of the triplets (C, r_0, δ) and the same method as in the proof of Lemma 3.7 of [6], we complete the proof. $\square$

Then we derive the following result:

THEOREM 5.16. *Let $N \geq 3$, $\alpha > 0$, and $p_b(N, \alpha) < p < p_s(N, \alpha)$. If there exists $R > 0$ such that $|u_\beta| > 0$ in $[R, \infty)$ and $u_\beta \notin H^1(\mathbb{H}^N)$, then the following holds*:

$$\lim_{r \to \infty} (\sinh r)^{\alpha/(p-1)} |u_\beta(r)| = \left\{ \frac{\alpha}{p-1} \left(N - 1 - \frac{\alpha}{p-1} \right) \right\}^{1/(p-1)}. \tag{5.17}$$

PROOF. Applying Lemmas 3.1, 5.15, and the same argument as in the proof of Theorem 3.2 in [6], we complete the proof. $\square$

Then we shall complete the proof of Theorem 1.3 (b).

Proof of Theorem 1.3 (b). Since the positive radial solution of Type R in Theorem 1.5 belongs to $H^1(\mathbb{H}^N)$, Theorem 5.1 implies that this positive solution equivalents to u_{β_H}, where β_H is defined in Theorem 5.1. Then the assertion (ii) holds. Moreover, it follows from Theorem 5.1 that if $\beta < \beta_H$, then $u_\beta \notin H^1(\mathbb{H}^N)$. Thus, using Theorem 5.16, we see that for $\beta < \beta_H$, u_β satisfies (5.17). Hence we obtain the assertion (i), where it holds that $\alpha/(p-1) < N-1$ when $p > p_b(N,\alpha)$. Furthermore, combining Theorem 5.12 with Theorem 5.13, we derive the assertion (iii). $\square$

Acknowledgements. The author would like to express his gratitude to Professor S. Okabe of Tohoku University for sincere advice and warm support.

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Manuscript received March 15, 2018

accepted April 30, 2018

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