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# EXISTENCE, UNIQUENESS AND PROPERTIES OF GLOBAL WEAK SOLUTIONS TO INTERDIFFUSION WITH VEGARD RULE

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Abstract. We consider the diffusional transport in an $r$-component solid solution. The model is expressed by the nonlinear system of strongly coupled parabolic differential equations with initial and nonlinear boundary conditions. The techniques involved are the local mass conservation law for fluxes, which are a sum of the diffusional and Darken drift terms, and the Vegard rule. The considered boundary conditions allow the physical system to be not only closed but also open. The theorems on existence, uniqueness and properties of global weak solutions are proved. The main tool used in the proof of the existence result is the Galerkin approximation method. The agreement between the theoretical results, numerical simulations and experimental data is shown.

## 1. Introduction

Quantitative description of the diffuse mass transport is particularly essential for materials processing and hydrodynamics. It is important for the Navier–Stokes problem, where it allows considering diffusion in multicomponent fluids [15]. An inspiring effort dedicated to the rigorous mathematical treatment

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of the flows occurring in multicomponent systems has begun with the work of Darken [11] on the modeling of diffusive flows. In the case of binary closed mixture with constant concentration, $c_1 + c_2 = \text{const}$, the Darken method allows to transform the system of two partial differential equations modeling the process

$$\partial_t c_i = -\partial_x(-\Theta_i(c_1, c_2)\partial_x c_i + c_i v^D) \quad \text{for } i = 1, 2 \tag{1.1}$$

to one quasi-linear diffusion equation

$$\partial_t c_1 = \partial_x\big(\widetilde{\Theta}(c_1)\partial_x c_1\big) \tag{1.2}$$

with the initial and the simple boundary conditions (semi-infinite only), where $v^D$ means a drift velocity. Equation (1.2) allows using the Boltzmann–Matano transformation [3]. It introduces a similarity parameter $\lambda = (x - x_0)/\sqrt{t}$, where $x_0$ is the position of the so-called Matano interface [18]. This ansatz transforms, in a not equivalent way, the governing partial differential diffusion equation (1.2) to a nonlinear ordinary differential equation. But in a multicomponent case (almost three components, $r \geq 3$) such Darken reduction is not effective, because it leads to a system of equations. Analogous procedure is used in the case of the Onsager phenomenological equations, where the fluxes are coupled by interdiffusion coefficients $\Theta_i(c_1, \ldots, c_{i-1})$, $i = 1, \ldots, r$.

The drift velocity $v^D$ is concerned with the Kirkendall effect [22]. It is the motion of the boundary layer between two metals that occurs as a consequence of the difference in diffusion rates of the metal atoms. The effect can be observed for example by placing insoluble markers at the interface between a pure metal and an alloy containing that metal, and heating to a temperature where atomic diffusion is possible; the boundary will move relative to the markers. The Kirkendall effect has important practical consequences. One of these is the prevention or suppression of voids formed at the boundary interface in various kinds of alloy to metal bonding. These are referred to as Kirkendall voids.

The Darken method was extended for multicomponent systems in [17], [2]. Later it was proved that it is self-consistent with the Onsager phenomenological description [4]. Several attempts to solve the problem in liquid mixtures were not very effective due to arbitrary selection of the reference frame for diffusion. The most fundamental approach is given in [5], where a volume transport is considered.

The model studied in our paper is expressed by the one-dimensional nonlinear system of strongly coupled parabolic differential equations

$$\partial_t \varrho_i = \partial_x\bigg(\Theta_i(\varrho_1, \ldots, \varrho_r)\partial_x \varrho_i - \varrho_i \sum_{j=1}^{r} \frac{\Omega_j \Theta_j(\varrho_1, \ldots, \varrho_r)}{M_j}\, \partial_x \varrho_j - K(t)\varrho_i\bigg) \tag{1.3}$$

for $i = 1, \ldots, r$, $\varrho_i = M_i c_i$, with initial and nonlinear coupled boundary conditions (see Section 2). It is obtained from the local mass conservation law for fluxes which are a sum of the diffusional and Darken drift terms, together with the Vegard rule. This rule is a straight application of the Euler homogeneous function theorem [14]. The strong coupling of the equations is caused by the drift velocity $v^D$. A detailed analysis of a concept of the drift velocity, a choice of the reference frame, as well as other physical, mathematical and numerical consequences of the proposed formalism can be found in [4], [9], [10], [13], [17], [21], [23]–[25] and in references therein. In these papers concentration of a mixture must be constant, while the Vegard rule used by us admits the overall concentration depending on time and space. We do not use the Darken reduction method and the not equivalent Boltzmann–Matano substitution mentioned above. Let us stress that such strongly coupled systems as (1.3) (i.e. by the second derivatives) are still insufficiently explored, they are not studied for example in [6]–[8], [12], [15], [16], [19], [20], [26].

The aim of this paper is to obtain existence, uniqueness, nonnegativity and estimates of global in time weak solutions (in suitable Sobolev spaces) of the nonlinear parabolic problem discussed above. Moreover, we show that if a physical system is closed, then an evolutional solution converges to the stationary one as time goes to infinity. The main tool used in the proof of the existence result is the Galerkin approximation method [26]. The existence, nonnegativity and estimates of solutions are obtained with the use of the properties of some family of automorphisms. We generalize the mathematical results for a similar differential problem given in [10] in the following sense: the weak version has a differential-integral form instead of the integral one, the regularity of the solution is much more stronger, the Galerkin system has a unique solution instead of the maximal only, we use the Aubin–Lions compactness lemma to pass to the limit which is a natural tool in the case of nonlinear problems and evolution triples.

The paper is organized as follows. In Section 2 the initial-boundary differential problem is formulated, and in Section 3 its weak version is given together with the assumptions that will be used in the further parts. Sections 4, 5 and 6 deal with existence, nonnegativity, estimates, uniqueness and asymptotic behavior of global weak solutions of the problem, respectively. In Section 7 examples of physical problems and numerical experiments are given.

## 2. Model of interdiffusion, a strong formulation

Let $\Omega = [-\Lambda, \Lambda] \subset \mathbb{R}$, $T > 0$ and $r \in \mathbb{N} \setminus \{1\}$ be fixed, and denote $\mathbb{R}_+ = (0, \infty)$. The following data are given:

(1) $M_i = \mathrm{const} \in \mathbb{R}_+$, the molecular mass of the $i$th component of the mixture, $i = 1, \dots, r$.

(2) $\Omega_i = \text{const} \in \mathbb{R}_+$, the partial molar volume of the $i$th component of the mixture, $i = 1, \ldots, r$.
(3) $\Theta_i \colon [0, M_1/\Omega_1] \times \ldots \times [0, M_r/\Omega_r] \to \mathbb{R}_+$, the diffusion coefficient of the $i$th component of the mixture, $i = 1, \ldots, r$.
(4) $\varrho_{0i} \colon \Omega \to \mathbb{R}_+$, the initial density of the $i$th component of the mixture, $i = 1, \ldots, r$.
(5) $j_{i,L}, j_{i,R} \colon [0, T] \to \mathbb{R}$, the evolution of a mass flow of the $i$th component of the mixture through the left and right boundaries, respectively $i = 1, \ldots, r$.

The following functions are unknown:

(1) $\varrho_i \colon [0, T] \times \Omega \to \mathbb{R}_+$, the density of the $i$th component of the mixture, $i = 1, \ldots, r$.
(2) $v^D \colon [0, T] \times \Omega \to \mathbb{R}$, the drift velocity.

The total mass of the $i$th component of the mixture at the fixed moment $t \in [0, T]$ is given by

$$m_i(t) = \int_\Omega \varrho_i(t, x)\, dx, \quad i = 1, \ldots, r, \tag{2.1}$$

while by

$$\overline{m}_i(t) = \frac{1}{2\Lambda} \int_\Omega \Omega_i \frac{\varrho_i(t, x)}{M_i}\, dx, \quad i = 1, \ldots, r, \tag{2.2}$$

the average value of the local volume fraction $\Omega_i \varrho_i / M_i$ is denoted. We assume that each component of the mixture is a continuous medium, i.e., it satisfies the local mass conservation low

$$\partial_t \varrho_i + \partial_x J_i = 0, \quad i = 1, \ldots, r, \tag{2.3}$$

where

$$J_i = -\Theta_i(\varrho_1, \ldots, \varrho_r) \partial_x \varrho_i + \varrho_i v^D, \quad i = 1, \ldots, r, \tag{2.4}$$

is a flux of the $i$th component of the mixture, and it is a sum of the diffusional and Darken drift fluxes. Note that (2.4) is a generalization of the Fick flux formula. Moreover, we postulate the Vegard rule

$$\Omega_1 \frac{\varrho_1}{M_1} + \ldots + \Omega_r \frac{\varrho_r}{M_r} = 1, \tag{2.5}$$

where $c_i = \varrho_i / M_i$ is the concentration of the $i$th component of the mixture. Consider the initial condition on the concentrations

$$\varrho_i(0, x) = \varrho_{0i}(x) \quad \text{for } x \in \Omega, \tag{2.6}$$

and the boundary conditions

$$\begin{cases} J_i(t, -\Lambda) = j_{i,L}(t) & \text{for } t \in [0, T],\ i = 1, \ldots, r, \\ J_i(t, \Lambda) = j_{i,R}(t) & \text{for } t \in [0, T],\ i = 1, \ldots, r. \end{cases} \tag{2.7}$$

Integrating (2.3) over the interval $\Omega$, using (2.1), and integrating once again over the interval $(0,t)$ we get

$$m_i(t) = \int_\Omega \varrho_{0i}(x)\,dx + \int_0^t (j_{i,L}(\tau) - j_{i,R}(\tau))\,d\tau \tag{2.8}$$

for $t \in [0,T]$, $i = 1,\dots,r$. Hence $\overline{m}_i$, $i = 1,\dots,r$, defined by (2.2) are known functions also.

Physical lows (2.3), (2.5) with the flux formula (2.4) lead to the nonlinear differentially algebraical system

$$\begin{cases} \partial_t \varrho_i + \partial_x(-\Theta_i(\varrho_1,\dots,\varrho_r)\partial_x \varrho_i + \varrho_i v^D) = 0, \quad i = 1,\dots,r, \\ \Omega_1 \dfrac{\varrho_1}{M_1} + \dots + \Omega_r \dfrac{\varrho_r}{M_r} = 1, \end{cases} \tag{2.9}$$

for $(t,x) \in [0,T] \times \Omega$ with the initial condition (2.6) and the coupled nonlinear boundary conditions

$$\begin{cases} (-\Theta_i(\varrho_1,\dots,\varrho_r)\partial_x \varrho_i + \varrho_i v^D)(t,-\Lambda) = j_{i,L}(t) \\ \qquad\qquad\qquad\qquad \text{for } t \in [0,T],\ i = 1,\dots,r, \\ (-\Theta_i(\varrho_1,\dots,\varrho_r)\partial_x \varrho_i + \varrho_i v^D)(t,\Lambda) = j_{i,R}(t) \\ \qquad\qquad\qquad\qquad \text{for } t \in [0,T],\ i = 1,\dots,r. \end{cases} \tag{2.10}$$

Note that the boundary conditions (2.10) generalize the Robin type ones. It remains to find the initial condition on the drift $v^D$ from the physical formalism (2.3)–(2.7). It follows from (2.3), (2.5) that

$$\partial_x\bigg(\sum_{i=1}^r \frac{\Omega_i J_i}{M_i}\bigg)(t,x) = 0 \quad \text{for } (t,x) \in [0,T] \times \Omega, \tag{2.11}$$

and in consequence

$$\sum_{i=1}^r \frac{\Omega_i J_i}{M_i}(t,x) = K(t) \quad \text{for} \quad (t,x) \in [0,T] \times \Omega, \tag{2.12}$$

where $K$ is an arbitrary function. By (2.7), we get the unique

$$K(t) = \sum_{i=1}^r \frac{\Omega_i j_{i,L}(t)}{M_i} = \sum_{i=1}^r \frac{\Omega_i j_{i,R}(t)}{M_i} \tag{2.13}$$

for $t \in [0,T]$. The second equality in (2.13) can be treated also as an assumption on the boundary evolutions $j_{i,L}$ and $j_{i,R}$. On the other hand, (2.4), (2.5) imply

$$\sum_{i=1}^r \frac{\Omega_i J_i}{M_i}(t,x) = -\sum_{i=1}^r \bigg(\frac{\Omega_i \Theta_i(\varrho_1,\dots,\varrho_r)}{M_i}\,\partial_x \varrho_i\bigg)(t,x) + v^D(t,x) \tag{2.14}$$

for $(t,x) \in [0,T] \times \Omega$. Formulas (2.12)–(2.14), for $(t,x) \in [0,T] \times \Omega$, give

$$v^D(t,x) = K(t) + \sum_{i=1}^r \bigg(\frac{\Omega_i \Theta_i(\varrho_1,\dots,\varrho_r)}{M_i}\,\partial_x \varrho_i\bigg)(t,x). \tag{2.15}$$

The solution to the problem of interdiffusion in the $r$-component solid solution are the functions $\varrho_i$, $i = 1, \ldots, r$, and $v^D$ which fulfill the differentially algebraical system (2.9), the initial condition (2.6) together with the initial condition

$$v^D(0,x) = K(0) + \sum_{i=1}^{r} \left( \frac{\Omega_i \Theta_i(\varrho_{01}, \ldots, \varrho_{0r})}{M_i} \, \partial_x \varrho_{0i} \right)(x) \tag{2.16}$$

and the boundary conditions (2.10). It is clear that this problem is equivalent to the nonlinear strongly coupled (i.e. by the second derivatives) differential system

$$\partial_t \varrho_i + \partial_x \biggl( - \Theta_i(\varrho_1, \ldots, \varrho_r) \partial_x \varrho_i + \varrho_i \sum_{j=1}^{r} \frac{\Omega_j \Theta_j(\varrho_1, \ldots, \varrho_r)}{M_j} \, \partial_x \varrho_j \biggr) + K(t) \partial_x \varrho_i = 0, \tag{2.17}$$

$i = 1, \ldots, r$, for $(t,x) \in [0,T] \times \Omega$ with the initial condition (2.6) and the boundary conditions

$$\begin{cases} \biggl( - \Theta_i(\varrho_1, \ldots, \varrho_r) \partial_x \varrho_i + \varrho_i \biggl( K(t) + \displaystyle\sum_{j=1}^{r} \frac{\Omega_j \Theta_j(\varrho_1, \ldots, \varrho_r)}{M_j} \, \partial_x \varrho_j \biggr) \biggr)(t, -\Lambda) = j_{i,L}(t) & \text{for } t \in [0,T], \\ \biggl( - \Theta_i(\varrho_1, \ldots, \varrho_r) \partial_x \varrho_i + \varrho_i \biggl( K(t) + \displaystyle\sum_{j=1}^{r} \frac{\Omega_j \Theta_j(\varrho_1, \ldots, \varrho_r)}{M_j} \, \partial_x \varrho_j \biggr) \biggr)(t, \Lambda) = j_{i,R}(t) & \text{for } t \in [0,T], \end{cases} \tag{2.18}$$

$i = 1, \ldots, r$, in which the unknowns are $\varrho_i$. We see that a diffusive matrix $D = [d_{ij}]_{i,j=1}^{r}$ has the form

$$d_{ii} = \Theta_i(\varrho_1, \ldots, \varrho_r) - \varrho_i \, \frac{\Omega_i \Theta_i(\varrho_1, \ldots, \varrho_r)}{M_i}, \qquad d_{ij} = -\varrho_i \, \frac{\Omega_j \Theta_j(\varrho_1, \ldots, \varrho_r)}{M_j}$$

for $i \neq j$. This matrix is not symmetric but it can be transformed to a symmetric $(r-1) \times (r-1)$ matrix by a change of the reference frame with respect to the $r$th component [4].

Remark 2.1. It is assumed in [9], [10] that the concentration of the mixture $c = c_1 + \ldots + c_r$, $c_i = \varrho_i / M_i$, $i = 1, \ldots, r$, is constant. The Vegard rule (2.5) generalizes this condition, it is more physical (see [14]) and now the concentration $c$ can be a function of $t$ and $x$.

## 3. Assumptions and a weak formulation

We introduce new variables as follows. The local deviation of volume fraction from its average value is

$$w_i(t,x) = \frac{\Omega_i \varrho_i(t,x)}{M_i} - \overline{m}_i(t), \quad i = 1,\dots,r, \tag{3.1}$$

for $(t,x) \in [0,T]\times\Omega$ (see (2.2)). Denote $w = (w_1,\dots,w_r)$ and $\overline{m} = (\overline{m}_1,\dots,\overline{m}_r)$. Note that

$$\sum_{i=1}^{r} \overline{m}_i(t) = 1, \qquad \sum_{i=1}^{r} w_i(t,x) = 0, \qquad \int_\Omega w_i(t,x)\,dx = 0 \tag{3.2}$$

for $(t,x) \in [0,T]\times\Omega$, $i = 1,\dots,r$.

Let

$$1^\perp = \{\xi = (\xi_1,\dots,\xi_r) \in \mathbb{R}^r : \xi_1 + \ldots + \xi_r = 0\}, \tag{3.3}$$

stands for the vector space orthogonal to the vector subspace $\{\alpha 1 : \alpha \in \mathbb{R}\}$, where $1 = (1,\dots,1) \in \mathbb{R}^r$. Define the Sobolev spaces

$$H = \left\{ f = (f_1,\dots,f_r) \in L^2(\Omega, 1^\perp) : \int_\Omega f_i(x)dx = 0,\ i = 1,\dots,r \right\}, \tag{3.4}$$

$$V = \{ f \in H^1(\Omega, 1^\perp) : f \in H\}. \tag{3.5}$$

The norm in $V$ is generated by the scalar product

$$(f,g)_V = \int_\Omega \partial_x f \cdot \partial_x g\,dx \tag{3.6}$$

for $f,g \in V$, while in $H$ by the scalar product

$$(f,g)_H = \int_\Omega f \cdot g\,dx \tag{3.7}$$

for $f,g \in H$. Then $V \subset H \subset V^*$ constitute an evolutional triple with the embeddings being dense, continuous and compact [26], [1].

Let

$$\mathcal{K} = \{\kappa = (\kappa_1,\dots,\kappa_r) \in \mathbb{R}^r : \kappa_1 + \ldots + \kappa_r = 1,\ k_i \geq 0,\ i = 1,\dots,r\}, \tag{3.8}$$

$\overline{\Theta}(w + \overline{m}) := \Theta(\varrho)$. Define the family of linear operators

$$A_\kappa \colon 1^\perp \mapsto 1^\perp, \qquad A_\kappa \xi = \sum_{i=1}^{r} \overline{\Theta}_i(\kappa)\xi_i e_i - (\overline{\Theta}(\kappa)\cdot\xi)\kappa \tag{3.9}$$

for $\kappa \in \mathcal{K}$, where $\xi \in 1^\perp$, $e_i = (0,\dots,0,1,0,\dots,0)$ with 1 in the $i$th entry, $i = 1,\dots,r$.

Remark 3.1. It follows from (3.1), (3.2) that $w(t,x) + \overline{m}(t) \in \mathcal{K}$ if and only if (2.5) holds and $\varrho(t,x) \geq 0$. Moreover, if (2.5) holds and $\varrho(t,x) \geq 0$, then $\varrho_i(t,x) \in [0, M_i/\Omega_i]$, $i = 1,\dots,r$.

Remark 3.2. If we assume that $w(t,x)+\overline{m}(t) \in \mathcal{K}$, then the linear operator $A_{w(t,x)+\overline{m}(t)}$ can be written in the matrix form

$$A_{w(t,x)+\overline{m}(t)}\xi = \overline{D}(w(t,x)+\overline{m}(t))\xi, \quad \xi \in 1^{\perp},$$

where $\overline{D}$ is the diffusive matrix in the variables (3.1) (compare with $D$ in Section 2).

We assume the following conditions.

Assumptions 3.3.

(H$_0$) $\varrho_0(x) = (\varrho_{01}(x),\ldots,\varrho_{0r}(x)) \geq 0$ and

$$\sum_{i=1}^{r} \frac{\Omega_i \varrho_{0i}(x)}{M_i} = 1 \quad \text{for } x \in \Omega,\ i=1,\ldots,r.$$

(H$_1$) $\displaystyle\int_\Omega \varrho_{0i}(x)\,dx + \int_0^t (j_{i,L}(\tau) - j_{i,R}(\tau))\,d\tau \geq 0$ for $t \in [0,T]$, $i=1,\ldots,r$.

(H$_2$) $\displaystyle\sum_{i=1}^{r} \frac{\Omega_i j_{i,L}(t)}{M_i} = \sum_{i=1}^{r} \frac{\Omega_i j_{i,R}(t)}{M_i}$ for $t \in [0,T]$.

(H$_3$) $\varrho_0 \in L^2(\Omega)$.

(H$_4$) $j_{i,L}, j_{i,R} \in L^\infty(0,T)$, $i=1,\ldots,r$.

(H$_5$) $\Theta_i$, $i=1,\ldots,r$, fulfill the Lipschitz condition in $[0,M_1/\Omega_1] \times \ldots \times [0,M_r/\Omega_r]$.

(H$_6$) The following *generalized parabolicity condition* holds:

$$\int_\Omega (A_g \partial_x f)\cdot \partial_x f\,dx \geq \mu\|f\|_V^2 - \nu\|f\|_H^2 \tag{3.10}$$

for some $\mu > 0$, $\nu \geq 0$ and for all $f \in V$, $g=(g_1,\ldots,g_r) \in H^1(\Omega,\mathbb{R}^r)$, $g_1+\ldots+g_r = 1$, $g_i \geq 0$, $i=1,\ldots,r$.

Remark 3.4. Note that (H$_0$) implies that $\varrho_{0i}(x) \in [0, M_i/\Omega_i]$, $i=1,\ldots,r$, and $w_0(x)+\overline{m}(0) \in \mathcal{K}$, where $w_0 \in H$ means $\varrho_0$ in the variables (3.1).

Let us observe that the generalized parabolicity condition (3.10) for all $f \in H^1(\Omega,\mathbb{R}^r)$ is not usually fulfilled, because there is $g=(g_1,\ldots,g_r)$, e.g. $g_1 = 1$, $g_2 = \ldots = g_r = 0$, such that a suitable $r\times r$ matrix has one zero on the diagonal. We give a criterion on (3.10) to be true for $f \in V \subset H^1(\Omega,\mathbb{R}^r)$, where $V$ is our natural space defined by (3.5). It was formulated for less general problems in [17], [9], [10]. This criterion works if $\Theta_i$, $i=1,\ldots,r$ are not too dispersed. It is true in most physical examples (see Section 7).

Lemma 3.5 (criterion of parabolicity). *Let $\Theta_i$, $i=1,\ldots,r$, be continuous and let $\delta = \min_{i=1,\ldots,r} \overline{\Theta}_i$,*

$$m = 0 \quad \textit{if} \quad \overline{\Theta}_1 = \ldots = \overline{\Theta}_r \quad \textit{or} \quad r = 2,$$

*otherwise*

$$m = \min\Big\{x < 0 : \ \exists j \in \{1,\dots,r\}\ 4r + \sum_{k=1,\, k\neq j}^{r} \frac{\alpha_j + (r-1)\alpha_k - 1}{\alpha_k - x} = 0\Big\},$$

*where* $\alpha_i = (\overline{\Theta}_i - \delta)\Big/\Big(\sum_{k=1}^{r}(\overline{\Theta}_k - \delta)\Big)$, $i = 1,\dots,r$. *If* $\delta + m\sum_{k=1}^{r}(\overline{\Theta}_k - \delta) > 0$, *then the generalized parabolicity condition* (3.10) *is fulfilled with* $\mu = \delta + m\sum_{k=1}^{r}(\overline{\Theta}_k - \delta)$ *and* $\nu = 0$. *Moreover,*

$$m \geq -\Big(\sqrt{\frac{r-1}{2r}} - \frac{1}{2}\Big), \tag{3.11}$$

*and if* $r = 3$, *then*

$$m = -\Big(\frac{1}{\sqrt{3}} - \frac{1}{2}\Big)\max\{\alpha_1, \alpha_2, \alpha_3\}. \tag{3.12}$$

To prove the above lemma it is sufficient to show that $m$ is a conditional minimum of a suitable polynomial of $2r$ variables.

Denote the functions

$$\begin{aligned}
\Gamma_{i,L}(t) &= K(t)\,\overline{m}_i(t) - \frac{\Omega_i j_{i,L}}{M_i}, \quad i = 1,\dots,r,\\
\Gamma_{i,R}(t) &= K(t)\,\overline{m}_i(t) - \frac{\Omega_i j_{i,R}}{M_i}, \quad i = 1,\dots,r,\\
\Gamma_L &= (\Gamma_{1,L},\dots,\Gamma_{r,L}),\\
\Gamma_R &= (\Gamma_{1,R},\dots,\Gamma_{r,R}),\\
\Gamma &= (\Gamma_L, \Gamma_R),
\end{aligned}$$

for $t \in [0,T]$. For any $w = (w_1,\dots,w_r) \in L^2(0,T;V)$ the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{V^*\times V}$ means a linear continuous functional of the form

$$\langle w'(t), v\rangle_{V^*\times V} = \sum_{i=1}^{r}\langle w_i'(t), v_i\rangle, \tag{3.13}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is a linear continuous functional acting on $L^2(0,T;H^1(\Omega,\mathbb{R}))$, $t \in (0,T)$, $v = (v_1,\dots,v_r) \in V$. Analogously, $\langle\,\cdot\,,\,\cdot\,\rangle_{V_n^*\times V_n}$ is defined.

The original initial-boundary value problem (2.17), (2.6), (2.18) has the following weak version.

PROBLEM 3.6. Find $w \in L^2(0,T;V)$ such that $w' \in L^2(0,T;V^*)$ and for almost every $t \in (0,T)$

$$\begin{aligned}\langle w'(t), v\rangle_{V^*\times V} + \int_\Omega (A_{w(t)+\overline{m}(t)}\partial_x w(t))\cdot\partial_x v\,dx - K(t)\int_\Omega w(t)\cdot\partial_x v\,dx&\\ = \Gamma_R(t)\cdot v(\Lambda) - \Gamma_L(t)\cdot v(-\Lambda)&\end{aligned} \tag{3.14}$$

for each $v \in V$, and the initial condition holds

$$w(0) = w_0. \tag{3.15}$$

Let $\{v_1, v_2, \ldots\}$ be an orthogonal basis in $V$, and let $(w_{n0})$ be a sequence with $w_{n0} \in V_n := \operatorname{span}\{v_1, \ldots, v_n\}$ for all $n \in \mathbb{N}$. In order to formulate the Galerkin method, we assume that $w_{n0} \to w_0$ in $H$. In practice, $\{v_1, v_2, \ldots\}$ is additionally an orthonormal basis in $H$, and then $w_{n0}$ is usually the orthogonal projection of $w_0$ onto $V_n$ in $H$ and $w_{n0} = \sum_{i=1}^{n} (v_i, w_0)_H v_i$. The basis can be constructed for example with the use of the eigenfunctions of the operator $-\triangle$ in $V$ (see Section 7).

Problem 3.7. Find $w_n \in L^2(0, T; V_n)$ absolutely continuous on $[0, T)$ of the form $w_n(t) = \sum_{k=1}^{n} d_{nk}(t) v_k$ for $t \in [0, T)$ such that $w_n' \in L^2(0, T; V^*)$ and for almost every $t \in (0, T)$,

$$\begin{aligned} \langle w_n'(t), v_j \rangle_{V_n^* \times V_n} &+ \int_\Omega (A_{w_n(t)+\overline{m}(t)} \partial_x w_n(t)) \cdot \partial_x v_j \, dx \\ &- K(t) \int_\Omega w_n(t) \cdot \partial_x v_j \, dx = \Gamma_R(t) \cdot v_j(\Lambda) - \Gamma_L(t) \cdot v_j(-\Lambda), \end{aligned} \tag{3.16}$$

for $j = 1, \ldots, n$, and the initial condition holds

$$w_n(0) = w_{n0}. \tag{3.17}$$

## 4. Existence, nonnegativity and estimates of weak solutions

In this section a weak solution to Problem 3.6 will be constructed using the Galerkin approximation in $V$ (see Problem 3.7).

Remark 4.1. It follows immediately from the definition (3.9) that each operator $A_\kappa$, $\kappa \in \mathcal{K}$, is linear, continuous and

$$\|A_\kappa\|_{L(1^\perp)} \le 2\sqrt{r} \|\overline{\Theta}(\kappa)\|.$$

Define the mapping

$$A \colon \mathcal{K} \ni \kappa \mapsto A_\kappa \in L(1^\perp). \tag{4.1}$$

Lemma 4.2. *If $\Theta_i$, $i = 1, \ldots, r$, satisfy the Lipschitz condition, then*

(a) *$A$ is bounded and*

$$\|A(\kappa)\|_{L(1^\perp)} \le A_\infty := 2\sqrt{r} \sup_{\widetilde{\kappa} \in \mathcal{K}} \|\overline{\Theta}(\widetilde{\kappa})\| \quad \textit{for } \kappa \in \mathcal{K}, \tag{4.2}$$

(b) *$A$ satisfies the Lipschitz condition*

$$\|A(\kappa_1) - A(\kappa_2)\|_{L(1^\perp)} \le L_A \|\kappa_1 - \kappa_2\| \quad \textit{for } \kappa_1, \kappa_2 \in \mathcal{K}. \tag{4.3}$$

Proof. The boundness of $A$ is implied by continuity of $\Theta_i$, compactness of $\mathcal{K}$ and Remark 4.1. After simple computations we obtain for any fixed $\kappa_1, \kappa_2 \in \mathcal{K}$ and each $\xi \in 1^\perp$

$$\|(A_{\kappa_1} - A_{\kappa_2})\xi\| \le (L_A \|\kappa_1 - \kappa_2\|)\|\xi\|,$$

where $L_A = 2\sum_{i=1}^{r} L_i + \sqrt{r}M$, $M = \max_{\kappa\in\mathcal{K}} \|\overline{\Theta}(\kappa)\|$ and $L_i$ is the Lipschitz constant for $\Theta_i$. Hence we get (4.3). □

Lemma 4.3. *Let* $\widetilde{A}\colon (f,g) \mapsto A_g \partial_x f \in L^2(\Omega, 1^\perp)$ *for* $(f,g) \in V \times \{g = (g_1, \ldots, g_r) \in H^1(\Omega, \mathbb{R}^r)\colon g_1 + \ldots + g_r = 1,\ g_i \ge 0,\ i = 1, \ldots, r\}$. *Then*

(a) $\|A_g \partial_x f\|_{L^2(\Omega,\mathbb{R}^r)} \le A_\infty \|\partial_x f\|_{L^2(\Omega,\mathbb{R}^r)}$ *for* $(f,g)$ *in the domain of* $\widetilde{A}$,

(b) $\widetilde{A}$ *satisfies the Lipschitz condition on each bounded set.*

Proof. To argument (a) it is enough to observe that Remark 4.1 and Lemma 4.2 imply that for each $\kappa \in \mathcal{K}$ we have

$$\|A_\kappa \xi\| \le A_\infty \|\xi\| \quad \text{for } \xi \in 1^\perp.$$

In consequence

$$\int_\Omega \|A_g \partial_x f\|^2 \, dx \le A_\infty^2 \int_\Omega \|\partial_x f\|^2 \, dx$$

for $(f,g)$ belonging to the domain of $\widetilde{A}$, and (a) is true.

To prove (b) we note firstly that for any $\kappa_1, \kappa_2 \in \mathcal{K}$

$$\|(A_{\kappa_1} - A_{\kappa_2})\xi\| \le L_A \|\kappa_1 - \kappa_2\| \|\xi\| \quad \text{for } \xi \in 1^\perp,$$

by Lemma 4.2. Hence

$$\|(A_{g_1} - A_{g_2})\partial_x f\|_{L^2(\Omega,\mathbb{R}^r)} \le L_A \|g_1 - g_2\|_{L^\infty(\Omega)} \|\partial_x f\|_{L^2(\Omega,\mathbb{R}^r)}$$

for $(f,g)$ belonging to the domain of $\widetilde{A}$. Using this inequality, (a) and the continuous imbedding $H^1(\Omega, \mathbb{R}^r) \subset C(\Omega, \mathbb{R}^r)$ we obtain

$$\begin{aligned}
&\|Ag_1 \partial_x f_1 - Ag_2 \partial_x f_2\|_{L^2(\Omega,\mathbb{R}^r)} \\
&\qquad \le \|Ag_1 \partial_x (f_1 - f_2)\|_{L^2(\Omega,\mathbb{R}^r)} + \|(A_{g_1} - A_{g_2})\partial_x f_2\|_{L^2(\Omega,\mathbb{R}^r)} \\
&\qquad \le A_\infty \|\partial_x (f_1 - f_2)\|_{L^2(\Omega,\mathbb{R}^r)} + L_A \|g_1 - g_2\|_{L^\infty(\Omega,\mathbb{R}^r)} \|\partial_x f_2\|_{L^2(\Omega,\mathbb{R}^r)} \\
&\qquad \le \max\big\{A_\infty, L_A \sqrt{2\Lambda/3}\|f_2\|_V\big\}\big(\|f_1 - f_2\|_V + \|g_1 - g_2\|_{H^1(\Omega,\mathbb{R}^r)}\big)
\end{aligned}$$

for any $(f_1, g_1)$, $(f_2, g_2)$ from the domain of $\widetilde{A}$, and the proof is complete. □

Let $Y$ be a Banach space and let $F\colon [0,T) \times Y \to Y$, $y_0 \in Y$ be given. Consider the initial problem

$$\begin{cases} y'(t) = F(t, y(t)) & \text{for a.e. } t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{4.4}$$

We formulate a lemma on existence and uniqueness of absolutely continuous solutions to (4.4). They are called *solutions in the extended sense*, see [8, p. 42].

Lemma 4.4. *Suppose that*

(a) *$F$ is strongly measurable* (*i.e. in the Bochner sense*) *in $t$ for all $y$,*
(b) *$F$ fulfills the local Lipschitz condition in $y$ for almost every $t \in (0,T)$,*
(c) *there exists $\alpha \in L^1_{\mathrm{loc}}(0,T)$ such that for almost every $t \in (0,T)$ and each $y \in Y$,*

$$\|F(t,y)\|_Y \le \alpha(t)(1+\|y\|_Y).$$

*Then there exists a unique global absolutely continuous solution to* (4.4), *i.e. on* $[0,T)$.

Proof. Note that absolute continuous solutions to (4.4) are strong ones (the Carathéodory solutions) by assumption (c). Hence they have an integral representation

$$y(t) = y_0 + \int_0^t F(s,y(s))\,ds \quad \text{for } t \in [0,T). \tag{4.5}$$

It follows from the classical Picard theorem and the Kuratowski–Zorn lemma that there exists an absolutely continuous maximal solution to (4.4) on $[0,T)$. Uniqueness is implied by the Gronwall lemma. □

Let $Y \subset V$ be a subspace of $H$ which has a finite dimension. Obviously $Y$ is a Hilbert space. For any $f \in L^2(0,T;\mathbb{R}^r)$ we define a linear and continuous functional $Bf \in Y^*$ by the formula

$$\langle Bf, v\rangle_{Y^*\times Y} = (f, \partial_x v)_{L^2(\Omega,\mathbb{R}^r)} \quad \text{for } v \in Y. \tag{4.6}$$

It follows from the Riesz–Fréchet theorem that for any $f \in L^2(0,T;\mathbb{R}^r)$ there exists the only vector $b(f) \in Y$ such that

$$\langle Bf, v\rangle_{Y^*\times Y} = (b(f), v)_{L^2(0,T;\mathbb{R}^r)} \quad \text{for } v \in Y. \tag{4.7}$$

Next, for any $\gamma = (\gamma_L, \gamma_R) \in \mathbb{R}^r \times \mathbb{R}^r$ we define a linear and continuous functional $G\gamma \in Y^*$ by the formula

$$\langle G\gamma, v\rangle_{Y^*\times Y} = \gamma_R \cdot v(\Lambda) - \gamma_L \cdot v(-\Lambda) \quad \text{for } v \in Y. \tag{4.8}$$

It follows from the Riesz–Fréchet theorem that for any $\gamma = (\gamma_L, \gamma_R) \in \mathbb{R}^r \times \mathbb{R}^r$ there exists the only vector $g(\gamma) \in Y$ such that

$$\langle G\gamma, v\rangle_{Y^*\times Y} = (g(\gamma), v)_{L^2(0,T;\mathbb{R}^r)} \quad \text{for } v \in Y. \tag{4.9}$$

The continuity of the functionals $Bf$ and $G\gamma$ follows from the equivalence of the norms $\|\cdot\|_V$, $\|\cdot\|_H$ in the finite dimensional space $Y$, the Schwartz inequality and the continuous imbedding $Y \subset C(\Omega,\mathbb{R}^r)$.

Define also the linear operators $\widetilde{B}\colon L^2(\Omega,\mathbb{R}^r) \to Y$ and $\widetilde{G}\colon \mathbb{R}^r \times \mathbb{R}^r \to Y$ which assign $b(f) \in Y$ to each $f \in L^2(\Omega,\mathbb{R}^r)$ and $g(\gamma) \in Y$ to each $\gamma = (\gamma_L, \gamma_R) \in$

$\mathbb{R}^r \times \mathbb{R}^r$, respectively. These operators are continuous due to the inequalities

$$\|\widetilde{B}f\|_Y = \|b(f)\|_Y = \|Bf\|_{Y^*} \le C_1 \|f\|_{L^2(\Omega,\mathbb{R}^r)},$$
$$\|\widetilde{G}\gamma\|_Y = \|g(\gamma)\|_Y = \|G\gamma\|_{Y^*} \le C_2(\|\gamma_L\| + \|\gamma_R\|),$$

for each $f \in L^2(\Omega, \mathbb{R}^r)$ and $\gamma = (\gamma_L, \gamma_R) \in \mathbb{R}^r \times \mathbb{R}^r$, where $C_1$, $C_2$ are some constants.

LEMMA 4.5. *Let $Y \subset V$ has finite dimension. If Assumption* 3.3 *is satisfied, then the variational Cauchy problem*

$$\tag{4.10} \begin{cases} \langle y'(t), v\rangle_{Y^*\times Y} + \displaystyle\int_\Omega (A_{y(t)+\overline{m}(t)}\partial_x y(t)) \cdot \partial_x v\,dx \\ \qquad\qquad -K(t)\displaystyle\int_\Omega y(t)\cdot \partial_x v\,dx = \Gamma_R(t)\cdot v(\Lambda) - \Gamma_L(t)\cdot v(-\Lambda) \\ \qquad\qquad \textit{for each } v \in Y \textit{ and for a.e. } t \in (0,T), \\ y(0) = y_0, \end{cases}$$

*and the Cauchy problem*

$$\tag{4.11} \begin{cases} y'(t) = -\widetilde{B}(A_{y(t)+\overline{m}(t)}\partial_x y(t)) + K(t)\widetilde{B}(y(t)) + \widetilde{G}(\Gamma(t)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for a.e. } t \in (0,T), \\ y(0) = y_0, \end{cases}$$

*where $y_0 \in Y$ is arbitrary, are equivalent in the class of absolutely continuous functions $y\colon [0,T) \to Y$. Moreover,*

(a) *the problem* (4.11) *has a unique global absolutely continuous solution $y\colon [0,T) \to Y$,*
(b) *for each $t \in [0,T)$,*

$$\tag{4.12} \|y(t)\|_H^2 \le C,$$
$$\tag{4.13} \|y\|_{L^2(0,T;V)}^2 \le C_1,$$
$$\tag{4.14} \|y'\|_{L^2(0,T;V^*)}^2 \le C_2,$$

*where*

$$C = \bigg(\|y_0\|_H^2 + \frac{4\Lambda}{3\mu}\big(\|\Gamma_R\|_{L^2(0,T)}^2 + \|\Gamma_L\|_{L^2(0,T)}^2\big)\bigg)\exp\bigg(2\nu T + \frac{1}{\mu}\,\|K\|_{L^2(0,T)}^2\bigg),$$
$$C_1 = \frac{1}{\mu}\bigg(\|y_0\|_H^2 + \frac{8\Lambda}{3\mu}\big(\|\Gamma_R\|_{L^2(0,T)}^2 + \|\Gamma_L\|_{L^2(0,T)}^2\big) + 2C\bigg(\nu T + \frac{1}{\mu}\,\|K\|_{L^2(0,T)}^2\bigg)\bigg),$$
$$C_2 = 2C_1(A_\infty + \|K\|_{L^\infty(0,T)})^2 + \frac{8\Lambda}{3}\big(\|\Gamma_R\|_{L^2(0,T)}^2 + \|\Gamma_L\|_{L^2(0,T)}^2\big),$$

(c) *$y \in C([0,T], H)$.*

PROOF. Firstly we will show the equivalence of (4.10) and (4.11). Note that

$$\langle y'(t), v\rangle_{Y^*\times Y} = (y'(t), v)_H \quad \text{for } v \in Y,$$

because $Y$ has finite dimension. Let $y\colon [0,T) \to Y$ be an absolutely continuous solution to (4.10). Then $y$ solves the variational equation

$$\big(y'(t) + \widetilde{B}(A_{y(t)+\overline{m}(t)}\partial_x y(t)) - K(t)\widetilde{B}(y(t)) - \widetilde{G}(\Gamma(t)), v\big)_H = 0$$

for each $v \in Y$ and almost every $t \in (0,T)$, and hence it solves (4.11). On the other hand, if $y\colon [0,T) \to Y$ is an absolutely continuous solution to (4.11), then

$$\begin{aligned}\big(-\widetilde{B}(A_{y(t)+\overline{m}(t)}\partial_x y(t)) &+ K(t)\widetilde{B}(y(t)) + \widetilde{G}(\Gamma(t)), v\big)_H \\ &= -\int_\Omega (A_{y(t)+\overline{m}(t)}\partial_x y(t)) \cdot \partial_x v\, dx \\ &\quad + K(t)\int_\Omega y(t)\cdot \partial_x v\, dx + \Gamma_R(t)\cdot v(\Lambda) - \Gamma_L(t)\cdot v(-\Lambda)\end{aligned}$$

for each $v \in Y$ and for almost every $t \in (0,T)$. This implies that $y$ solves (4.10).

To prove (a) we will apply Lemma 4.4. Define $F\colon [0,T)\times Y \to Y$ by the formula

$$F(t,y) = -\widetilde{B}(A_{y+\overline{m}(t)}\partial_x y) + K(t)\widetilde{B}(y) + \widetilde{G}(\Gamma(t))$$

for $(t,y) \in [0,T)\times Y$. It is clear that $F$ is strongly measurable in $t$. The following estimate holds:

$$\begin{aligned}\|F(t,y_1) - F(t,y_2)\|_V \le \big(2\max\big\{A_\infty, L_A\sqrt{2\Lambda/3}\|y_2\|_V\big\} + |K(t)|\big)& \\ \times \|\widetilde{B}\|_{L(L^2(\Omega,\mathbb{R}^r),Y)}\|y_1 - y_2\|_V&\end{aligned}$$

for any $(t,y_1),(t,y_2) \in [0,T)\times Y$. Therefore $F$ is locally a Lipschitz function in $y$. The grow condition is also satisfied

$$\|F(t,y)\|_V \le (A_\infty + |K(t)|)\|\widetilde{B}\|_{L(L^2(\Omega,\mathbb{R}^r),Y)}\|y\|_V + \|\widetilde{G}\|_{L(\mathbb{R}^r\times\mathbb{R}^r,Y)}\|\Gamma(t)\|$$

for all $(t,y) \in [0,T)\times Y$. It follows from Lemma 4.4 that the Cauchy problem (4.11) has a unique global absolutely continuous solution.

The a priori estimates (4.12) and (4.13) can be proved with the use of the Gronwall lemma, analogously as in [10]. We will show (4.14). Let $Y = \operatorname{span}\{v_1,\ldots,v_n\}$. Fix any $v \in V$. Since $H$ is a Hilbert space, we can write $v = v^1 + v^2$, where $v^1 \in Y$ and $(v^2, v_k)_H = 0$, $k = 1,\ldots,n$. Note that $\|v^1\|_V \le \|v\|_V$, because the basis $\{v_1, v_2, \ldots\}$ is orthogonal in $V$. Moreover, from (2.13) and under $(\mathrm{H}_4)$, $K \in L^\infty(0,T)$. We will use the continuous embedding $Y \subset C(\Omega,\mathbb{R}^r)$. Taking into account Lemma 4.2, the following estimate is true:

$$
\begin{aligned}
(4.15)\quad |\langle y'(t),v\rangle_{V^*\times V}| &= |(y'(t),v)_H| = |(y'(t),v^1)_H| \\
&\le \|A_{y(t)+\overline{m}(t)}\partial_x y(t)\|_{L^2(\Omega,\mathbb{R}^r)}\|\partial_x v^1\|_{L^2(\Omega,\mathbb{R}^r)} \\
&\quad + |K(t)|\|y(t)\|_{L^2(\Omega,\mathbb{R}^r)}\|\partial_x v^1\|_{L^2(\Omega,\mathbb{R}^r)} \\
&\quad + \|\Gamma_R(t)\|\|v^1(\Lambda)\| + \|\Gamma_L(t)\|\|v^1(-\Lambda)\| \\
&\le \big[(A_\infty + \|K\|_{L^\infty(0,T)})\|y(t)\|_V + \sqrt{2\Lambda/3}\,(\|\Gamma_R(t)\| + \|\Gamma_L(t)\|)\big]\|v\|_V
\end{aligned}
$$

for almost every $t \in (0,T)$. Hence

$$
(4.16)\quad \|y'(t)\|_{V^*} \le (A_\infty + \|K\|_{L^\infty(0,T)})\|y(t)\|_V + \sqrt{2\Lambda/3}\,(\|\Gamma_R(t)\| + \|\Gamma_L(t)\|)
$$

for almost every $t \in (0,T)$. In consequence

$$
\begin{aligned}
(4.17)\quad \int_0^T \|y'(t)\|_{V^*}^2\,dt &\le 2(A_\infty + \|K\|_{L^\infty(0,T)})^2 \int_0^T \|y(t)\|_V^2\,dt \\
&\quad + \frac{8\Lambda}{3}\int_0^T (\|\Gamma_R(t)\|^2 + \|\Gamma_L(t)\|^2)\,dt.
\end{aligned}
$$

The inequalities (4.17) and (4.13) imply (4.14).

Point (c) follows from [26, Proposition 23.23], because $y \in L^2(0,T;V)$ and $y' \in L^2(0,T;V^*)$ by (b). □

Theorem 4.6. *If Assumption* 3.3 *is satisfied, then Problem* 3.6 *has a solution* $w$ *such that* $w(t,x) + \overline{m}(t) \in \mathcal{K}$.

Proof. It follows from Lemma 4.5 that Problem 3.7 has a unique solution $w_n$ and $w_n \in C([0,T],H)$, $n \in \mathbb{N}$. Moreover, there exists a constant $C_1$ independent of $n$ such that $\|w_n\|_{L^2(0,T;V)} \le C_1$, for $n \in \mathbb{N}$, since $w_{n0} \to w_0$ in $H$ as $n \to \infty$. It follows from the Banach–Alaoglu theorem that there exists a subsequence, not renumbered $w_n \to w$ weakly in $L^2(0,T;V)$ as $n \to \infty$.

Fix any real function $\varphi \in C^1([0,T])$ with $\varphi(T) = 0$. Multiplying the Galerkin equation (3.16) by $\varphi$ and using intergration by parts, we obtain the integral identity

$$
\begin{aligned}
(4.18)\quad -(w_{n0},v_j)_H\varphi(0) &- \int_0^T (w_n(t),v_j)_H\varphi'(t)\,dt \\
&+ \int_0^T \bigg(\int_\Omega (A_{w_n(t)+\overline{m}(t)}\partial_x w_n(t))\cdot\partial_x v_j\,dx\bigg)\varphi(t)\,dt \\
&- \int_0^T \bigg(K(t)\int_\Omega w_n(t)\cdot\partial_x v_j\,dx\bigg)\varphi(t)\,dt \\
&= \int_0^T (\Gamma_R(t)\cdot v_j(\Lambda))\varphi(t)\,dt - \int_0^T (\Gamma_L(t)\cdot v_j(-\Lambda))\varphi(t)\,dt,
\end{aligned}
$$

for $j = 1, \ldots, n$. Integration by parts is admitted since $w_n \in C([0,T], H)$. We want to pass to the limit in (4.18). Observe that the Aubin–Lions compactness lemma implies existence of a subsequence, which we denote again by $w_n$, strongly converging in $L^2(0,T;H)$ to $w$ as $n \to \infty$. Applying the strong limit $w_{n0} \to w_0$ in $H$, the weak limit $w_n \to w$ in $L^2(0,T;V)$ and the strong limit $w_n \to w$ in $L^2(0,T;H)$ as $n \to \infty$ to equation (4.18), for $j \in \mathbb{N}$ we get

$$
\begin{aligned}
(4.19)\qquad -(w_0,v_j)_H\varphi(0) &- \int_0^T (w(t), v_j)_H \varphi'(t)\,dt \\
&+ \int_0^T \bigg(\int_\Omega (A_{w(t)+\overline{m}(t)}\partial_x w(t)) \cdot \partial_x v_j\,dx\bigg)\varphi(t)\,dt \\
&- \int_0^T \bigg(K(t)\int_\Omega w(t) \cdot \partial_x v_j\,dx\bigg)\varphi(t)\,dt \\
&= \int_0^T (\Gamma_R(t) \cdot v_j(\Lambda))\varphi(t)\,dt - \int_0^T (\Gamma_L(t) \cdot v_j(-\Lambda))\varphi(t)\,dt.
\end{aligned}
$$

In order to justify this limit we will first consider the nonlinear integral with $\partial_x w_n(t)$ on the left in (4.18). Note that the space $H^2(\Omega, 1^\perp) \cap V$ is dense in $V$. Hence there are sequences $v_{kj} \in H^2(\Omega, 1^\perp) \cap V$, $j = 1, \ldots, n$, $v_{kj} \to v_j$ as $k \to \infty$ strongly in $V$. We see that

$$
\begin{aligned}
(4.20)\qquad &\int_0^T \bigg(\int_\Omega (A_{w_n(t)+\overline{m}(t)}\partial_x w_n(t)) \cdot \partial_x v_{kj} dx\bigg)\varphi(t)\,dt \\
&= \int_0^T \bigg(\int_\Omega \bigg(\sum_{i=1}^r \overline{\Theta}_i(w_n(t)+\overline{m}(t))\partial_x w_{ni}(t) e_i\bigg) \cdot \partial_x v_{kj}\,dx\bigg)\varphi(t)\,dt \\
&- \int_0^T \bigg[\int_\Omega \big[\overline{\Theta}(w_n(t)+\overline{m}(t)) \cdot \partial_x w_n(t)\big]\big[(w_n(t)+\overline{m}(t)) \cdot \partial_x v_{kj}\big]\,dx\bigg]\varphi(t)\,dt.
\end{aligned}
$$

We shall study the second integral on the right in (4.20) only, as an idea with the first one is very similar. The following inequality holds

$$
\begin{aligned}
(4.21)\qquad \bigg| &\int_0^T \bigg[\int_\Omega (\overline{\Theta}(w_n(t)+\overline{m}(t)) \cdot \partial_x w_n(t)) \\
&\qquad\qquad \times ((w_n(t)+\overline{m}(t)) \cdot \partial_x v_{kj})\,dx\bigg]\varphi(t)\,dt \\
&- \int_0^T \bigg[\int_\Omega (\overline{\Theta}(w(t)+\overline{m}(t)) \cdot \partial_x w(t)) \\
&\qquad\qquad \times ((w(t)+\overline{m}(t)) \cdot \partial_x v_{kj})\,dx\bigg]\varphi(t)\,dt\bigg| \\
\le \|\varphi\|_{L^\infty(0,T)} &\int_0^T \bigg|\int_\Omega \big[(\overline{\Theta}(w_n(t)+\overline{m}(t)) - \overline{\Theta}(w(t)+\overline{m}(t))) \cdot \partial_x w_n(t)\big] \\
&\qquad\qquad \times \big[(w_n(t)+\overline{m}(t)) \cdot \partial_x v_{kj}\big]\,dx\bigg|\,dt
\end{aligned}
$$

$$+ \|\varphi\|_{L^\infty(0,T)} \int_0^T \left| \int_\Omega \big[(w_n(t) - w(t)) \cdot \partial_x v_{kj}\big] \times \big[\overline{\Theta}(w(t) + \overline{m}(t)) \cdot \partial_x w_n(t)\big]\, dx \right| dt$$

$$+ \|\varphi\|_{L^\infty(0,T)} \int_0^T \left| \int_\Omega \big[(\partial_x w_n(t) - \partial_x w(t)) \cdot \overline{\Theta}(w(t) + \overline{m}(t))\big] \times \big[(w(t) + \overline{m}(t)) \cdot \partial_x v_{kj}\big]\, dx \right| dt.$$

Further, we have the estimates

$$\begin{aligned} &\int_0^T \left| \int_\Omega \big[(\overline{\Theta}(w_n(t) + \overline{m}(t)) - \overline{\Theta}(w(t) + \overline{m}(t))) \cdot \partial_x w_n(t)\big] \times \big[(w_n(t) + \overline{m}(t)) \cdot \partial_x v_{kj}\big]\, dx \right| dt \\ &\qquad \le c_1 \|\partial_x v_{kj}\|_V \|w_n\|_{L^2(0,T;V)} \|w_n - w\|_{L^2(0,T;H)}, \\ &\int_0^T \left| \int_\Omega \big[(w_n(t) - w(t)) \cdot \partial_x v_{kj}\big] \big[\overline{\Theta}(w(t) + \overline{m}(t)) \cdot \partial_x w_n(t)\big]\, dx \right| dt \\ &\qquad \le c_2 \|\partial_x v_{kj}\|_V \|w_n\|_{L^2(0,T;V)} \|w_n - w\|_{L^2(0,T;H)}, \end{aligned} \tag{4.22}$$

where $c_1$, $c_2$ are some constants. Moreover, the integral

$$\int_0^T \int_\Omega \big[\partial_x w_n(t) \cdot \overline{\Theta}(w(t) + \overline{m}(t))\big] \big[(w(t) + \overline{m}(t)) \cdot \partial_x v_{kj}\big]\, dx\, dt$$

represents a linear continuous functional in the space $L^2(0,T;V)$ with respect to $w_n$. Indeed, we have that

$$\begin{aligned} \int_0^T \left| \int_\Omega \big[\partial_x w_n(t) \cdot \overline{\Theta}(w(t) + \overline{m}(t))\big] \big[(w(t) + \overline{m}(t)) \cdot \partial_x v_{kj}\big]\, dx \right| dt \\ \le c_3 \|\partial_x v_{kj}\|_V \|w_n\|_{L^2(0,T;V)}, \end{aligned} \tag{4.23}$$

where $c_3$ is some constant. From (4.21)–(4.23) we get

$$\begin{aligned} &\int_0^T \left[ \int_\Omega \big(\overline{\Theta}(w_n(t) + \overline{m}(t)) \cdot \partial_x w_n(t)\big)\big((w_n(t) + \overline{m}(t)) \cdot \partial_x v_{kj}\big)\, dx \right] \varphi(t)\, dt \\ &\to \int_0^T \left[ \int_\Omega \big(\overline{\Theta}(w(t) + \overline{m}(t)) \cdot \partial_x w(t)\big)\big((w(t) + \overline{m}(t)) \cdot \partial_x v_{kj}\big)\, dx \right] \varphi(t)\, dt \end{aligned} \tag{4.24}$$

as $n \to \infty$. To pass to the limit with $k \to \infty$ note that the integral

$$\int_0^T \left[ \int_\Omega (\overline{\Theta}(w(t) + \overline{m}(t)) \cdot \partial_x w(t))((w(t) + \overline{m}(t)) \cdot \partial_x v)\, dx \right] \varphi(t)\, dt$$

represents a linear continuous functional in the space $V$ with respect to $v$. It is implied immediately by the estimate

$$\tag{4.25} \begin{aligned} \bigg|\int_0^T\bigg[\int_\Omega\big(\overline{\Theta}(w(t)+\overline{m}(t))\cdot\partial_x w(t)\big)\big((w(t)+\overline{m}(t))\cdot\partial_x v\big)\,dx\bigg]\varphi(t)\,dt\bigg| \\ \le c_4\|w\|_{L^2(0,T;V)}\|v\|_V, \end{aligned}$$

where $c_4$ is some constant.

Note that the first and third integrals on the left in (4.18) represent linear continuous functionals in the space $L^2(0,T;V)$ with respect to $w_n$. Indeed, we have that

$$\tag{4.26} \begin{aligned} \bigg|\int_0^T(w_n(t),v_j)_H\varphi'(t)\,dt\bigg| &\le c_5\|v_j\|_V\|w_n\|_{L^2(0,T;V)}, \\ \bigg|\int_0^T\bigg(K(t)\int_\Omega w_n(t)\cdot\partial_x v_j\,dx\bigg)\varphi(t)\,dt\bigg| &\le c_6\|v_j\|_V\|w_n\|_{L^2(0,T;V)}, \end{aligned}$$

where $c_5$, $c_6$ are some constants.

In order to finish our argument let $v\in V$. Then there exists a sequence $p_n\to v$ strongly in $V$ as $n\to\infty$, where each $p_n$ is a finite linear combination of certain basis elements $v_j$. Letting $n\to\infty$ we get that equation (4.19) is also valid if we replace $v_j$ with $v$. This gives

$$\tag{4.27} \begin{aligned} -(w_0,v)_H\varphi(0) &- \int_0^T(w(t),v)_H\varphi'(t)\,dt \\ &+\int_0^T\bigg(\int_\Omega(A_{w(t)+\overline{m}(t)}\partial_x w(t))\cdot\partial_x v\,dx\bigg)\varphi(t)\,dt \\ &-\int_0^T\bigg(K(t)\int_\Omega w(t)\cdot\partial_x v\,dx\bigg)\varphi(t)\,dt \\ &=\int_0^T(\Gamma_R(t)\cdot v(\Lambda))\varphi(t)\,dt-\int_0^T(\Gamma_L(t)\cdot v(-\Lambda))\varphi(t)\,dt \end{aligned}$$

for all $v\in V$ and all $\varphi\in C^1([0,T])$ with $\varphi(T)=0$. To justify this limit we need the fact that the terms in (4.19) represent linear continuous functionals in the space $V$ with respect to $v_j$. This follows from (4.26) and

$$\begin{aligned} \bigg|\int_0^T\bigg(\int_\Omega(A_{w_n(t)+\overline{m}(t)}\partial_x w_n(t))\cdot\partial_x v_j\,dx\bigg)\varphi(t)\,dt\bigg| &\le c_7\|v_j\|_V\|w_n\|_{L^2(0,T;V)}, \\ \bigg|\int_0^T(\Gamma_R(t)\cdot v_j(\Lambda))\varphi(t)\,dt\bigg| &\le c_8\|v_j\|_V, \\ \bigg|\int_0^T(\Gamma_L(t)\cdot v_j(\Lambda))\varphi(t)\,dt\bigg| &\le c_9\|v_j\|_V, \end{aligned}$$

where $c_7$, $c_8$, $c_9$ are some constants, by Lemma 4.3 and the continuous imbedding $V_n\subset C(\Omega,\mathbb{R}^r)$.

Now we will prove that $w$ satisfies (3.14). For each $t \in [0,T]$ such that $w(t)$ exists, we define operators $B$ and $b$ by the formulas

$$\begin{aligned}\langle B(t), v\rangle_{V^*\times V} &= \int_\Omega (A_{w(t)+\overline{m}(t)}\partial_x w(t)) \cdot \partial_x v\,dx - K(t)\int_\Omega w(t)\cdot \partial_x v\,dx,\\ \langle b(t), v\rangle_{V^*\times V} &= \Gamma_R(t)\cdot v(\Lambda) - \Gamma_L(t)\cdot v(-\Lambda),\end{aligned}$$

where $v \in V$. Let us show that they are well defined. The linearity of $B(t)$ and $b(t)$ is obvious. Moreover, $B(t), b(t) \in V^*$ because

$$\begin{aligned}\|B(t)\|_{V^*} &\le (A_\infty + \|K\|_{L^\infty(0,T)})\|w(t)\|_V,\\ \|b(t)\|_{V^*} &\le \sqrt{2\Lambda/3}(\|\Gamma_R(t)\| + \|\Gamma_L(t)\|).\end{aligned} \tag{4.28}$$

From (4.27) we obtain that

$$\left\langle -\int_0^T w(t)\varphi'(t)\,dt + \int_0^T (B(t)-b(t))\varphi(t)\,dt,\, v\right\rangle_{V^*\times V} = 0$$

for all $v \in V$ and all $\varphi \in C_0^\infty(0,T)$. Hence

$$-\int_0^T w(t)\varphi'(t)\,dt + \int_0^T (B(t)-b(t))\varphi(t)\,dt = 0$$

for all $\varphi \in C_0^\infty(0,T)$. This shows that $w$ has a generalized derivative on $(0,T)$ and fulfills the equation

$$w'(t) + B(t) = b(t) \quad \text{for a.e. } t \in (0,T). \tag{4.29}$$

The inequalities (4.28) imply that $B \in L^2(0,T;V^*)$ and $b \in L^2(0,T;V^*)$. Hence $w' \in L^2(0,T;V^*)$ and (3.14) holds.

It remains to show that $w$ fulfills the initial condition (3.15). Since by [26, Proposition 23.23], $w \in C([0,T],H)$ we can apply the integration by parts formula. This yields

$$\begin{aligned}&(w(T), \varphi(T)v)_H - (w(0), \varphi(0)v)_H\\ &\qquad = \int_0^T \big(\langle w'(t), \varphi(t)v\rangle_{V^*\times V} + \langle \varphi'(t)v, w(t)\rangle_{V^*\times V}\big)\,dt\end{aligned}$$

for all $\varphi \in C^1([0,T])$ and all $v \in V$. In particular, if $\varphi(0) = 1$ and $\varphi(T) = 0$, the equation (4.27) along with (4.29) yields

$$(w(0) - w_0, v)_H = 0 \quad \text{for all } v \in V. \tag{4.30}$$

Since $V$ is dense in $H$, we get (3.15). □

Remark 4.7. In [10] it is assumed that $j_{i,L}, j_{i,R} \in L^2(0,T)$ instead of $j_{i,L}, j_{i,R} \in L^\infty(0,T)$ (see (H$_4$)). But under that assumption we are not able to prove that $w \in L^2(0,T;V)$ and $w' \in L^2(0,T;V^*)$.

## 5. Uniqueness of weak solutions

In this section we prove that Problem 3.6 cannot have more than one weak solution. However, we can do it in the stronger space $L^4(0,T;V)$. It is a type of the weak-strong uniqueness valid also for the incompressible Navier–Stokes system [15].

THEOREM 5.1. *If Assumption* 3.3 *is satisfied, then Problem* 3.6 *has at most one solution in* $L^4(0,T;\,V)\cap\{w\colon[0,T]\times\Omega\to\mathbb{R}^r : w(t,x)+\overline{m}(t)\in\mathcal{K}\}$.

PROOF. Suppose that Problem 3.6 has two solutions $w_1,w_2\in L^4(0,T;V)\cap\{w\colon[0,T]\times\Omega\to\mathbb{R}^r : w(t,x)+\overline{m}(t)\in\mathcal{K}\}$. We will show that they are equal. By putting $v=w_1-w_2$ in (3.14) we get for almost every $t\in(0,T)$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(w_1-w_2)(t)\|_H^2 &\\ &+\int_\Omega(A_{w_1(t)+\overline{m}(t)}\partial_x(w_1-w_2)(t))\cdot\partial_x(w_1-w_2)(t)\,dx\\ &+\int_\Omega\big[(A_{w_1(t)+\overline{m}(t)}-A_{w_2(t)+\overline{m}(t)})\partial_x w_2(t)\big]\cdot\partial_x(w_1-w_2)(t)\,dx\\ &-K(t)\int_\Omega(w_1-w_2)(t)\cdot\partial_x(w_1-w_2)(t)\,dx=0.\end{aligned}\tag{5.1}$$

We will estimate each integral term. By assumption ($\mathrm{H}_6$) we obtain for almost every $t\in(0,T)$,

$$\begin{aligned}\int_\Omega(A_{w_1(t)+\overline{m}(t)}\partial_x(w_1-w_2)(t))\cdot\partial_x(w_1-w_2)(t)\,dx&\\ \ge\mu\|(w_1-w_2)(t)\|_V^2-\nu\|(w_1-w_2)(t)\|_H^2.&\end{aligned}\tag{5.2}$$

It follows from the Hölder inequality that, for almost every $t\in(0,T)$,

$$\begin{aligned}\bigg|\int_\Omega\big[(A_{w_1(t)+\overline{m}(t)}-A_{w_2(t)+\overline{m}(t)})\partial_x w_2(t)\big]\cdot\partial_x(w_1-w_2)(t)\,dx\bigg|&\\ \le\sup_{x\in\Omega}\big\{\|A_{w_1(t,x)+\overline{m}(t)}-A_{w_2(t,x)+\overline{m}(t)}\|_{L(1^\perp)}\big\}&\\ \times\|\partial_x w_2(t)\|_{L^2(\Omega,1^\perp)}\|\partial_x(w_1-w_2)(t)\|_{L^2(\Omega,1^\perp)}.&\end{aligned}\tag{5.3}$$

Lemma 4.2 implies that for $x\in\Omega$ and a.e. $t\in(0,T)$,

$$\begin{aligned}\|A_{w_1(t,x)+\overline{m}(t)}-A_{w_2(t,x)+\overline{m}(t)}\|_{L(1^\perp)}&\\ \le L_A\|w_1(t,x)-w_2(t,x)\|\le L_A\|(w_1-w_2)(t)\|_{L^\infty(\Omega)}.&\end{aligned}\tag{5.4}$$

Note that the interpolating inequality holds for almost every $t\in(0,T)$,

$$\|(w_1-w_2)(t)\|_{L^\infty(\Omega)}\le\sqrt{2}\,\|(w_1-w_2)(t)\|_V^{1/2}\|(w_1-w_2)(t)\|_H^{1/2}\tag{5.5}$$

(see [26, p. 285]). Using (5.3)–(5.5) we have for almost every $t \in (0,T)$,

$$\tag{5.6} \begin{aligned} &\left| \int_\Omega \big[(A_{w_1(t)+\overline{m}(t)} - A_{w_2(t)+\overline{m}(t)})\partial_x w_2(t)\big] \cdot \partial_x (w_1 - w_2)(t)\, dx \right| \\ &\qquad \le \big(\sqrt{2} L_A \|(w_1 - w_2)(t)\|_H^{1/2} \|w_2(t)\|_V\big) \|(w_1 - w_2)(t)\|_V^{3/2}. \end{aligned}$$

Applying the Young inequality with 4 and 4/3 to the right-hand side of (5.6) we get for almost every $t \in (0,T)$,

$$\tag{5.7} \begin{aligned} &\left| \int_\Omega \big[(A_{w_1(t)+\overline{m}(t)} - A_{w_2(t)+\overline{m}(t)})\partial_x w_2(t)\big] \cdot \partial_x (w_1 - w_2)(t)\, dx \right| \\ &\qquad \le C\|w_2(t)\|_V^4 \|(w_1 - w_2)(t)\|_H^2 + \varepsilon \|(w_1 - w_2)(t)\|_V^2. \end{aligned}$$

The Hölder inequality and the Young inequality with 2 and 2 yield for almost every $t \in (0,T)$,

$$\tag{5.8} \begin{aligned} &\left| K(t) \int_\Omega (w_1 - w_2)(t) \cdot \partial_x (w_1 - w_2)(t)\, dx \right| \\ &\qquad \le (|K(t)| \|(w_1 - w_2)(t)\|_H) \|(w_1 - w_2)(t)\|_V \\ &\qquad \le C(K(t))^2 \|(w_1 - w_2)(t)\|_H^2 + \varepsilon \|(w_1 - w_2)(t)\|_V^2. \end{aligned}$$

We take $\varepsilon = \mu/2$ in the estimates (5.7), (5.8), and using these estimates together with (5.2) in (5.1) we get for almost every $t \in (0,T)$,

$$\tag{5.9} \frac{d}{dt} \|(w_1 - w_2)(t)\|_H^2 \le 2[\nu + C(\|w_2(t)\|_V^4 + (K(t))^2)] \|(w_1 - w_2)(t)\|_H^2.$$

Hence the Gronwall lemma implies that for all $t \in [0,T]$,

$$\tag{5.10} \|(w_1 - w_2)(t)\|_H^2 = 0,$$

and in consequence $w_1 = w_2$. □

## 6. Asymptotic behavior

Theorem 6.1. *Let $w\colon [0,\infty) \to H$ be a solution to Problem 3.6 on each interval $[0,T]$ for $T \in \mathbb{R}_+$ such that $w(t,x) + \overline{m}(t) \in \mathcal{K}$. If Assumption 3.3 is satisfied with $j_{i,L}(t) = j_{i,R} \equiv 0$, $t \in [0,\infty)$, $i = 1,\dots,r$ and $\nu = 0$ in (3.10), then $w \in L^2(0,\infty;V) \cap L^\infty(0,\infty;H)$, $w' \in L^2(0,\infty;V^*)$, the function $[0,\infty) \ni t \mapsto \|w(t)\|_H^2$ is nonincreasing and $\lim_{t\to\infty} \|w(t)\|_H^2 = 0$.*

Proof. The following inequalities follow from (3.10):

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2 \le -\mu \|w(t)\|_V^2,$$

$$\|w(t)\|_H^2 + 2\mu \int_0^t \|w(\tau)\|_V^2\, d\tau \le \|w_0\|_H^2$$

for $t \in [0,\infty)$. Hence $w \in L^2(0,\infty;V) \cap L^\infty(0,\infty;H)$. Moreover, $w' \in L^2(0,\infty;V^*)$ because $B \in L^2(0,\infty;V^*)$ and $b(t) \equiv 0$ for $t \in [0,\infty)$ in (4.29). It is clear that

$w \in C([0,\infty), H)$. Note that the function $[0,\infty) \ni t \mapsto \|w(t)\|_H^2$ is continuous, nonnegative, nonincreasing and

$$\int_0^\infty \|w(t)\|_H^2 \, dt < \infty. \tag{6.1}$$

The integral criterion for series implies

$$\lim_{t\to\infty} \|w(t)\|_H^2 = 0. \qquad \square$$

The above theorem is consistent with the physical properties of a mixture. Since $w(t)$ is the local deviation of the volume fraction of the mixture components, consequently $\|w(t)\|_H^2$ is the random deviation of this volume fraction. Thus, the random deviation in a closed system is monotonically decreasing to zero if time $t$ goes to infinity. It means that for long times the mixture becomes homogeneous.

## 7. Examples and numerical experiments

To illustrate a class of problems which can be treated with our method, we consider an example of ternary mixture of nickel (Ni), copper (Cu) and iron (Fe) with constants $\Theta_i$, $i = 1,2,3$. The examples with $\Theta_i$ depending on densities $\varrho_i$ (the dependence is of a polynomial type) are given in [23]. All the assumptions of our theorems are fulfilled. To solve Problem 3.7 we will use the finite element method (FEM). Set

$$\widetilde{e}_s = \sqrt{\frac{s}{s+1}}\left(\frac{1}{s}\sum_{i=1}^{s} e_i - e_{s+1}\right), \quad s = 1,\dots,r-1. \tag{7.1}$$

Observe that $\{\widetilde{e}_1,\dots,\widetilde{e}_{r-1}\}$ is an orthonormal basis in $1^\perp$. Let $N \in \mathbb{N}$ be fixed and define nodal points $x_k = -\Lambda + kh$ on $\Omega$, $k = 0,\dots,N$, where $h = 2\Lambda/N$ is a step. Define functions $\varphi_k \colon \Omega \to \mathbb{R}$, $k = 1,\dots,r$, as affine functions on each interval $[x_{m-1}, x_m]$, $m = 1,\dots,N$, and $\varphi_k(x_l) = \delta_{kl}$, $k,l = 0,\dots,N$.

Remark 7.1. If $f \in \operatorname{span}\{\varphi_k : k = 0,\dots,N\}$, then $f = \sum_{k=0}^{N} f(x_k)\varphi_k$. Moreover, $\varphi_k \in H^1(\Omega,\mathbb{R})$, $k = 0,\dots,N$, and $\bigcup_{N\in\mathbb{N}} \operatorname{span}\{\varphi_k\widetilde{e}_s : k = 0,\dots,N,\ s = 1,\dots,r-1\}$ is dense in $V$.

Define the subspace

$$V_n = \operatorname{span}\{\varphi_k\widetilde{e}_s : k = 0,\dots,N,\ s = 1,\dots,r-1\}$$

of dimension $n = (N+1)(r-1)$ of the Sobolev space $V$. We look for a solution to Problem 3.7 in the form

$$w_n(t) = \sum_{k=0}^{N}\sum_{p=1}^{r-1} \lambda_{nkp}(t)\varphi_k\widetilde{e}_p. \tag{7.2}$$

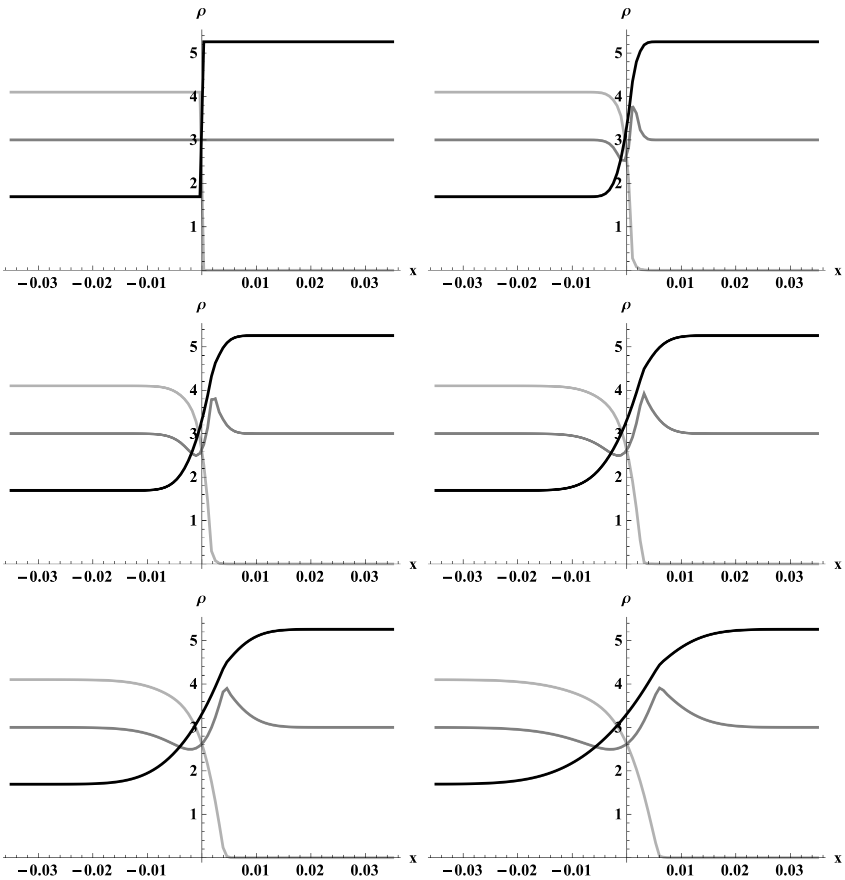


FIGURE 1. Distributions of density: $\rho_1$ (light gray), $\rho_2$ (dark gray), $\rho_3$ (black).

REMARK 7.2. If $f \in V_n$, then

$$f = \sum_{k=0}^{N} f(x_k)\varphi_k = \sum_{k=0}^{N}\sum_{p=1}^{r-1}(f(x_k)\cdot \widetilde{e}_p)\varphi_k\widetilde{e}_p.$$

An optimal element for $f \in V$ with respect to the subspace $V_n$ is the spline

$$f_n = \sum_{k=0}^{N}\sum_{p=1}^{r-1}(f(x_k)\cdot \widetilde{e}_p)\varphi_k\widetilde{e}_p.$$

Hence

$$f_n = \sum_{k=0}^{N} f(x_k)\varphi_k.$$

In the special case if $w_0 \in V$, then

$$w_{0n} = \sum_{k=0}^{N}\sum_{p=1}^{r-1}(w_0(x_k)\cdot \widetilde{e}_p)\varphi_k\widetilde{e}_p = \sum_{k=0}^{N} w_0(x_k)\varphi_k.$$

Moreover, $\lambda_{nkp}(t) = w_n(t,x_k)\cdot \widetilde{e}_p$ and putting $w_{n0} := w_{0n}$ in Problem 3.7 we have $\lambda_{nkp}(0) = w_0(x_k)\cdot \widetilde{e}_p$.

Now put

$$\varphi_{ks} = \frac{1}{\sqrt{\Lambda}}\cos\frac{k\pi}{2\Lambda}(x+\Lambda)\widetilde{e}_s, \quad k\in\mathbb{N},\ s=1,\dots,r-1.$$

The set $\{\varphi_{ks} : k\in\mathbb{N},\ s=1,\dots,r-1\}$ is an orthogonal basis in $V$ and orthonormal one in $H$. Define the subspace

$$V_n = \operatorname{span}\{\varphi_{ks} : k=1,\dots,N,\ s=1,\dots,r-1\}$$

of dimension $n = N(r-1)$ of the Sobolev space $V$. We look for a solution to Problem 3.7 in the form

$$w_n(t) = \sum_{k=1}^{N}\sum_{p=1}^{r-1}\lambda_{nkp}(t)\varphi_{kp}.$$

Remark 7.3. If $w_0 \in H$, then

$$w_0 = \sum_{k=1}^{\infty}\sum_{p=1}^{r-1}(\varphi_{kp}, w_0)_H\varphi_{kp},$$

where the right-hand side is a sum of the Fourier series of $w_0$ with respect to the basis given. Putting

$$w_{n0} := \sum_{k=1}^{N}\sum_{p=1}^{r-1}(\varphi_{kp}, w_0)_H\varphi_{kp}$$

in Problem 3.7 we have $\lambda_{nkp}(0) = (\varphi_{kp}, w_0)_H$.

Let the physical data be given:

$$\begin{aligned} r=3, \qquad & Ni-Cu-Fe, \qquad \Lambda = 0.035,\\ (M_1,M_2,M_3) &= (58.7, 63.5, 55.8),\\ (\Omega_1,\Omega_2,\Omega_3) &= (6.5, 7.0, 7.1),\\ (\Theta_1,\Theta_2,\Theta_3) &= (1.58\cdot 10^{-13}, 5.73\cdot 10^{-12}, 2.99\cdot 10^{-11}),\\ (\varrho_{01},\varrho_{02},\varrho_{03}) &= (4.1, 3.0, 1.68377) \quad \text{for } x<0,\\ (\varrho_{01},\varrho_{02},\varrho_{03}) &= (0.0, 3.0, 5.25185) \quad \text{for } x>0. \end{aligned}$$

For the times 0, 1, 3, 7, 14 and 28 days we obtain with approximation (7.2) the results displayed in Figure 1.

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