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GENERIC DOMAIN DEPENDENCE FOR NON-SMOOTH EQUATIONS AND THE OPEN SET PROBLEM FOR JUMPING NONLINEARITIES

E. N. Dancer

(Submitted by K. Gęba)

Dedicated to the memory of Karol Borsuk

Saut and Teman [19] proved that, if $g \in C^1$ and $g(0) = 0$ then for a generic smooth bounded domain Ω the equation

$$\Delta u = g(u) \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

has only isolated solutions and each of these solutions is non-degenerate (that is, the linearization is invertible). They also allow g to depend on $x \in \Omega$ but for simplicity we avoid this. In this short paper, we show that the result remains true even if g' has some discontinuities. Note that non-differentiable nonlinearities frequently occur as limit problems or in singular perturbations of more regular nonlinearities (even analytic nonlinearities). Some examples of this appear in [6] and [10]. They also occur in plasma problems.

The main application of our techniques is to the open set problem for jumping nonlinearities. The problem is as follows. If Ω is a smooth bounded domain in R^n, we consider $A_{-1} = \{(a,d) \in R^2 : -\Delta u = au^+ + du^- \text{ has a non-trivial solution in } \dot{W}^{1,2}(\Omega)\}$.

The main question then is whether A_{-1} can contain an open set. While a good deal is known on the structure of A_{-1} (see [2], [5], [7], [10], and [11], where further references can be found), it is not even known if there are any domains

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Ω with $\dim \Omega > 1$ while A_{-1} does not contain an open set. If $\dim \Omega = 1$, A_{-1} can easily be calculated explicitly and A_{-1} does not contain an open set. (In fact if $\dim \Omega = 1$, analyticity arguments can be used to show that A_{-1} does not contain an open set when $-\Delta$ is replaced by a rather more general second order differential operator.) Here we use genericity arguments to show that for a dense set of $\Omega's$ (in appropriate topology) A_{-1} does not contain an open set. By our comments above, these are the first examples where $\dim \Omega > 1$ and where A_{-1} is known not to contain an open set. We also make some other remarks on the open set problem.

In §1, we obtain the genericity result while in §2 we discuss the open set problem. We discuss the genericity in some detail because there are a number of minor errors in [19].

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1. Genericity under Domain Perturbation

In this section, we prove our main genericity result. We consider the problem

$$\begin{aligned} \Delta u &= g(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Here g is locally Lipschitz on R, C^1 except at $\tilde{a}$ and $g(0) = 0$. We prove that for most domains Ω, all the nontrivial solutions of (1) are non-degenerate. We prove this by modifying the arguments in Saut and Teman [19]. Suppose that Ω_0 is a fixed domain with smooth boundary, $p > n$, $\alpha \in (0,1)$ and Q is an open neighbourhood of $\overline{\Omega}_0$. It is easy to see that there is a $c_n > 0$ such that if $\theta \in C^{3,\alpha}(\overline{Q})^n$ and $\|\theta\|_{2,\alpha} \leq c_n$, then $(I + \theta)$ maps $\overline{\Omega}_0$ in a 1–1 way onto a C^3 manifold with boundary $(I + \theta)(\partial\Omega)$. We will only consider sets of this form. Let $T = I + \theta$, $\Omega = T(\Omega_0)$, $\Gamma = T(\partial\Omega_0)$. For future reference note that $\Omega = \Omega_0$ if $\theta = 0$ on $\partial\Omega_0$. In this case we have not changed the domain at all. This will be useful later.

Define

$$\begin{aligned} X &= \{u \in W^{2,p}(\Omega_0) : u = 0 \text{ on } \partial\Omega_0\}, \\ U &= X \backslash \{0\}, \\ Y &= C^{3,\alpha}(Q)^n, \\ V &= \{\theta \in C^{3,\alpha}(Q)^n : \|\theta\|_{2,\alpha} < c_n\}, \\ Z &= L^p(\Omega_0). \end{aligned}$$

We will define a map $F : U \times V \to Z$ and apply Sard's theorem. The main new observation in the proof is encapsuled in the following simple lemma. Note that strict diffentiability is defined in Cartan [3].

LEMMA 1. *Assume that $\tilde{F} : U \times V \to Z$ is continuous and is strictly differentiable at (u, v) whenever $\tilde{F}(u, v) = 0$. Moreover assume that zero is a regular value of $\tilde{F}$ and the kernel of $\tilde{F}'(u, v)$ is complemented. Then $\tilde{F}^{-1}(0)$ is a C^1 manifold.*

PROOF. If $(u_0, v_0) \in U \times V$ and $\tilde{F}(u_0, v_0) = 0$, then since $\tilde{F}'(u_0, v_0)$ is onto and $\tilde{F}$ is strictly differentiable at (u_0, v_0), a simple contraction mapping argument shows that there is a neighbourhood W of (u_0, v_0) in $U \times V$, a complement M to $N = N(\tilde{F}'(u_0, v_0))$ and a function r from a neighbourhood S of zero in N to M such that $r(0) = 0$ and

$$\{(u, v) \in W : \tilde{F}(u, v) = 0\} = \{(u_0, v_0) + s + r(s) : s \in S\}.$$

Moreover, by the strict differentiability of $\tilde{F}$, one easily sees that, given $\epsilon > 0$, there is a $\delta > 0$ such that $\|r(s_2) - r(s_1)\| \leq \epsilon \|s_2 - s_1\|$ if $s_1, s_2 \in S$, and $\|s_1\|$, $\|s_2\| \leq \delta$. This implies that $r'(0)$ exists and is zero and that $\|r'(s)\| \leq \epsilon$ whenever $\|s\| \leq \delta$ and r is differentiable at s. Thus r' will be continuous at 0 if we prove that r is differentiable at every s with $\|s\| \leq \delta$. We know that for every x near (u_0, v_0), where $\tilde{F}'(x) = 0$, $\tilde{F}'(x)$ exists. Since $\tilde{F}(x) - \tilde{F}'(u_0, v_0)(x - x_0)$ satisfies a small Lipschitz condition near (u_0, v_0) (by the strict differentiability), it follows that $\|\tilde{F}'(x) - \tilde{F}'(u_0, v_0)\|$ is small whenever x is near (u_0, v_0) and $\tilde{F}(x) = 0$. Since $\tilde{F}'(u_0, v_0)$ is onto, and N is complemented, it follows easily by a Liapounov-Schmidt type argument that $N(\tilde{F}'(x)) = \{s + L_x(s) : s \in N\}$, where L_x is linear and $\|L_x\|$ is small if x is near (u_0, v_0) and $\tilde{F}(x) = 0$. Now M must be a complement to $N(\tilde{F}'(x))$ for all x near (u_0, v_0) with $\tilde{F}(x) = 0$. Since $\tilde{F}$ is strictly differentiable at any such x we see that, as before, the zeros of $\tilde{F}$ near x are of the form $x + z + a(z)$, where $z \in N(\tilde{F}'(x))$, $a(z) \in M$, $a(0) = 0$, and a satisfies a small Lipschitz condition near zero. In the following argument, we assume for *simplicity* that $(u_0, v_0) = 0$. Now

$$s + r(s) = x + z + a(z). \tag{2}$$

But $N(\tilde{F}'(x)) = \{w + L_x(w) : w \in N\}$. Hence our equation becomes

$$\begin{aligned} s &= Q(x + w + L_x w + a(w + L_x(w))), && \text{where } z = w + L_x(w), \\ &= Qx + w, && \text{since } L_x w,\ a(z) \in M, \\ &\equiv \tilde{T}(x). \end{aligned}$$

Here Q is the projection onto N parallel to M. By (2),

$$\begin{aligned} r(s) &= (I - Q)(x + z + a(z)) \\ &= (I - Q)(x + w + L_x(w) + a(w + L_x w)) \\ &= (I - Q)(s + \tilde{T}^{-1}(s) + L_x(\tilde{T}^{-1} s) + a(\tilde{T}^{-1}(s) + L_x \tilde{T}^{-1}(s))). \end{aligned}$$

It is clear that the right hand side is differentiable at Qx (since $\tilde{T}^{-1}$ is continuous and affine and a is differentiable at zero). Hence r is differentiable at Qx and our claim follows. □

REMARK.

1) We do need to check carefully that the two parametrizations are differentiably related especially because we are in infinite dimensions.

2) Essentially, the lemma implies that strict differentiability is sufficient to justify the argument in [19]. Note that our argument implies that locally near (u_0, v_0)
$$\{x \in U \times V : \tilde{F}(x) = 0\} = \{(u_0, v_0) + s + r(s)\},$$
where r is C^1.

We now return to the Saut-Teman argument. We define $F : U \times V \to Z$ by
$$F(\tilde{u}, \theta) = -\Delta_x \tilde{u}(T\tilde{x}) + g(\tilde{u}(\tilde{x})).$$
This needs some explanation. Here x is the generic point of Ω, $\tilde{x}$ the generic point of Ω_0, $x = T\tilde{x}$, Δ_x denotes the Laplacian in x coordinates and $T = I + \theta$. The point of F is that $F(\tilde{u}, 0) = 0$ if and only if $\tilde{u} \circ T^{-1}$ is a solution of (1) on $\Omega = T(\Omega_0)$ satisfying the boundary conditions on $\partial\Omega$. As in [19], we see that

$$F(u, \theta) = -\operatorname{div}_{\tilde{x}}\{(\det T')(T')^{-1}({}^tT')^{-1} \operatorname{grad}_{\tilde{x}} u\} + \det T' \cdot g(u). \tag{3}$$

(This is most easily derived from the Laplacian written in weak form. Note that we have corrected a misprint in [19].) To apply our lemma we need to prove that F is strictly differentiable. For the first term, this is straightforward but tedious. It is easiest to first expand the divergence. (In fact, one easily sees that the first term is C^1.) Similar arguments appear in [16] and [19]. Note that the second term is the product of a smooth function of θ and $g(u)$, which is independent of θ and hence we need only prove the differentiability of $g(u)$ in u. It suffices to prove that any non-trivial solution u of the equation $F(u, \theta) = 0$ only takes the value $\tilde{a}$ on a set of zero measure because we can then use the argument in [9]. Note that $\hat{v} = u \circ T^{-1}$ is a solution of (1) on $(I + \theta)\Omega_0$. If $\hat{v}$ equals $\tilde{a}$ on a set of positive measure A then by Stampacchia [20], $\Delta\hat{v} = 0$ ae on A . Thus, by the equation $g(\tilde{a}) = 0$. Then $\tilde{v} - \tilde{a}$ is a solution of a linear equation $-\Delta h = c(x)h$, where $c(x) = (\hat{v}(x) - \tilde{a})^{-1} g(\hat{v}(x))$ if $\hat{v}(x) \neq \tilde{a}$) with c bounded. (Here we use that g is locally Lipschitz.) Hence by Aronszajn et al. [1], $\hat{v}(x) \equiv \tilde{a}$ on Ω. This is only possible if $\tilde{a} = 0$ and $\hat{v}$ is the trivial solution. Hence we have proved the strict differentiability. For future reference, note that we also have the strict differentiability at zero if $\tilde{a} \neq 0$.

As the next step, we need to prove that zero is a regular value of F. Here we follow [19] but correct some errors. Firstly as in [19], we see that we need only check the ontoness at F' at a point $(u^0, 0)$, where $F(u^0, 0) = 0$. This simplifies the calculations. As in [19],

$$F'_u(u^0, 0)v = -\Delta v + g'(u^0)v.$$

(Note that, by our comments above $u^0(x) = \tilde{a}$ only on a set of measure zero and hence $g'(u^0)$ makes sense.) Moreover,

$$F'_u(u^0, 0)\zeta = -\sum_{i=1}^{n}((\operatorname{div}\zeta)u^0_{x_i})_{x_i} + \operatorname{div}((^t(\zeta') + \zeta')\nabla u^0) + g(u^0)\operatorname{div}\zeta.$$

Here we have corrected some minor errors in the calculations in [19]. (We discuss below why the calculations in [19] cannot be quite correct. This was also realized by Henry [13].) As in [19, p. 312–313], to prove that $F'(u^0, 0)$ is onto, we must prove that, if

$$\begin{aligned} -\Delta w + g'(u^0)w &= 0 \text{ on } \Omega_0, \\ w &= 0 \text{ on } \partial\Omega_0, \end{aligned} \tag{4}$$

(where u^0 is as before) and if

$$\langle F'_\theta(u^0, 0)\zeta, w\rangle = 0$$

for all $\zeta \in C^{3,\alpha}(\overline{Q})^n$, then $w = 0$. As in [19], we can use that u^0 is a solution of (1) to simplify the formula for $F'_\theta(u^0, 0)\zeta$. The only point to note is that the seventh line on p. 313 should have a minus sign in front of the first term. We find that

$$F'_\theta(u^0, 0)\zeta = \operatorname{div}(\zeta(-\Delta u^0 + g(u^0))) - \zeta \cdot \nabla(g(u^0)) + \Delta(\zeta \cdot \nabla u^0).$$

(The changes come from the earlier errors in [19].) There is one minor point here. We can use standard regularity theory to ensure that $u^0 \in W^{3,p}(\Omega)$ and hence all the terms make sense. It is here that it is convenient to use $C^{3,\alpha}$ for V rather than $C^{2,\alpha}$. Since u^0 is a solution of (1), the first term vanishes and hence

$$F'_\theta(u^0, 0)\zeta = -\zeta \cdot \nabla(g(u^0)) + \Delta(\zeta \cdot \nabla u^0).$$

Thus, by a simple integration by parts using that $w = 0$ on $\partial\Omega_0$, we see that

$$\begin{aligned} \langle F'_\theta(u^0, 0)\zeta, w\rangle &= \int_{\Omega_0} -\zeta g'(u^0)\nabla u^0 w + \zeta \cdot \nabla u^0 \Delta w - \int_{\partial\Omega_0} \zeta \cdot \nabla u^0 \frac{\partial w}{\partial n} \\ &= -\int_{\partial\Omega_0} \zeta \cdot \nabla u^0 \frac{\partial w}{\partial n}, \end{aligned} \tag{5}$$

since w is a solution of (4). At this stage, we realize that the formula for $\langle F'_\theta(u^0, 0)\zeta, w\rangle$ in [19] must be in error. If $\zeta = 0$ on $\partial\Omega_0$, the boundary of Ω_0 is not changed (at least infinitesimally and not at all if the correct small variation is chosen) and the change in θ corresponds to looking at the same problem on the same domain Ω_0 under different coordinates. We would not expect this to yield transversality. (The original formula in [19] would give transversality for the variations which fix $\partial\Omega_0$.)

By (5), we see that, if $\langle F'_\theta(u^0,0)\zeta, w\rangle = 0$ for every $\zeta \in C^{3,\alpha}(\overline{Q})^n$, then

$$\nabla u^0 \frac{\partial w}{\partial n} \equiv 0 \text{ on } \partial\Omega_0. \tag{6}$$

Now w satisfies a linear equation and $w = 0$ on $\partial\Omega_0$. Thus by a result of Aronszajn et al. (cf Landis [14]), either $w \equiv 0$ or $\{x \in \partial\Omega_0 : \partial w/\partial n(x) \neq 0\}$ is dense in $\partial\Omega_0$. Thus, if w does not vanish identically, it follows from (6) that $\{x \in \partial\Omega_0 : \nabla u^0(x) = 0\}$ is dense in $\partial\Omega_0$. Hence by continuity, $\nabla u^0 \equiv 0$ on $\partial\Omega_0$. Since $g(0) = 0$, u^0 satisfies a linear elliptic equation with bounded coefficients. As before, results of Aronszajn et al. and Landis imply that $u^0 \equiv 0$ on Ω_0. This contradicts our assumptions. Hence $w \equiv 0$ on Ω_0 and $F'(u^0,0)$ is onto as claimed.

We now complete the proof. We can argue as in Lemma A.1 of [19] to check that $N(F'(u_0,v_0))$ is complemented when $F(u_0,v_0) = 0$. Hence we can now apply Lemma 1 to check that $F^{-1}(0)$ is a C^1 manifold modeled locally on $N(F'(u_0,v_0))$ (where $F(u_0,v_0) = 0$). As in the appendix to [19], it follows easily that $\tau \equiv \pi_v\,|_{F^{-1}(0)}$ is a C^1 Fredholm mapping of index zero of $F^{-1}(0)$ to Y and that 0 is a regular value of $F(\,\cdot\,,v_0)$ if and only if v_0 is a regular value of τ. Here π_v is the natural projection of $U \times V$ onto the second factor. Moreover, by the argument on p. 311 of [19], τ is proper when u is restricted to any set $\{u : \epsilon \leq \|u\| \leq R\}$. Hence τ is σ-proper in the sense of [18] and hence we can apply Sard's theorem as in [19]. We need to use the version of Sard's theorem in [18]. (Note that τ is σ-proper and not necessarily proper.) Hence we see that for a dense set if v's in V, 0 is a regular value of τ and hence of $F(\,\cdot\,,v)$. Hence we have established the following theorem.

THEOREM 1. *There is a dense subset B of V such that if $\theta \in B$ and $\Omega = (I+\theta)(\Omega_0)$, then every non-zero solution of (1) on Ω is non-degenerate (and hence isolated).*

REMARKS.

1) By the result in [18] and the σ-properness, B is in fact a countable intersection of dense open sets in V. This will be useful in §2. Note that our parametrization of domains is not 1–1 but if we choose a good local 1–1 parametrization of domains it is easy to see that density still holds.

2) If $\tilde{a} \neq 0$, g is differentiable at zero. In this case it is easy to prove that zero is not an eigenvalue of $-\Delta + g'(0)I$ (for Dirichlet boundary conditions on Ω) for a dense open set of θ's in V (where $\Omega = (I+\theta)(\Omega_0)$). Hence, if $\tilde{a} \neq 0$ we can improve the theorem to show that for a dense set of θ's in V every solution of (1) is non-degenerate.

3) If $\tilde{a} \neq 0$, but g has right and left limits at zero, it is possible to use some of the ideas in §2 to show that for a dense open set of $\theta's$, zero is an isolated solution of (1) (and hence for a dense set of θ's every solution is isolated).

4) As in [13], the condition that $g(0) = 0$ can be removed with some care. We have not included this here because all the applications we have in mind have $g(0) = 0$. Note that we only used $g(0) = 0$ to ensure that there are no solutions u of (1) on Ω with $\partial u/\partial n \equiv 0$ on $\partial\Omega$. The idea here is to construct $\hat{F}$ similarly to the way we constructed F before, except that we replace X by

$$\left\{u \in W^{2,p}(\Omega_0) : u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_0\right\}.$$

One shows that for fixed θ, $\hat{F}$ is an injective semi-Fredholm map of index $-\infty$ and that, if $\hat{F}(u^0, 0) = 0$, then there exists $\tilde{\zeta}$ such that $\hat{F}_\theta(u^0, 0)\tilde{\zeta} \notin \mathcal{R}(\hat{F}_u(u^0, 0))$. It follows that for most small θ, the equation $\hat{F}(u^0, \theta) = 0$ has no solution near u^0. Here we use that for u near u^0,

$$\|\hat{F}(u, 0) - \hat{F}(u^0, 0)\|_p \geq K\|u - u^0\|_{2,p}.$$

The result follows easily from this. A similar idea appears to be used in [13]. Our methods could also be used if g depends suitably on $x \in \Omega$ and $g(x, 0) \equiv 0$ (or if $g(x, 0)$ does not vanish identically on $\partial\Omega$).

5) Because of the (small) errors in [19], the argument in [8, p. 320–321] also needs some minor corrections. However, the results are correct.

6) If g is C^2, it is easy to modify the argument of Saut and Teman to prove that zero is a regular value of the mapping $\tilde{F}$ on $(X\backslash\{0\}) \times V \times R$, where $\tilde{F}(\tilde{u}, \theta, \lambda) = -\Delta_x \tilde{u}(T\tilde{x}) - \lambda g(\tilde{u}(\tilde{x}))$. Hence, by a result of Crandall and Rabinowitz [4], for most domains Ω the solutions of

$$-\Delta u = \lambda g(u) \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$

form a smooth curve in $(X\backslash\{0\}) \times R$. It is unclear if this is still true if g is only C^1 but our techniques imply that the result is true if g is C^1, g' is locally Lipschitz and g' is differentiable except at one point. Such a nonlinearity occurs naturally in [9] (when $a = d$ in the notation there).

2. The Open Set Problem for Jumping Nonlinearities

In this section, we show how to adapt the methods of §1 to show that for 'most' smooth domains Ω, $A_{-1} = \{(a, d) \in R^2 : -\Delta u = au^+ + du^-$ has a non-trivial solution in $\dot{W}^{1,2}(\Omega)\}$ has empty interior in R^2. Note that, as in [5], it is easy to prove that A_{-1} is closed in R^2. As we mentioned in the introduction, these are the first cases when $\dim\Omega > 1$ and A_{-1} is shown to have empty interior.

More precisely, following the notation of §1, we prove the following theorem.

THEOREM 2. *For a dense subset $\tilde{Z}$ of V, A_{-1} has empty interior when $\Omega = (I+\theta)(\Omega_0)$ provided that $\theta \in \tilde{Z}$.*

First note that, if there is a non-trivial solution u of

$$-\Delta u = au^+ + du^- \text{ in } \Omega_0,$$
$$u = 0 \text{ on } \partial\Omega_0,$$

then there is one with $\|u\|_2 = 1$. Hence it suffices to look at our problem on the surface $\hat{T} = \{u \in W^{2,p}(\Omega_0) : u = 0 \text{ on } \partial\Omega_0, \|u\|_2 = 1\}$. Since $\|u\|_2^2$ is a continuous polynomial, it is easy to see that $\hat{T}$ is a smooth (unbounded) surface in $\{u \in W^{2,p}(\Omega_0) : u = 0 \text{ on } \partial\Omega_0\}$. Here we choose $p > n$. We define the mapping $F(u,\theta)$ as in §1 (for $g(y) = -ay^+ - by^-$) and we define $\hat{F}$ to be a map of $U \times V \times R^2 \to Z$ (with U, V as there) by also considering it to be a function of a and d. Then $\hat{F}$ is strictly differentiable at any point $(u^0, \theta_0, a_o, d_0) \in U \times V \times R^2$, with $\hat{F}(u^0, \theta_0, a_0, d_0) = 0$. To prove this, we see as in §1 that it suffices to assume that $\theta_0 = 0$. By (3), $\hat{F}$ is the sum of two terms. The first term is independent of a and d and hence is strictly differentiable by the arguments of §1. The second term is, except for a smooth factor $\det T'$, $-au^+ - du^-$, which is independent of θ. As in §1, this is strictly differentiable in u (when $\hat{F} = 0$). Since it is smooth in a and d, it follows easily that it is strictly differentiable in (u, a, d) and hence our claim follows.

For fixed d, we consider the map $\hat{F}|_{\hat{T} \times V \times R \times \{d\}}$. We will prove that $\hat{F}'(z)$ is onto whenever $\hat{F}'(z) = 0$ (when $\hat{F}$ is considered restricted to $\hat{T} \times V \times R \times \{d\}$. We then argue much as in §1 to deduce for most $\theta \in V$, the map

$$(h, \tau) \longrightarrow -\Delta h - (a\chi_{u>0} + d\chi_{u<0})h - \tau u^+$$

is onto Z (where $h \in X$) when $u \neq 0$, $u \in X$, and $-\Delta u = au^+ + du^-$ on $\Omega = (I+\theta)(\Omega_0)$. The result will follow easily from this.

To prove this formally, first note that the map $-\Delta h - (a\chi_{u>0} + d\chi_{u<0})h$ is Fredholm of index zero considered as a map on X into Z and hence it is Fredholm of index -1 when considered as a map of $T_u(\hat{T})$ into Z (by Lemma V.1.5 in Goldberg [12]). Here $T_u(\hat{T})$ is the tangent space to $\hat{T}$ at u. We use that $T_u(\hat{T})$ is of codimension 1 in X. Hence, by the same result, the map

$$(h, \tau) \longrightarrow -\Delta h - (a\chi_{u>0} + d\chi_{u<0})h - \tau u^+$$

is Fredholm of index zero considered as a mapping of $T_u(\hat{T}) \times R$ to Z. Hence we are back to the situation in §1. If we prove that $\hat{F}'(u, \theta, a, d)$ maps $T_u(\hat{T}) \times Y \times R \times \{d\}$ onto Z whenever $\hat{F}(u, \theta, a, d) = 0$, we can use the same argument as in §1 to prove our claim. Note that here we are using the map $(u, \theta, a) \to \hat{F}(u, \theta, a, d)$.

Note also that we easily obtain the σ-properness because as in [19] it is easy to use standard regularity theory to prove that

$$\{(u,\theta,a) : \hat{F}(u,\theta,a,d) = 0 : \theta \in K,\ \epsilon \leq \|u\|_{2,p} \leq \epsilon^{-1},\ |a| \leq \epsilon^{-1}\}$$

is compact in X for every $\epsilon > 0$ if K is compact in Y. Hence it remains to prove the ontoness condition on $\hat{F}'$. First note that $\hat{F}$ is positive homogeneous in u when θ, a, d are fixed. Hence $\hat{F}'_u(u,\theta,a,d)u = 0$ whenever $\hat{F}(u,\theta,a,d) = 0$. Since u is clearly not in $T_u(\hat{T})$, we see that $\mathcal{R}(\hat{F}')$ is the same whether considered as a map of $T_u(\hat{T}) \times Y \times R \times \{0\}$ into Z or as a map of $X \times Y \times R \times \{0\}$ into Z. Thus it suffices to prove that $\hat{F}'$ maps $X \times Y \times R \times \{0\}$ onto Z. A sufficient condition for this is to prove that $\hat{F}'$ maps $X \times Y \times \{0\} \times \{0\}$ onto Z. This follows by the argument in §1. Hence there is a dense subset T_d of V which is a countable intersection of open sets such that, if $\theta \in T_d$, $(h,\tau) \to \hat{F}'_u(u,\theta,a,d)h + \hat{F}'_a(u,\theta,a,d)\tau$ is onto Z whenever $\hat{F}(u,\theta,a,d) = 0$. In other words, if $\theta \in T_d$, $T = I + \theta$ and $\Omega = (I+\theta)\Omega_0$, then the map

$$(h,\tau) \to -\Delta h - (a\chi_{u>0} + d\chi_{u<0})h - \tau u^+$$

maps $T_u(\hat{T}) \times R$ onto Z whenever $u \in \hat{T}$, $a \in R$ and $-\Delta u = au^+ + du^-$. Note that the construction of V is independent of d. Choose a countable dense subset $\{d_n\}_{n=1}^{\infty}$ of R. Then $T_\infty = \bigcap_{n=1}^{\infty} T_{d_n}$ is dense in V by the Baire category theory since each T_{d_n} is a countable intersection of dense open sets. On the other hand, we will prove in a moment that,

$$\text{(7)} \qquad \text{if } \theta \in T_d, \qquad \{a : (a,d) \in A_{-1}\} \text{ is countable.}$$

Hence we see that if $\theta \in T_\infty$, $\{a : (a,d) \in A_{-1}\}$ is countable for every n. This implies that A_{-1} does not contain an open set because if $(a_0,d_0) \in \operatorname{int} A_{-1}$, then $A_{-1} \cap (R \times \{d\})$ would contain an open vertical segment for all d close to d_0. In particular, since $\{d_n\}$ is dense in R, $A_{-1} \cap (R \times \{d_n\})$ would contain a vertical line segment for some n. This contradicts our countability claim (7) above.

Hence our result is proved if we prove (7) for $\theta \in T_d$. It suffices to prove that, if $\theta \in T_d$, then $S = \{(a,u) \in R \times \hat{T} : -\Delta u - au^+ - du^- = 0 \text{ in } \Omega\}$ consists of isolated points in $R \times W^{1,p}(\Omega)$ if $p > n$. This suffices because standard regularity theory implies that $S_n \equiv \{(a,u) \in S : |a| \leq n\}$ is bounded in $R \times W^{2,p}(\Omega)$ and hence is compact in $R \times W^{1,p}(\Omega)$. Thus by the isolatedness, S_n is finite and hence $S = \bigcup_{n=1}^{\infty} S_n$ is countable.

To prove our isolatedness claim, we first note that it follows easily from standard regularity theory that it suffices to prove isolatedness in $R \times W^{2,p}(\Omega)$. The required result follows from a simple contraction mapping theorem argument. The details appear in Pope [17, Theorem 3.3.1]. (If $(a_0,u_0) \in S$ one shows as before that the map $(a,u) \to -\Delta u - au^+ - du^-$ is strictly differentiable at

(a_0, u_0) and the derivative maps $R \times T_{u_0}(\hat{T})$ bijectively onto $L^p(\Omega)$. Hence the result follows from the contraction mapping theorem.) This completes the proof.

REMARKS.

1) A similar, but easier, argument implies that if a and d are fixed then, for most Ω, $(a, d) \notin A_{-1}$. This shows that an assumption in [9] holds for generic Ω. Indeed Dr. Yihong Du has noted that this can be used to give a slightly shorter proof of Theorem 2. The argument above has the advantage that it gives more information on the structure of A_{-1} for "generic" Ω.

2) Some of our ideas have other uses. It can be proved that if $W \subseteq R^2$ is open and if $-\Delta - (a\chi_{u>0} + d\chi_{u<0})I$ has at most a two-dimensional kernel (for Dirichlet boundary conditions) whenever $(a, d) \in W$, $u \in W^{2,p}(\Omega) \cap \dot{W}^{1,2}(\Omega)$, and

$$\{x \in \Omega : u(x) = 0\}$$

has zero measure, then $A_{-1} \cap W$ does not contain an open set. This shows that the difficulties in the open set problem are caused by multiple eigenvalues. Micheletti [15] already had results close to, but slightly weaker than, this. The idea is to use the strict differentiability and ideas similar to that in the above proof to locally reduce our problem to one for a mapping of R^1 to R^1 and then use Sard's theorem and Pope's result. More precisely, one can show that, if u_0 is a solution of $-\Delta u = au^+ + du^-$ vanishing on $\partial\Omega$, then, for fixed d greater than λ_1 the set of possible solutions $\tilde{W}$ nearby on $\hat{T} \times R^2$ is contained in $\{(u(s), a(s), d)\}$, where u and a are C^1 in s and $s \in R$. One proves by differentiating the original equation in s that $a'(s_0) = 0$ whenever $(u(s_0), a(s_0), d)$ is non-isolated in $\tilde{W}$ and thus by Sard's theorem applied to $a(s)$ the set

$$\{a : -\Delta u = au^+ + du^- \text{ has a nontrivial solution } u \text{ near } u_0 \text{ with } a \text{ near } a(0)\}$$

has measure zero. It follows easitly that $W \cap \{R \times \{d\}) \cap A_{-1}$ has measure zero and hence $A_{-1} \cap W$ has measure zero. In particular, $A_{-1} \cap W$ contains no open set. This proof does not seem to generalize to a larger dimensional kernel because a seems to be only C^1 and we are then unable to apply Sard's theorem. If a large symmetry group acts on Ω, one can sometimes permit even larger kernels effectively by factoring out the symmetry in some way. For example, one can prove that, if Ω is a ball in R^n and if $\alpha < \beta$ are successive eigenvalues of $-\Delta$ such that one of them is simple and the symmetry group $O(n)$ acts transitively on the sphere in the eigenspace of the other eigenvalue, then

$$\{(a, d) : \alpha_0 < a, d < \beta_0, (a, d) \in A_{-1}\}$$

contains no open set where

$$\alpha_0 = \sup\{\lambda : \lambda \in \sigma(-\Delta),\, \lambda < \alpha,\, \lambda < \beta\}$$

and

$$\beta_0 = \inf\{\lambda : \lambda \in \sigma(-\Delta),\, \lambda > \alpha,\, \lambda > \beta\}.$$

3) One possible way one might try to construct an example where A_{-1} contains an open set would be to start with a case where $-\Delta u = au^- + du^-$ has a smooth manifold of solutions for fixed a and d due to symmetries of Ω and then try to perturb Ω so that it is less symmetric but for which we still have a nearby manifold M of solutions of our equation but such that a and d vary on M.

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E. N. DANCER
Department of Mathematics, Statistics and Computing Science
The University of New England, Armidale, NSW 2351, AUSTRALIA