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NIVELOIDS

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Dedicated to Ky Fan

Let X be a nonempty set. A functional $T : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is a *niveloid* if, for each $f, g \in \overline{\mathbb{R}}^X$,

$$f \leq g \Rightarrow T(f) \leq T(g), \tag{0.1}$$

$$\forall_{r \in \mathbb{R}} \quad T(f + r) = T(f) + r. \tag{0.2}$$

The functionals satisfying (0.1) are called *isotone*; those fulfilling (0.2) will be called *vertically invariant*.

Niveloids occur among functionals used in convex analysis. For example, for each fixed element x_0 of a vector space X the convex hull on $\overline{\mathbb{R}}^X$ evaluated at x_0 is a niveloid. If X is a topological vector space, then the lower semicontinuous convex hull evaluated at x_0 is a niveloid. This is also the case of biconjugations. Actually, the latter exceeds the framework of classical as well as generalized convexity.

Suppose that $\langle\, , \,\rangle : X \times Y \to \overline{\mathbb{R}}$. Then the *lower* and *upper biconjugates* at x_0 of a function $f : X \to \overline{\mathbb{R}}$ (with respect to $\langle\, , \,\rangle$) are

$$f^{**}(x_0) = \sup_{y \in Y} [\langle x_0, y\rangle \underset{\cdot}{+} \inf_{x \in X} (f(x) \dot{-} \langle x, y\rangle)], \tag{0.3}$$

$$f_{**}(x_0) = \inf_{y \in Y} [\langle x_0, y\rangle \dot{+} \sup_{x \in X} (f(x) \underset{\cdot}{-} \langle x, y\rangle)]. \tag{0.4}$$

These notions are the extensions due to J. J. Moreau [7, 8] for arbitrary "coupling" functions $\langle\, , \,\rangle$ of the biconjugates of Fenchel (and Legendre) introduced

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in the framework of bilinear coupling functions. The symbols $\underset{\cdot}{+}$, $\dot{+}$ denote the extensions of addition to $\overline{\mathbb{R}}$ such that $(+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty$ and $(+\infty) \dot{+} (-\infty) = (+\infty) \dot{+} (-\infty) = +\infty$. The following convention is adopted: $r \underset{\cdot}{-} s = r \underset{\cdot}{+} (-s)$ and $r \dot{-} s = r \dot{+} (-s)$.

Sure enough, $f \mapsto f^{**}(x_0)$ and $f \mapsto f_{**}(x_0)$ are niveloids. Other examples of niveloids arise in analysis and topology. For any family $\mathcal{A}$ of subsets of X,

$$\operatorname{liminf}_{\mathcal{A}} f = \sup_{A \in \mathcal{A}} \inf_{A} f, \tag{0.5}$$

$$\operatorname{limsup}_{\mathcal{A}} f = \inf_{A \in \mathcal{A}} \sup_{A} f \tag{0.6}$$

are niveloids. In the case where $\mathcal{A}$ is a semifilter base (i.e. $\emptyset \notin \mathcal{A} \neq \emptyset$), these are precisely *limitoids* of G. H. Greco [6], i.e., those functionals that fulfil (0.1) and

$$L(\varphi(f)) = \varphi(L(f)) \tag{0.7}$$

for every homomorphism φ of $\overline{\mathbb{R}}$ and each $f \in \overline{\mathbb{R}}^X$. Recall that the lattice homomorphisms of $\overline{\mathbb{R}}$ are precisely the continuous nondecreasing functions into $\overline{\mathbb{R}}$.

In particular, Γ-functionals of E. De Giorgi [2] are limitoids, hence niveloids. These are functionals on $X = X_1 \times \ldots \times X_n$ constructed, by induction, with the aid of semifilter bases $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ (on $X_1, X_2, \ldots, X_n$, respectively) and signs $\alpha_1, \alpha_2, \ldots, \alpha_n \in \{-, +\}$ such that

$$\Gamma(\mathcal{A}^-)f = \operatorname{liminf}_{\mathcal{A}} f, \qquad \Gamma(\mathcal{A}^+)f = \operatorname{limsup}_{\mathcal{A}} f$$

and

$$\begin{aligned}\Gamma(\mathcal{A}_1^{\alpha_1}, \ldots, \mathcal{A}_{n-1}^{\alpha_{n-1}}, \mathcal{A}_n^{\alpha_n})f \\ = \operatorname{ext}^{-\alpha_n}_{A_n \in \mathcal{A}_n} \Gamma(\mathcal{A}_1^{\alpha_1}, \ldots, \mathcal{A}_{n-1}^{\alpha_{n-1}}) \operatorname{ext}^{\alpha_n}_{x_n \in A_n} f(\ldots, x_n),\end{aligned} \tag{0.8}$$

where $\operatorname{ext}^+ = \sup$, $\operatorname{ext}^- = \inf$, $-- = +$, $-+ = -$.

Even more involved functionals constructed with the aid of successive extremization (see e.g. [5]) are limitoids and thus representable in the form (0.5) (and (0.6)). They find applications in the theory of general convergences.

Γ-functionals and their extensions may be described as mixtures of upper and lower limits. But the sense of this statement is broader, as (0.5) and (0.6) may well differ from classical topological unilateral limits. For instance, if $\mathcal{A}$ denotes the family of all concave subsets (of a vector space X) that contain a given point x_0, then (0.5) becomes the quasi-convex regularization of f evaluated at x_0 and (0.6) its quasi-concave regularization.

Some mixtures of limitoids and biconjugation-like niveloids may be found in the calculus of variations (L. Cesari [1]) under the name of Q-limits.

This (not exhaustive) list of examples explains our interest in niveloids. We shall investigate their representation in terms of some familiar constructions. We shall study as well the structure of the complete lattice of niveloids on a given set.

It turns out that every niveloid T may be represented as a lower (and as an upper) Γ-*fuzzy functional* (as introduced by S. Dolecki in [3]), namely there exist S and R mapping $\overline{\mathbb{R}}^X$ to $\overline{\mathbb{R}}$ such that, for each $f \in \overline{\mathbb{R}}^X$,

$$T(f) = \sup_{w \in \overline{\mathbb{R}}^X} [S(w) \underset{\cdot}{+} \inf_{x \in X} (f(x) \dot{-} w(x))], \tag{0.9}$$

$$T(f) = \inf_{w \in \overline{\mathbb{R}}^X} [R(w) \dot{+} \sup_{x \in X} (f(x) \underset{\cdot}{-} w(x))] \, . \tag{0.10}$$

Actually, every niveloid satisfies (0.9) and (0.10) itself, that is, we may put $S = R = T$.

1. First representation theorem

Let X be a nonempty set. We shall denote by $\mathcal{N}(X)$ the set of all the niveloids on $\overline{\mathbb{R}}^X$. The functional identically equal to $+\infty$ and that identically equal to $-\infty$ are niveloids. They will be referred to as *degenerate*.

If a niveloid T is *nondegenerate*, then, in view of properties (0.1) and (0.2), $T(-\infty) = -\infty$ and $T(+\infty) = +\infty$. The niveloids taking only infinite values are called *improper*, all the others are *proper*. In other words, T is proper whenever there exists $f \in \overline{\mathbb{R}}^X$ such that $T(f) \in \mathbb{R}$.

If S is an arbitrary functional on $\overline{\mathbb{R}}^X$, then its *dual functional* S^* is defined by

$$S^*(f) = -S(-f).$$

The dual of a niveloid is a niveloid.

A key property is that the set of niveloids, considered in the complete lattice of all the functionals $\mathcal{F}(X)$ ordered pointwise, is closed under the operations of least upper bound and of greatest lower bound. Namely, if $\{T_i\}_{i \in I}$ is a family of niveloids indexed by a (possibly empty) set I, then

$$\bigvee_{i \in I} T_i \quad \text{and} \quad \bigwedge_{i \in I} T_i \tag{1.1}$$

are niveloids. Indeed, if $\{T_i\}_{i \in I}$ are vertically invariant (resp. isotone), then (1.1) are vertically invariant (resp. isotone).

More precisely, for every functional F, there exists the least isotone functional (resp. vertically invariant functional, niveloid) that is greater than F. We shall denote them by

$$\mathbf{I}^+(F), \quad \mathbf{V}^+(F), \quad \mathbf{N}^+(F),$$

and call them *upper projections*.

As well, there exists the greatest isotone functional (resp. vertically invariant functional, niveloid) that is smaller than F:

$$\mathbf{I}^-(F), \quad \mathbf{V}^-(F), \quad \mathbf{N}^-(F).$$

They will be called *lower projections*.

Theorem 1.1. *Let F be an arbitrary functional. Then*

$$\mathbf{N}^{+}(F)(f) = \sup_{w\in\overline{\mathbb{R}}^X} [F(w) \underset{\cdot}{+} \inf (f \dot{-} w)], \tag{1.2}$$

$$\mathbf{N}^{-}(F)(f) = \inf_{w\in\overline{\mathbb{R}}^X} [F(w) \dot{+} \sup(f \underset{\cdot}{-} w)]. \tag{1.2*}$$

Proof. In order to prove (1.2), we first observe that

$$\mathbf{I}^{+}(F)(f) = \sup_{g\leq f} F(g) \quad \text{and} \quad \mathbf{V}^{+}(G)(f) = \sup_{r\in\mathbb{R}}(G(f+r)-r). \tag{1.3}$$

One notes that if F is vertically invariant, then $\mathbf{I}^{+}(F)$ is also vertically invariant. Similarly, if G is isotone, then $\mathbf{V}^{+}(G)$ is also isotone. Therefore, $\mathbf{N}^{+} = \mathbf{I}^{+}\mathbf{V}^{+} = \mathbf{V}^{+}\mathbf{I}^{+}$. Hence,

$$\mathbf{N}^{+}(F)(f) = \sup_{r\in\mathbb{R}} \sup_{g\leq f}(F(g-r)+r) = \sup_{r\in\mathbb{R}} \sup_{w+r\leq f}(F(w)+r)$$

where $w \in \overline{\mathbb{R}}^X$. In other words, we maximize over the relation $\{(r,w) : w + r \leq f\}$. Consequently,

$$\mathbf{N}^{+}(F)(f) = \sup_{w\in\overline{\mathbb{R}}^X} \sup_{r\leq\inf (f \dot{-} w)} (F(w)+r) = \sup_{w\in\overline{\mathbb{R}}^X} (F(w) \underset{\cdot}{+} \sup_{r\leq\inf (f \dot{-} w)} r),$$

proving (1.2). The equality (1.2*) follows from $\mathbf{N}^{-}(F) = [\mathbf{N}^{+}(F^{*})]^{*}$. □

Of course, a functional T is a niveloid if and only if it is equal to $\mathbf{N}^{+}(T)$ (resp. $\mathbf{N}^{-}(T)$). Therefore we have

Theorem 1.2 (First Representation Theorem). *A functional T is a niveloid if and only if one of the following formulae hold:*

$$T(f) = \sup_{w}[T(w) \underset{\cdot}{+} \inf (f \dot{-} w)] \quad \textit{and} \quad T(f) = \inf_{w}[T(w) \dot{+} \sup(f \underset{\cdot}{-} w)].$$

Here the extremizations with respect to w are carried over $\overline{\mathbb{R}}^X$. They amount to $\sup_{\{w:T(w)>-\infty\}}$ and $\inf_{\{w:T(w)<+\infty\}}$ respectively. The first representation theorem immediately yields

Corollary 1.3. *The following statements are equivalent:*

(1.4) T *is a niveloid,*

(1.5) $\forall_{f,g\in\overline{\mathbb{R}}^X} \quad T(f) \dot{-} T(g) \geq \inf (f \dot{-} g),$

(1.6) $\forall_{f,g\in\overline{\mathbb{R}}^X} \quad \sup(f \underset{\cdot}{-} g) \geq T(f) \underset{\cdot}{-} T(g).$

A function $f \in \overline{\mathbb{R}}^X$ is said to be *proper* if there exists x for which $f(x) \in \mathbb{R}$. Note that this definition of properness is less restrictive than the usual one, in that it does not require $f(x) > -\infty$ for every x.

A family of functions is said to be *proper* whenever its elements are proper functions. For every niveloid T, the family $\{f : -\infty < T(f) < +\infty\}$ is proper. In fact, if $T(f) \in \mathbb{R}$, then, by (1.5), $0 \geq \inf (f \dot{-} f)$, which amounts to the properness of f.

If T is a niveloid, then

$$T(g) \geq 0 \Leftrightarrow \forall_{f \in \overline{\mathbb{R}}^X} \; \inf\,(f \dot{-} g) \leq T(f). \tag{1.7}$$

Indeed, if $T(g) \geq 0$, then by Corollary 1.3, (1.7) holds. Conversely, (1.7) entails that $T(g) \geq \inf\,(g \dot{-} g) \geq 0$.

Given $x \in X$, $E_x(f) = f(x)$ is a niveloid. The niveloids E_x will be referred to as *evaluations* on X. The First Representation Theorem immediately yields

COROLLARY 1.4. *The set $\mathcal{N}(X)$ of all niveloids is the smallest collection $\mathcal{C}$ of functionals on $\overline{\mathbb{R}}^X$ such that:*

(1.8) $\forall_{x \in X}\, E_x \in \mathcal{C}$,
(1.9) $r \in \mathbb{R}$ *and* $T \in \mathcal{C} \Rightarrow T + r \in \mathcal{C}$,
(1.10) $(T_i)_{i \in I} \subset \mathcal{C} \Rightarrow \bigvee_{i \in I} T_i$ *and* $\bigwedge_{i \in I} T_i \in \mathcal{C}$.

The fact that $\mathcal{N}(X)$ is closed both under the least upper and the greatest lower bounds entails

THEOREM 1.5. *Let $\{S_i\}_{i \in I}$ be a family of arbitrary functionals. Then*

$$\mathbf{N}^+\Big(\bigvee_{i \in I} S_i\Big) = \bigvee_{i \in I} \mathbf{N}^+(S_i) \quad \textit{and} \quad \mathbf{N}^-\Big(\bigwedge_{i \in I} S_i\Big) = \bigwedge_{i \in I} \mathbf{N}^-(S_i). \tag{1.11}$$

REMARK 1.6. The reader can use Theorem 1.5 to obtain several representations of the lower and upper projections $\mathbf{N}^+$ and $\mathbf{N}^-$. For example, one can recover (1.2) and (1.2*) in this way. Take an arbitrary functional S on $\overline{\mathbb{R}}^X$. For every $p \in \overline{\mathbb{R}}^X$, define the functionals S_p^- and S_p^+ by

$$S_p^-(f) = \begin{cases} S(p) & \text{if } f = p, \\ -\infty & \text{otherwise,} \end{cases} \qquad \text{and} \qquad S_p^+(f) = \begin{cases} S(p) & \text{if } f = p, \\ +\infty & \text{otherwise.} \end{cases}$$

Of course, $S = \bigvee_{p \in \mathcal{P}} S_p^-$ and $S = \bigwedge_{p \in \mathcal{P}} S_p^+$. Thus, in view of Theorem 1.5, we have

$$\mathbf{N}^+(S)(f) = \sup_{p \in \mathcal{P}} [\mathbf{N}^+(S_p^+)(f)] \quad \text{and} \quad \mathbf{N}^-(S)(f) = \inf_{p \in \mathcal{P}} [\mathbf{N}^+(S_p^-)(f)]. \tag{1.12}$$

An immediate computation yields

$$\begin{aligned} &\mathbf{N}^+(S_p^-)(f) = S(p) \underset{\cdot}{+} \inf\,(f \dot{-} p) \quad \text{and} \\ &\mathbf{N}^-(S_p^+)(f) = S(p) \dot{+} \sup(f \underset{\cdot}{-} p); \end{aligned} \tag{1.13}$$

hence, by (1.12) and (1.13), we have recovered Theorem 1.1. $\square$

2. The operators ∇ and Δ

Let $f, p \in \overline{\mathbb{R}}^X$. We define the *inf-* and *sup-convolutions* of f and $-p$ by

$$f\,\nabla(-p) = \inf(f \dot{-} p) \quad \text{and} \quad f\,\Delta(-p) = \sup(f \,\underset{\cdot}{-}\, p). \tag{2.1}$$

For a fixed p, (2.1) defines two functionals which are niveloids. They are proper if and only if p is a proper function.

Let $\mathcal{P} \subset \overline{\mathbb{R}}^X$. Then the functionals $\nabla_{\mathcal{P}}, \Delta_{\mathcal{P}}$ defined by

$$\nabla_{\mathcal{P}} f = \sup_{p\in\mathcal{P}} \inf(f \dot{-} p) \quad \text{and} \quad \Delta_{\mathcal{P}} f = \inf_{p\in\mathcal{P}} \sup(f \,\underset{\cdot}{-}\, p) \tag{2.2}$$

are niveloids. In view of Theorem 1.1 we have

Proposition 2.1. *The operator $\nabla_{\mathcal{P}}$ is the upper projection (on $\mathcal{N}(X)$) of the functional equal to 0 on $\mathcal{P}$ and $-\infty$ elsewhere. Similarly, $\Delta_{\mathcal{P}}$ is the lower projection of the functional equal to 0 on $\mathcal{P}$ and $+\infty$ elsewhere.*

Observe that

$$\begin{aligned} &\nabla^*_{\mathcal{P}} := (\nabla_{\mathcal{P}})^* = \Delta_{(-\mathcal{P})}, \qquad &&\Delta^*_{\mathcal{P}} := (\Delta_{\mathcal{P}})^* = \nabla_{(-\mathcal{P})}, \\ &\nabla_{\emptyset} = -\infty, &&\Delta_{\emptyset} = +\infty. \end{aligned} \tag{2.3}$$

Throughout this paper, the usual conventions about extrema are respected: $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$.

The functionals $\nabla^\infty_{\mathcal{P}}$, $\Delta^\infty_{\mathcal{P}}$ defined by

$$\nabla^\infty_{\mathcal{P}} f = \sup_{p\in\mathcal{P}}[+\infty \,\underset{\cdot}{+}\, \inf(f \dot{-} p)] \quad \text{and} \quad \Delta^\infty_{\mathcal{P}} f = \inf_{p\in\mathcal{P}}[-\infty \dot{+} \sup(f \,\underset{\cdot}{-}\, p)] \tag{2.4}$$

are niveloids. They are always improper: $\nabla^\infty_{\mathcal{P}}$ is the smallest niveloid equal to $+\infty$ on $\mathcal{P}$ and $\Delta^\infty_{\mathcal{P}}$ is the greatest niveloid equal to $-\infty$ on $\mathcal{P}$. One has the following relationships:

$$(\nabla^\infty_{\mathcal{P}})^* = \Delta^\infty_{(-\mathcal{P})}, \qquad (\Delta^\infty_{\mathcal{P}})^* = \nabla^\infty_{(-\mathcal{P})}, \tag{2.5}$$

$$\nabla_{\mathcal{P}} \text{ is improper} \Leftrightarrow \mathcal{P} \subset \{\nabla_{\mathcal{P}} = +\infty\} \Leftrightarrow \nabla_{\mathcal{P}} = \nabla^\infty_{\mathcal{P}}, \tag{2.6}$$

$$\Delta_{\mathcal{P}} \text{ is improper} \Leftrightarrow \mathcal{P} \subset \{\Delta_{\mathcal{P}} = -\infty\} \Leftrightarrow \Delta_{\mathcal{P}} = \Delta^\infty_{\mathcal{P}}. \tag{2.6*}$$

Proposition 2.2. $\nabla^\infty_{\mathcal{P}} = \nabla_{\mathcal{P}+\mathbb{R}}$ *and* $\Delta^\infty_{\mathcal{P}} = \Delta_{\mathcal{P}+\mathbb{R}}$. *Hence* $\nabla^\infty_{\mathcal{P}} = \nabla_{\mathcal{P}}$ *and* $\Delta^\infty_{\mathcal{P}} = \Delta_{\mathcal{P}}$ *whenever* $\mathcal{P}+\mathbb{R} \subset \mathcal{P}$.

Proof. In fact, in this case

$$\nabla_{\mathcal{P}+\mathbb{R}}\, f = \sup_{p\in\mathcal{P}} \sup_{r\in\mathbb{R}} \inf(f \dot{-} (p+r)) = \sup_{p\in\mathcal{P}}[(\sup_{r\in\mathbb{R}} r) \,\underset{\cdot}{+}\, \inf(f \dot{-} p)] = \nabla^\infty_{\mathcal{P}} f,$$

proving the first equality. The second one follows by duality. $\square$

It is clear that every niveloid T is completely determined by the families of functions $\{T=0\}$ and $\{T=+\infty\}$ (or $\{T=0\}$ and $\{T=-\infty\}$). An analytical version of this elementary fact is given by the following theorem.

Theorem 2.3 (Second Representation Theorem). *Let T be a niveloid. Then* $\nabla_{\{T=0\}} \vee \nabla_{\{T=+\infty\}} = T = \Delta_{\{T=0\}} \wedge \Delta_{\{T=-\infty\}}$.

Proof. By the First Representation Theorem (Theorem 1.2),

$$T(f) = \sup_{T(w)>-\infty} [T(w) \dotplus \inf(f \dot{-} w)].$$

This equality yields

$$\begin{aligned} T(f) &= (\sup_{r\in\mathbb{R}} \sup_{T(w)=r} (r + \inf(f \dot{-} w))) \vee (\sup_{T(w)=+\infty} [+\infty \dotplus \inf(f \dot{-} w)]) \\ &= (\sup_{r\in\mathbb{R}} \sup_{T(v)=0} \inf(f \dot{-} v)) \vee \nabla^{\infty}_{\{T=+\infty\}}(f) \\ &= (\sup_{T(v)=0} \inf(f \dot{-} v)) \vee \nabla^{\infty}_{\{T=+\infty\}}(f). \end{aligned}$$

Now, $T(w)=+\infty$ and $t\in\mathbb{R}$ imply that $T(w+t)=+\infty$, so that $\{T=+\infty\}+\mathbb{R}\subset\{T=+\infty\}$. Therefore, by Proposition 2.2, the above equality becomes $T=\nabla_{\{T=0\}} \vee \nabla_{\{T=+\infty\}}$. The second equality of the theorem is obtained by duality. □

3. Uniform and improper topologies

The *upper uniform topology* τ^+ on $\overline{\mathbb{R}}^X$ is such that the sets

$$\{g : g \le f + \varepsilon\}_{\varepsilon>0} \tag{3.1}$$

constitute a neighbourhood base of f. A neighbourhood base of f for the *lower uniform topology* τ^- is

$$\{h : h \ge f - \varepsilon\}_{\varepsilon>0}. \tag{3.1*}$$

Observe that, for every $\mathcal{P}\subset\overline{\mathbb{R}}^X$,

$$\begin{aligned} \mathrm{cl}_{\tau^+}\mathcal{P} &= \{f : \forall_{\varepsilon>0}\ \exists_{p\in\mathcal{P}}\ p-\varepsilon\le f\}, \\ \mathrm{cl}_{\tau^-}\mathcal{P} &= \{f : \forall_{\varepsilon>0}\ \exists_{q\in\mathcal{P}}\ f\le q+\varepsilon\}; \end{aligned} \tag{3.2}$$

in other words, using the ∇ and Δ operators

$$\mathrm{cl}_{\tau^+}\mathcal{P} = \{\nabla_{\mathcal{P}} \ge 0\} \quad \text{and} \quad \mathrm{cl}_{\tau^-}\mathcal{P} = \{\Delta_{\mathcal{P}} \le 0\}; \tag{3.3}$$

hence all the τ^+-closed (resp. τ^--closed) sets are of the form $\{\nabla_{\mathcal{P}}\ge 0\}$ (resp. $\{\Delta_{\mathcal{P}}\le 0\}$), where $\mathcal{P}\subset\overline{\mathbb{R}}^X$.

Theorem 3.1. *The upper* (*resp. lower*) *uniform topology is the coarsest topology for which every niveloid is upper* (*resp. lower*) *semicontinuous.*

Proof. Let T be a niveloid and f a function such that $T(f) < +\infty$. Fix $\varepsilon > 0$. Then, for every $g \le f + \varepsilon$, $T(g) \le T(f+\varepsilon) = T(f) + \varepsilon$, that is, T does not exceed $T(f) + \varepsilon$ on the τ^+-neighbourhood $\{g : g \le f + \varepsilon\}$ of f; hence T is τ^+-upper semicontinuous. Consider now a function f and the niveloid $\Delta_{\{f\}}$. Then $\Delta_{\{f\}} f \le 0$, so that, for every $\varepsilon > 0$, $\{\Delta_{\{f\}} \le \varepsilon\} = \{g : \sup(g \dot{-} f) \le \varepsilon\} = \{g : g \le f + \varepsilon\}$ is a neighbourhood of f in each topology for which $\Delta_{\{f\}}$ is upper semicontinuous. By duality, the second part of the statement is valid. □

The least upper bound $\tau = \tau^+ \vee \tau^-$ (in the lattice of all topologies) is strictly stronger than the uniform topology on $\overline{\mathbb{R}}^X$, induced by the usual uniformity of $\overline{\mathbb{R}}$. Indeed, the family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = n \wedge x$, is τ-closed, but not closed in the uniform topology of $\overline{\mathbb{R}}^X$.

Let $\mathrm{cl}_{\tau^\pm}$ denote $\mathrm{cl}_{\tau^+} \cap \mathrm{cl}_{\tau^-}$; that is, by (3.2) and (3.3), for every $\mathcal{P} \subset \overline{\mathbb{R}}^X$,

$$\mathrm{cl}_{\tau^\pm}\mathcal{P} = \{\nabla_{\mathcal{P}} \ge 0\} \cap \{\Delta_{\mathcal{P}} \le 0\} = \{f : \forall_{\varepsilon>0}\ \exists_{p,q\in\mathcal{P}}\ p - \varepsilon \le f \le q + \varepsilon\}.$$

Observe that $\mathrm{cl}_\tau \subset \mathrm{cl}_{\tau^\pm}$, but $\mathrm{cl}_\tau \ne \mathrm{cl}_{\tau^\pm}$. For example, for a family $\mathcal{F}$ composed of two functions f and g such that $f \ne g$ and $f \le g$ we have $\mathcal{F} = \mathrm{cl}_\tau\mathcal{F} \ne \mathrm{cl}_{\tau^\pm}\mathcal{F} = \{h : f \le h \le g\}$.

The operator $\mathrm{cl}_{\tau^\pm}$ is a *closure operator*, that is, by definition: *isotone* (i.e. $\mathrm{cl}_{\tau^\pm}\mathcal{P} \subset \mathrm{cl}_{\tau^\pm}\mathcal{B}$ for all $\mathcal{B} \subset \mathcal{P} \subset \overline{\mathbb{R}}^X$), *expansive* (i.e. $\mathcal{P} \subset \mathrm{cl}_{\tau^\pm}\mathcal{P}$ for all $\mathcal{P} \subset \overline{\mathbb{R}}^X$) and *idempotent* (i.e. $\mathrm{cl}_{\tau^\pm}\mathrm{cl}_{\tau^\pm} = \mathrm{cl}_{\tau^\pm}$). Moreover, it is easy to check that

$$\mathrm{cl}_{\tau^+} = \mathrm{cl}_{\tau^+}\mathrm{cl}_{\tau^\pm} = \mathrm{cl}_{\tau^\pm}\mathrm{cl}_{\tau^+} \quad \text{and} \quad \mathrm{cl}_{\tau^-} = \mathrm{cl}_{\tau^-}\mathrm{cl}_{\tau^\pm} = \mathrm{cl}_{\tau^\pm}\mathrm{cl}_{\tau^-}. \tag{3.4}$$

We shall call $\mathcal{P}$ *$\tau^\pm$-closed* if $\mathrm{cl}_{\tau^\pm}\mathcal{P} = \mathcal{P}$; $\tau^\pm$-closed families are called *superconvex* in [4]. The families that are either τ^+-or τ^--closed, are $\tau^\pm$-closed. Moreover, the intersection of $\tau^\pm$-closed families is $\tau^\pm$-closed. In particular, if T is a niveloid then $\{T > -\infty\}$, $\{T < +\infty\}$, $\{T = 0\}$ and $\{f : r \le T(f) \le s\}$ are all $\tau^\pm$-closed for each $r, s \in \overline{\mathbb{R}}$.

A family $\mathcal{P}$ is said to be *order-convex* if $f, g \in \mathcal{P}$ and $f \le h \le g$ imply $h \in \mathcal{P}$. A $\tau^\pm$-closed family is both order-convex and τ-closed. Observe that an order-convex τ-closed family is not necessarily $\tau^\pm$-closed. For example, consider the family $\{1/n - \psi_{\{n\}} : 0 \ne n \in \mathbb{N}\} \cup \{-1/n + \psi_{\{-n\}} : 0 \ne n \in \mathbb{N}\}$, where $\psi_{\{n\}} : \mathbb{R} \to \overline{\mathbb{R}}$ is equal to 0 at n and $+\infty$ elsewhere.

As we know, $\inf(f \dot{-} g) \ge 0$ is tantamount to the usual pointwise order: $f \ge g$. This order is used to define upper and lower uniformities. Now

$$f \sqsupseteq g \Leftrightarrow \inf(f \dot{-} g) > -\infty \tag{3.5}$$

defines a preorder on $\overline{\mathbb{R}}^X$ which will be used to define the upper and lower improper topologies. Of course, "$f \sqsupseteq g$" means that there exists $r \in \mathbb{R}$ such that $f \geq g + r$. Two functions f, g are equivalent with respect to this preorder (i.e., $f \sqsupseteq g$ and $f \sqsubseteq g$) if and only if there exists a bounded function b such that $f = g + b$.

A subset $\mathcal{H}$ of $\overline{\mathbb{R}}^X$ is called a $(+\infty)$*family* whenever $h \in \mathcal{H}$ and $f \sqsupseteq h$ imply that $f \in \mathcal{H}$. Dually, $\mathcal{G}$ is called a $(-\infty)$*family* if $h \in \mathcal{H}$ and $f \sqsubseteq h$ imply $f \in \mathcal{G}$. Note that $\mathcal{H}$ is a $(+\infty)$family if and only if $\mathcal{H}^c$ is a $(-\infty)$family.

If T is a niveloid, then $\{T = +\infty\}$ is a $(+\infty)$family and $\{T = -\infty\}$ is a $(-\infty)$family. Conversely, for every $(+\infty)$family $\mathcal{H}$ (resp. $(-\infty)$family $\mathcal{G}$), one has $\mathcal{H} = \{\nabla_{\mathcal{H}} = +\infty\}$ (resp. $\mathcal{G} = \{\Delta_{\mathcal{G}} = -\infty\}$).

Since every intersection and every union of $(+\infty)$families is a $(+\infty)$family, all the $(+\infty)$families are the closed sets of a topology that we shall call the *upper improper topology* and denote by ξ^+. Dually, the collection of all the $(-\infty)$families constitutes the collection of closed sets of a topology called the *lower improper topology* and denoted by ξ^-.

Hence, for every $\mathcal{P} \subset \overline{\mathbb{R}}^X$, we have

$$\mathrm{cl}_{\xi^+} \mathcal{P} = \{f : \exists_{\, h \in \mathcal{P}}\ f \sqsupseteq h\} = \{\nabla_{\mathcal{P}} > -\infty\}, \tag{3.6}$$

$$\mathrm{cl}_{\xi^-} \mathcal{P} = \{f : \exists_{\, h \in \mathcal{P}}\ f \sqsubseteq h\} = \{\Delta_{\mathcal{P}} < +\infty\}. \tag{3.6*}$$

PROPOSITION 3.2. *The upper* (*resp. lower*) *improper topology is the coarsest topology for which every improper niveloid is upper* (*resp. lower*) *semicontinuous.*

Let $\mathrm{cl}_{\xi^\pm}$ denote $\mathrm{cl}_{\xi^+} \cap \mathrm{cl}_{\xi^-}$. Hence, by (3.6) and (3.6*), for every $\mathcal{P} \subset \overline{\mathbb{R}}^X$,

$$\mathrm{cl}_{\xi^\pm}\mathcal{P} = \{\nabla_{\mathcal{P}} > -\infty\} \cap \{\Delta_{\mathcal{P}} < +\infty\} = \{f : \exists_{p,q \in \mathcal{P}}\, p \sqsubseteq f \sqsubseteq q\}.$$

The operator $\mathrm{cl}_{\xi^\pm}$ is a closure operator. A family $\mathcal{P}$ is said to be $\xi^\pm$-*closed* if $\mathrm{cl}_{\xi^\pm}\mathcal{P} = \mathcal{P}$. A family $\mathcal{P}$ is $\xi^\pm$-closed if and only if $\mathcal{P}$ is order-convex and $\mathcal{P}+\mathbb{R} \subset \mathcal{P}$. The families that are either ξ^+- or ξ^--closed are $\xi^\pm$-closed. Moreover, the intersection of $\xi^\pm$-closed families is $\xi^\pm$-closed. In particular, if T is a niveloid then $\{T > -\infty\}$, $\{T < +\infty\}$ and $\{-\infty < T < +\infty\}$ are all $\xi^\pm$-closed.

Of course, ξ^+ is coarser than τ^+ (and ξ^- is coarser than τ^-); hence each $\xi^\pm$-closed set is $\tau^\pm$-closed.

4. Lattice structure of the set of niveloids

For every $\mathcal{P} \subset \overline{\mathbb{R}}^X$, $\nabla_{\mathcal{P}}$ (resp. $\Delta_{\mathcal{P}}$) is a niveloid. We shall see that every niveloid is of this form.

THEOREM 4.1. *For every niveloid T, $T = \nabla_{\{T \geq 0\}}$ and $T = \Delta_{\{T \leq 0\}}$.*

PROOF. By Theorem 1.2,

$$T(f) \geq \sup_{T(w) \geq 0} [T(w) \dotplus \inf (f \dot{-} w)] \geq \sup_{T(w) \geq 0} \inf (f \dot{-} w) \geq \nabla_{\{T \geq 0\}}.$$

On the other hand,

$$\nabla_{\{T\geq 0\}} \geq \nabla_{\{T=0\}} \vee \nabla_{\{T=+\infty\}}.$$

Hence, by Theorem 2.3, we have $T = \nabla_{\{T\geq 0\}}$. The second equality follows by duality. □

Corollary 4.2. *For each* $\mathcal{P} \subset \overline{\mathbb{R}}^X$,

$$\nabla_{\mathcal{P}} = \nabla_{\mathrm{cl}_{\tau^+}\mathcal{P}} \quad \text{and} \quad \Delta_{\mathcal{P}} = \Delta_{\mathrm{cl}_{\tau^-}\mathcal{P}}, \tag{4.1}$$

$$\nabla_{\mathcal{P}} = \nabla_{\mathrm{cl}_{\tau^\pm}\mathcal{P}} \quad \text{and} \quad \Delta_{\mathcal{P}} = \Delta_{\mathrm{cl}_{\tau^\pm}\mathcal{P}}. \tag{4.2}$$

Proof. Apply (3.3) and Theorem 4.1 with $T = \nabla_{\mathcal{P}}$ (resp. $T = \Delta_{\mathcal{P}}$) to obtain (4.3). To prove (4.2), combine (4.1) and the equalities $\mathrm{cl}_{\tau^+}\mathcal{P} = \mathrm{cl}_{\tau^+}\mathrm{cl}_{\tau^\pm}\mathcal{P}$, $\mathrm{cl}_{\tau^-}\mathcal{P} = \mathrm{cl}_{\tau^-}\mathrm{cl}_{\tau^\pm}\mathcal{P}$. □

We are now in a position to prove that the complete lattice of niveloids is isomorphic to the complete lattice of closed sets with respect to the upper (resp. lower) uniform topology.

Theorem 4.3. *The mapping* $T \mapsto \{T \geq 0\}$ (*resp.* $T \mapsto \{T \leq 0\}$) *is an isomorphism of the complete lattice of niveloids onto the complete lattice of closed sets with respect to* τ^+ (*resp.* τ^-). *The inverse isomorphism associates every closed set* $\mathcal{P}$ *with* $\nabla_{\mathcal{P}}$ (*resp.* $\Delta_{\mathcal{P}}$).

Proof. Clearly $S \leq T$ implies $\{S \geq 0\} \subset \{T \geq 0\}$ and, by Theorem 3.1, $T \mapsto \{T \geq 0\}$ ranges over all the τ^+-closed sets. On the other hand, $\mathcal{P} \subset \mathcal{B}$ entails $\nabla_{\mathcal{P}} \leq \nabla_{\mathcal{B}}$ and, by Theorem 4.1, ∇ is a bijection of the τ^+-closed sets onto the set of niveloids. By duality the second part of the statement is valid. □

5. Niveloidal extensions

Consider a functional F defined on a subset $\mathcal{F}$ of $\overline{\mathbb{R}}^X$. We shall say that F is *extensible* if there exists a niveloid T (on $\overline{\mathbb{R}}^X$) called an *extension* of F such that $T_{|\mathcal{F}} = F$.

Define the following functionals $F^\diamondsuit, F_\diamondsuit$ on $\overline{\mathbb{R}}^X$:

$$F^\diamondsuit(f) = \sup_{p\in\mathcal{F}}[F(p) \mathbin{\underset{\cdot}{+}} \inf(f \mathbin{\dot{-}} p)], \tag{5.1}$$

$$F_\diamondsuit(f) = \inf_{p\in\mathcal{F}}[F(p) \mathbin{\dot{+}} \sup(f \mathbin{\underset{\cdot}{-}} p)]. \tag{5.1*}$$

Proposition 5.1. *The functional* $F^\diamondsuit$ (*resp.* $F_\diamondsuit$) *is the least* (*resp. greatest*) *niveloid that majorizes* (*minorizes*) F *on* $\mathcal{F}$. *If* F *is extensible, then* $F^\diamondsuit$ (*resp.* $F_\diamondsuit$) *is the least* (*resp. greatest*) *niveloid that extends* F *on* $\mathcal{F}$.

PROOF. The niveloid $F^\diamondsuit$ (resp. $F_\diamondsuit$) is the upper (resp. lower) projection of the functional F_- (resp. F_+) defined by

$$F_-(f) = \begin{cases} F(f) & \text{if } f \in \mathcal{F}, \\ -\infty & \text{if } f \notin \mathcal{F}, \end{cases} \qquad \text{and} \qquad F_+(f) = \begin{cases} F(f) & \text{if } f \in \mathcal{F}, \\ +\infty & \text{if } f \notin \mathcal{F}. \end{cases}$$

Every niveloid T which majorizes (resp. minorizes) F on $\mathcal{F}$ fulfils $T \geq F_-$ (resp. $T \leq F_+$); hence $F_- \leq F^\diamondsuit \leq T$ (resp. $T \leq F_\diamondsuit \leq F_+$). □

THEOREM 5.2 (Extensibility). *Let F be a functional defined on $\mathcal{F}$. The following statements are equivalent:*

(5.2) F *is extensible,*

(5.3) $F^\diamondsuit = F_\diamondsuit$ *on* $\mathcal{F}$,

(5.4) $F^\diamondsuit \leq F_\diamondsuit$,

(5.5) $\forall_{f,g\in\mathcal{F}} \quad \inf(f \dot{-} g) \leq F(f) \dot{-} F(g)$,

(5.5*) $\forall_{f,g\in\mathcal{F}} \quad F(f) \dot{\div} F(g) \leq \sup(f \dot{\div} g)$.

PROOF. (5.2)⇒(5.3): by the second part of Proposition 5.1. (5.2)⇒(5.3): for every $f \in F$, $F_\diamondsuit(f) \leq F(f) \leq F^\diamondsuit(f)$. (5.2)⇒(5.4): if T is an extension of F, then, by Proposition 5.1, $F^\diamondsuit \leq T \leq F_\diamondsuit$. (5.4)⇒(5.5): by Proposition 5.1, (5.4) implies that $F^\diamondsuit, F$ and $F_\diamondsuit$ coincide on $\mathcal{F}$. Thus, by replacing $F^\diamondsuit(f)$ by $F(f)$ in (5.1) when $f \in \mathcal{F}$, we get (5.5). (5.5)⇒(5.5*): by multiplying (5.5) by -1 and by changing the roles of f and g. (5.5*)⇒(5.2): actually (5.5*) amounts to

$$\forall_{f\in\mathcal{F}} \quad F(f) \leq \inf_{g\in\mathcal{F}} [F(g) \dot{+} \sup(f \dot{\div} g)];$$

thus, in view of the definition (5.1*), (5.5*) entails $F(f) \leq F_\diamondsuit(f)$ for each $f \in \mathcal{F}$. By virtue of Proposition 5.1, this means that $F = F_{\diamondsuit|\mathcal{F}}$. □

PROPOSITION 5.3. *Let S be an extensible functional, $f \in \overline{\mathbb{R}}^X$ and $r \in \overline{\mathbb{R}}$. There exists an extension of S equal to r at f if and only if*

$$S^\diamondsuit(f) \leq r \leq S_\diamondsuit(f), \tag{5.6}$$

$$\textit{either } f \textit{ is proper or } r \in \{-\infty, +\infty\}. \tag{5.7}$$

PROOF. If f is in the domain $\mathcal{F}$ of S, then, by Theorem 5.2 ((5.2)⇒(5.3)), S is extensible and equal to r at f if and only if $S^\diamondsuit(f) = r = S_\diamondsuit(f)$. In this case $r \dot{-} r \geq \inf(f \dot{-} f)$, which amounts to (5.7).

If $f \notin \mathcal{F}$, then applying Theorem 5.2 ((5.2)⇔(5.5)) to the functional equal to S on $\mathcal{F}$ and equal to r at f we get (because of the extensibility of S), for every $g \in \mathcal{F}$,

$$\inf(f \dot{-} g) \leq r \dot{-} F(g), \quad \inf(g \dot{-} f) \leq F(g) \dot{-} r \quad \text{and} \quad \inf(f \dot{-} f) \leq r \dot{-} r.$$

Now the first inequality amounts to $S^\diamondsuit(f) \leq r$, the second to $r \leq S_\diamondsuit(f)$ and the last to (5.7). □

6. Admissibility and maximality

Let $\mathcal{B} \subset \overline{\mathbb{R}}^X$ and $A \subset \overline{\mathbb{R}}$. The family $\mathcal{B}$ is called A-*admissible* if there exists a niveloid T such that $T(\mathcal{B}) \subset A$. We shall say that $\mathcal{B}$ is 0-*admissible* in case it is $\{0\}$-admissible. Of course a 0-admissible family is proper (i.e. each element of it is a proper function).

Theorem 6.1. *The following statements are equivalent:*

(6.1) $\mathcal{B}$ *is* 0-*admissible*,
(6.2) $\nabla_{\mathcal{B}} = \Delta_{\mathcal{B}}$ *on* $\mathcal{B}$,
(6.3) $\nabla_{\mathcal{B}} \le \Delta_{\mathcal{B}}$,
(6.4) $\mathcal{B} \subset \{\nabla_{\mathcal{B}} \le 0\}$(*i.e.* $\forall_{b,w\in\mathcal{B}}$ $\inf(w \dot{-} b) \le 0$),
(6.4*) $\mathcal{B} \subset \{\Delta_{\mathcal{B}} \ge 0\}$(*i.e.* $\forall_{b,w\in\mathcal{B}}$ $\sup(w \dot{-} b) \ge 0$).

Proof. One notes that $\mathcal{B}$ is 0-admissible if and only if the null functional (defined on $\mathcal{B}$) is extensible. Therefore (5.5) and (5.5*) of Theorem 5.2 specialize to (6.4) and (6.4*) respectively. In view of (5.1) and (5.1*), properties (5.3) and (5.4) of Theorem 5.2 specialize to (6.2) and (6.3) respectively. $\square$

One infers that if $\mathcal{B}$ is 0-admissible, then $\nabla_{\mathcal{B}}$ (resp. $\Delta_{\mathcal{B}}$) is *the least* (resp. *the greatest*) *niveloid equal to* 0 *on* $\mathcal{B}$. Observe that (6.4) is tantamount to

$$b, w \in \mathcal{B} \text{ and } b \le w \Rightarrow \inf(w \dot{-} b) = 0. \tag{6.5}$$

Corollary 6.2. *The following statements are equivalent:*

(6.6) $\mathcal{B}$ *is* 0-*admissible,*
(6.7) $\mathrm{cl}_{\tau^\pm}\mathcal{B}$ *is* 0-*admissible,*
(6.8) $\mathrm{cl}_{\tau^\pm}\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \{\Delta_{\mathcal{B}} = 0\}$.

In particular, $\mathcal{B}$ *is a* $\tau^\pm$-*closed* 0-*admissible family if and only if*

$$\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \{\Delta_{\mathcal{B}} = 0\}.$$

Proof. Apply (4.2). $\square$

For every family $\mathcal{B}$, define

$$\mathcal{B}^\alpha := \{\nabla_{\mathcal{B}} \le 0\} \cap \{0 \le \Delta_{\mathcal{B}}\}.$$

By (6.4) and (6.4*) it follows that $\mathcal{B} \subset \mathcal{B}^\alpha$ if and only if $\mathcal{B}$ is 0-admissible.

As a direct consequence of Proposition 5.3 we have the following criterion for the 0-admissibility of $\mathcal{B} \cup \{f\}$ under the assumption that $\mathcal{B}$ is 0-admissible.

Lemma 6.3. *Let* $\mathcal{B}$ *be* 0-*admissible and let* $f \in \overline{\mathbb{R}}^X$. *Then* $\mathcal{B} \cup \{f\}$ *is* 0-*admissible if and only if* f *is proper and* $f \in \mathcal{B}^\alpha$.

0-admissibility is an inductive property. Thus every 0-admissible family is included in a maximal 0-admissible family. Lemma 6.3 entails

$$\mathcal{B} = \mathcal{B}^\alpha \cap \{f \in \overline{\mathbb{R}}^X : f \text{ proper}\} \Leftrightarrow \mathcal{B} \text{ is maximal 0-admissible.} \tag{6.9}$$

By (6.7) a maximal 0-admissible family is $\tau^{\pm}$-closed. The maximality may be characterized by a *coincidence condition* for the operators ∇ and Δ.

THEOREM 6.4. *$\mathcal{B}$ is a maximal 0-admissible family if and only if $\mathcal{B}$ is a $\tau^{\pm}$-closed proper family and*

$$\forall_{f\ proper}\quad \nabla_{\mathcal{B}} f = \Delta_{\mathcal{B}} f. \tag{6.10}$$

PROOF. Let $\mathcal{B}$ be maximal 0-admissible. By Corollary 6.2, the 0-admissibility of $\mathcal{B}$ implies that $\mathrm{cl}_{\tau^{\pm}}\mathcal{B}$ is 0-admissible. Hence, since $\mathcal{B} \subset \mathrm{cl}_{\tau^{\pm}}\mathcal{B}$, by the maximality of $\mathcal{B}$, we have $\mathcal{B} = \mathrm{cl}_{\tau^{\pm}}\mathcal{B}$, that is, $\mathcal{B}$ is a $\tau^{\pm}$-closed family of proper functions. Suppose that there exists a proper function f such that $\nabla_{\mathcal{B}} f \neq \Delta_{\mathcal{B}} f$. It follows that there exists $r \in \mathbb{R}$ such that $\nabla_{\mathcal{B}} f < r < \Delta_{\mathcal{B}} f$, that is,

$$\nabla_{\mathcal{B}}(f - r) < 0 < \Delta_{\mathcal{B}}(f - r). \tag{6.11}$$

Since the 0-admissibility of $\mathcal{B}$ implies that $\nabla_{\mathcal{B}}$ equals 0 on $\mathcal{B}$, by (6.11) we have $f - r \notin \mathcal{B}$. On the other hand, by Lemma 6.3, (6.11) implies that $\mathcal{B} \cup \{f - r\}$ is 0-admissible, contradicting the maximality.

Now, suppose that $\mathcal{B}$ is a $\tau^{\pm}$-closed family of proper functions and (6.10) holds. Since $\nabla_{\mathcal{B}}$ and $\Delta_{\mathcal{B}}$ coincide on $\mathcal{B}$, by (6.2), the family $\mathcal{B}$ is 0-admissible. If f is a proper function such that $\mathcal{B} \cup \{f\}$ is 0-admissible, then (6.9) and (6.10) imply $\nabla_{\mathcal{B}} f = \Delta_{\mathcal{B}} f = 0$. Hence, since $\mathcal{B}$ is $\tau^{\pm}$-closed, it follows from (6.8) that $f \in \mathcal{B}$. Therefore $\mathcal{B}$ is maximal 0-admissible. $\square$

A family $\mathcal{B}$ is said to be a *base* of a maximal 0-admissible family if $\mathrm{cl}_{\tau^{\pm}}\mathcal{B}$ is a maximal 0-admissible family.

COROLLARY 5.6. *Let $\mathcal{B}$ be a proper family. The coincidence condition* (6.10) *holds if and only if $\mathcal{B}$ is a base of a maximal* 0*-admissible family.*

PROOF. Apply (4.2) and Theorem 6.4. $\square$

A $\tau^{\pm}$-closed family $\mathcal{B}$ is said to be *autodual* if $\Delta_{\mathcal{B}} = \nabla_{\mathcal{B}}$.

COROLLARY 6.6. *A family $\mathcal{B}$ is autodual if and only if it is a maximal* 0*-admissible family and $\overline{\mathbb{R}}^X = \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$. Hence $\mathcal{B}$ is autodual if and only if $\mathcal{B} = \mathcal{B}^{\alpha}$.*

PROOF. Apply Theorem 6.4 and (6.9). $\square$

COROLLARY 6.7. *A family $\mathcal{B}$ is autodual if and only if it is a maximal* 0*-admissible family such that $\mathcal{B}^{\alpha}$ is proper. Hence, if $\mathcal{A}$ is a* 0*-admissible family such that $\mathcal{A}^{\alpha}$ is proper, then every maximal* 0*-admissible family including $\mathcal{A}$ is autodual.*

PROOF. Observe that the properness of $\mathcal{B}^{\alpha}$ amounts to $\{-\infty, +\infty\}^X \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$. Hence, apply Corollary 6.6 and the fact that $\mathcal{B}^{\alpha} \subset \mathcal{A}^{\alpha}$ whenever $\mathcal{A}$ and $\mathcal{B}$ are 0-admissible families with $\mathcal{A} \subset \mathcal{B}$. $\square$

Examples 6.8. **(A)** For a fixed point $x_0 \in X$, the family of all functions f with $f(x_0) = 0$ is autodual. **(B)** Let g be a finite function (i.e. $g(X) \subset \mathbb{R}$). Then $\mathcal{B} = \{f : \sup(f-g) = 0\}$ is autodual. (Hint: prove that (a) $\mathcal{B}$ is 0-admissible, (b) for every $h \notin \mathcal{B}$, the family $\mathcal{B} \cup \{h\}$ is not 0-admissible, (c) $\overline{\mathbb{R}}^X = \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$, and finally apply Corollary 6.6). **(C)** Let g be a finite function on X and, for every $x \in X$, let $\psi_{\{x\}}$ be the indicator function of $\{x\}$ (i.e. $\psi_{\{x\}}$ is 0 at x and $+\infty$ elsewhere). Then the family $\mathcal{B} = \{\psi_{\{x\}} + g(x) : x \in X\}$ is such that $\mathcal{B}^\alpha$ is proper. Actually, $\mathcal{B}^\alpha = \mathcal{B} \cup \{f \in \overline{\mathbb{R}}^X : f \geq g \text{ and } \mathrm{card}(\{f < +\infty\}) \geq 2\}$. Since $\mathcal{B}$ is 0-admissible, by Corollary 6.7 every maximal 0-admissible family including $\mathcal{B}$ is autodual. Note that if X contains at least two elements, $\mathcal{B}$ is not maximal. □

7. Coincidence conditions and maximality

This section is intended to improve the understanding of the role of coincidence conditions (like (6.2), (6.10), definition of autodual families) in the study of various types of 0-admissible families.

Theorem 7.1 (Coincidence). *Let $\mathcal{B} \subset \mathcal{P} \subset \overline{\mathbb{R}}^X$ be families such that $\mathcal{B}$ is $\tau^\pm$-closed and $\mathcal{P} + \mathbb{R} \subset \mathcal{P}$. The following properties are equivalent:*

(7.1) $\nabla_{\mathcal{B}} = \Delta_{\mathcal{B}}$ *on* $\mathcal{P}$,

(7.2) $\mathcal{B} = \mathcal{B}^\alpha \cap \mathcal{P}$,

(7.3) $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$ *and* $\mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$.

Observe that for $\mathcal{P}$ equal to $\overline{\mathbb{R}}^X$ or to the family of all proper functions the equality (7.1) characterizes autodual families and maximal 0-admissible families, respectively.

Proof. (7.1)⇒(7.2): In virtue of Corollary 6.2, since $\mathcal{B}$ is $\tau^\pm$-closed, it is enough to prove that

$$(7.2^*) \qquad \mathcal{B} \subset \mathcal{B}^\alpha \cap \mathcal{P} \subset \{\nabla_{\mathcal{B}} = 0\} \cap \{\Delta_{\mathcal{B}} = 0\}.$$

As $\mathcal{B} \subset \mathcal{P}$, by Theorem 6.1 we see that $\mathcal{B}$ is 0-admissible, that is, $\mathcal{B} \subset \mathcal{B}^\alpha$. Hence the first inclusion of (7.2*) holds; the second inclusion follows directly from (7.1) and the definition of $\mathcal{B}^\alpha$.

(7.2)⇒(7.3): Since (7.2) implies $\mathcal{B} \subset \mathcal{B}^\alpha$, the family $\mathcal{B}$ is 0-admissible. Hence $\mathcal{B} \subset \{\nabla_{\mathcal{B}} = 0\} \subset \mathcal{B}^\alpha$ and $\mathcal{B} \subset \{\Delta_{\mathcal{B}} = 0\} \subset \mathcal{B}^\alpha$; therefore, since $\mathcal{B} \subset \mathcal{P}$, (7.2) shows that $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$. To prove $\mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$, let $f \in \mathcal{P}$. If $f \in \mathcal{B}^\alpha$, then, by (7.2), $f \in \mathcal{B}$; hence $f \in \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$. Otherwise, if $f \notin \mathcal{B}^\alpha$, then, by definition of $\mathcal{B}^\alpha$, we have either $\nabla_{\mathcal{B}} f > 0$ or $\Delta_{\mathcal{B}} f < 0$; therefore in either case $f \in \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$.

(7.3)⇒(7.1): Let $f \in \mathcal{P}$. By (7.3), $\mathcal{B}$ is included in the 0-admissible family $\{\nabla_{\mathcal{B}} = 0\}$, hence $\mathcal{B}$ is 0-admissible and, consequently, by Theorem 6.1, $\nabla_{\mathcal{B}} \leq \Delta_{\mathcal{B}}$. Therefore, since $\mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$, we have to consider three possibilities:

(a) $\nabla_{\mathcal{B}} f = \Delta_{\mathcal{B}} f = -\infty$ or $\nabla_{\mathcal{B}} f = \Delta_{\mathcal{B}} f = +\infty$, (b) $\nabla_{\mathcal{B}} f \in \mathbb{R}$ and (c) $\Delta_{\mathcal{B}} f \in \mathbb{R}$. In case (a) there is nothing to prove. In case (b), $f - \nabla_{\mathcal{B}} f \in \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$. On the other hand, by (7.3), $\{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} \subset \{\Delta_{\mathcal{B}} = 0\}$; hence we obtain $\Delta_{\mathcal{B}}(f - \nabla_{\mathcal{B}} f) = 0$, that is, $\Delta_{\mathcal{B}} f = \nabla_{\mathcal{B}} f$. Similarly, in case (c) we can prove that $\Delta_{\mathcal{B}} f = \nabla_{\mathcal{B}} f$. □

THEOREM 7.2 (Maximality). *Under the hypotheses of Theorem* 7.1, *the following properties are equivalent:*

(7.4) $\mathcal{B}$ *is maximal among the* 0-*admissible families included in* $\mathcal{P}$,

(7.5) $\nabla_{\mathcal{B}} = \Delta_{\mathcal{B}}$ *on* $\mathcal{P} \cap \{f \in \overline{\mathbb{R}}^X : f$ *proper*$\}$ *and* $\mathcal{B}$ *is proper.*

PROOF. Use Lemma 6.3 to prove that (7.4) amounts to $\mathcal{B} = \mathcal{B}^{\alpha} \cap \mathcal{P} \cap \{f \in \overline{\mathbb{R}}^X : f \text{ proper}\}$ and, finally, apply Theorem 7.1. □

The inclusion $\mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$ of (7.3) means that $\mathcal{P}$ "depends" on $\mathcal{B}$. A function f is called *independent* of $\mathcal{B}$ if for every niveloid T and for every $r \in \overline{\mathbb{R}}$ such that f is $\{r\}$-admissible, there exists a niveloid S such that $S = T$ on $\mathcal{B}$ and $S(f) = r$. Otherwise f is called *dependent* on $\mathcal{B}$. A family $\mathcal{F}$ is said to be *dependent* on $\mathcal{B}$ if each f in $\mathcal{F}$ is dependent on $\mathcal{B}$. Then we have

PROPOSITION 7.3. *Let $\mathcal{B}$ and $\mathcal{P}$ be arbitrary families of functions. Then the following properties are equivalent:*

(7.6) $\mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$,

(7.7) $\mathcal{P}$ *is dependent on* $\mathcal{B}$.

PROOF. Since $f \notin \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$ if and only if $\nabla_{\mathcal{B}} f = -\infty$ and $\Delta_{\mathcal{B}} f = +\infty$, that is,

$$\forall_{g \in \mathcal{B}} \quad \inf(f \dot{-} g) = \inf(g \dot{-} f) = -\infty, \tag{7.8}$$

it is enough to prove that the independence of a function f of $\mathcal{B}$ amounts to (7.8). Suppose that f is independent of $\mathcal{B}$. Since every function is both $\{+\infty\}$- and $\{-\infty\}$-admissible, from the definition of independence it follows that there exists a niveloid T_1 (resp. T_2) such that $T_1 = +\infty$ (resp. $T_2 = -\infty$) on $\mathcal{B}$ and $T_1(f) = -\infty$ (resp. $T_1(f) = +\infty$). Consequently, by Corollary 1.3, we have, for every $g \in \mathcal{B}$, $\inf(f \dot{-} g) \le T_1(f) \dot{-} T_1(g) = (-\infty) \dot{-} (+\infty) = -\infty$ and $\sup(f \underset{\cdot}{-} g) \ge T_2(f) \underset{\cdot}{-} T_2(g) = (+\infty) \underset{\cdot}{-} (-\infty) = +\infty$. Therefore (7.8) holds. Now suppose (7.8). Let T be a niveloid and let $r \in \overline{\mathbb{R}}$ be such that f is $\{r\}$-admissible. Set $\mathcal{F} = \mathcal{B} \cup \{f\}$ and define a functional $S : \mathcal{F} \to \overline{\mathbb{R}}$ by $S(f) = r$ and $S = T$ on $\mathcal{B}$. Since the $\{r\}$-admissibility amounts to $\inf(f \dot{-} f) \le (r \dot{-} r)$, in virtue of (7.8), the functional S satisfies (5.5). Hence, by Theorem 5.2, there exists a niveloid which extends S. Therefore f is independent of $\mathcal{B}$. □

Corollary 7.4. *Under the hypotheses of Theorem* 7.1*,* $\mathcal{B} = \mathcal{B}^\alpha \cap \mathcal{P}$ *if and only if* (7.4) *holds and one of the following equivalent properties is satisfied:*

(7.7′) *every improper function in* $\mathcal{P}$ *is dependent on* $\mathcal{B}$,

(7.7″) $\mathcal{B}^\alpha \cap \mathcal{P}$ *is proper.*

To provide an insight into property (7.3) we give the following proposition.

Proposition 7.5. *Under the hypotheses of Theorem* 7.1*, the following properties are equivalent:*

(7.9) $\mathcal{B}$ *is* 0*-admissible and* $\{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$,

(7.10) $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$,

(7.11) $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$ *and* $\{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B}$,

(7.12) $\mathcal{B} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$ *and* $\{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} \subset \mathrm{cl}_{\xi^-}\mathcal{B}$.

Proof. Property (7.10) implies that the niveloid $\nabla_{\mathcal{B}}$ is 0 on $\mathcal{B}$; that is, by definition, $\mathcal{B}$ is 0-admissible. Hence (7.10)⇒(7.9). Conversely, suppose (7.9) holds; we will prove (7.10). Since $\mathcal{B}$ is a 0-admissible family, the niveloid $\nabla_{\mathcal{B}}$ is 0 on $\mathcal{B}$; hence $\mathcal{B} \subset \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$. On the other hand, by the equality $\{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$, we have $\{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} \subset \{\nabla_{\mathcal{B}} = 0\} \cap \{\Delta_{\mathcal{B}} = 0\}$. Therefore, $\mathcal{B}$ being $\tau^{\pm}$-closed, Corollary 6.2 shows that $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$. Hence (7.9)⇒(7.10).

From the definition $\mathrm{cl}_{\xi^+}\mathcal{B} := \{\nabla_{\mathcal{B}} > -\infty\}$ it follows that (7.10)⇒(7.11). Now, assuming that (7.11) holds, we will prove (7.10). By $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$, the niveloid $\nabla_{\mathcal{B}}$ is 0 on $\mathcal{B}$, hence, by definition, $\mathcal{B}$ is 0-admissible. The 0-admissibility of $\mathcal{B}$ implies that the niveloid $\Delta_{\mathcal{B}}$ is 0 on $\mathcal{B}$; hence

$$\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P} \subset \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}. \tag{7.13}$$

Therefore, since $\mathcal{B}$ is $\tau^{\pm}$-closed, in order to prove (7.10) it is enough to verify that

$$\{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P} \subset \{\nabla_{\mathcal{B}} = 0\} \cap \{\Delta_{\mathcal{B}} = 0\}. \tag{7.14}$$

Pick $f \in \mathcal{P}$ such that $\Delta_{\mathcal{B}} f = 0$. Then the set inclusion of (7.11) implies $\nabla_{\mathcal{B}} f > -\infty$. On the other hand, by the 0-admissibility of $\mathcal{B}$, from Theorem 6.1 it follows that $\nabla_{\mathcal{B}} f \le \Delta_{\mathcal{B}} f$; hence $\nabla_{\mathcal{B}} f$ is finite in view of $\Delta_{\mathcal{B}} f = 0$. Therefore, since $f \in \mathcal{P}$ and, by hypothesis, $\mathcal{P} + \mathbb{R} \subset \mathcal{P}$, we obtain $f - \nabla_{\mathcal{B}} f \in \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$. Thus, by the set equality of (7.11), $f - \nabla_{\mathcal{B}} f \in \mathcal{B}$. Hence, by the 0-admissibility of $\mathcal{B}$, $\Delta_{\mathcal{B}}(f - \nabla_{\mathcal{B}}(f)) = 0$, that is, $\Delta_{\mathcal{B}} f = \nabla_{\mathcal{B}} f$. Therefore, since $\Delta_{\mathcal{B}} f = 0$, we have $\nabla_{\mathcal{B}} f = 0$; thus (7.14) holds. The proof of (7.11)⇒(7.10) is complete. Dually, the equivalence (7.10)⇔(7.12) holds. □

Examples 7.6. Consider the following three properties:

(a) $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathcal{P}$, (b) $\mathcal{B} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathcal{P}$, (c) $\mathcal{P} \subset \mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$.

Anticipating Section 9, we say that a family $\mathcal{B}$ is *inf-convolutive* (resp. *sup-convolutive*) if $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\}$ (resp. $\mathcal{B} = \{\Delta_{\mathcal{B}} = 0\}$). The three properties (a), (b) and (c) constitute (7.3). We will show their independence.

(A) Let $\mathcal{B} = \emptyset$ and $\mathcal{P} = \overline{\mathbb{R}}^X$. Then (a) and (b) hold, but not (c), because $\nabla_\emptyset = -\infty$, $\Delta_\emptyset = +\infty$ and $\mathrm{cl}_{\xi^+}\emptyset = \mathrm{cl}_{\xi^-}\emptyset = \emptyset$. Observe that $\mathcal{B}$ is both inf- and sup-convolutive; but it is not a maximal 0-admissible family.

(B) Now let X contain at least two points; denote them by x_0 and x_1. Set $\mathcal{B} := \{p \in \overline{\mathbb{R}}^X : p(x_0) \dot{+} p(x_1) = 0\}$. Since for every function f,

$$\nabla_{\mathcal{B}} f = \sup_{s\in\mathbb{R}}((f(x_0) - s) \wedge (f(x_1) + s))$$

and

$$\Delta_{\mathcal{B}} f = \inf_{s\in\mathbb{R}}((f(x_0) - s) \vee (f(x_1) + s)),$$

it is easy to verify that $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\}$ and $\mathcal{B} = \{\Delta_{\mathcal{B}} = 0\}$.

First case: X contains more than two points. Then $\mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$ does not contain all proper functions (e.g. the function f defined by $f(x_0) = -\infty$, $f(x_1) = +\infty$ and $f(x) = 0$ elsewhere). Hence, for $\mathcal{P} = \{f \in \overline{\mathbb{R}}^X : f \text{ proper}\}$, (a) and (b) hold, but (c) does not. As in example (A), in this case $\mathcal{B}$ is both inf- and sup-convolutive, but it is not a maximal 0-admissible family.

Second case: X contains exactly two points. Then $\mathrm{cl}_{\xi^+}\mathcal{B} \cup \mathrm{cl}_{\xi^-}\mathcal{B}$ contains all proper functions but not all improper functions (e.g. the function f defined by $f(x_0) = -\infty$ and $f(x_1) = +\infty$). Hence, for $\mathcal{P} = \overline{\mathbb{R}}^X$, (a) and (b) hold, but not (c). In this case $\mathcal{B}$ is a maximal 0-admissible family, but it is not autodual. The reader can verify that all other maximal 0-admissible families are autodual.

(C) Let X contain at least two points; denote them by x_0 and x_1. Set $\mathcal{B} := \{p \in \overline{\mathbb{R}}^X : p(x_0) = 0 \text{ and } p(x_1) = +\infty\}$ and $\mathcal{P} := \mathrm{cl}_{\xi^+}\mathcal{B}$. As for every function f,

$$\nabla_{\mathcal{B}} f = f(x_0) \wedge (f(x_1) \dot{+} (-\infty)) \quad \text{and} \quad \Delta_{\mathcal{B}} f = f(x_0) \vee (f(x_1) \dot{+} (-\infty)),$$

we have $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\}$ and $\{\Delta_{\mathcal{B}} = 0\} = \{f \in \overline{\mathbb{R}}^X : f(x_0) = 0\}$. Hence (a) and (c) hold, but (b) is not valid. The family $\mathcal{B}$ is inf-convolutive, but it is not sup-convolutive.

(D) Let X contain at least two points; denote them by x_0 and x_1. Set $\mathcal{B} := \{p \in \overline{\mathbb{R}}^X : p(x_0) = 0 \text{ and } p(x_1) = -\infty\}$ and $\mathcal{P} := \mathrm{cl}_{\xi^-}\mathcal{B}$. Then $\mathcal{B} = \{\Delta_{\mathcal{B}} = 0\}$ and $\{\nabla_{\mathcal{B}} = 0\} = \{f \in \overline{\mathbb{R}}^X : f(x_0) = 0\}$. Hence (b) and (c) hold, but (a) does not. The family $\mathcal{B}$ is sup-convolutive, but not inf-convolutive. $\square$

8. 0-families

A subset $\mathcal{B}$ of $\overline{\mathbb{R}}^X$ is called a 0-*family* if there exists a niveloid T such that $\mathcal{B} = \{T = 0\}$. Every 0-family is 0-admissible and $\tau^{\pm}$-closed. Maximal 0-admissible families are 0-families. Special classes of 0-families (inf- and sup-convolutive families) will be discussed in the following section.

Lemma 8.1. *Let $\mathcal{P}$ and $\mathcal{B}$ be 0-admissible families such that $\mathcal{P} \supset \mathcal{B}$. Then $\mathcal{B} = \mathcal{P} \cap (\mathcal{B} + \mathbb{R})$.*

Proof. Suppose $g \in \mathcal{P} \cap (\mathcal{B}+\mathbb{R})$. In particular, there exists a real number r such that $g + r \in \mathcal{B}$. Since $\mathcal{P} \supset \mathcal{B}$, we have $g + r \in \mathcal{P}$. Hence, by 0-admissibility of $\mathcal{P}$ we obtain $\nabla_{\mathcal{P}} g = 0 = \nabla_{\mathcal{P}}(g + r)$; so $r = 0$. Thus $g = g + r \in \mathcal{B}$. Hence $\mathcal{P} \cap (\mathcal{B} + \mathbb{R}) \subset \mathcal{B}$. This completes the proof, because the reverse inclusion $\mathcal{P} \cap (\mathcal{B} + \mathbb{R}) \supset \mathcal{B}$ is trivial. □

Lemma 8.2. *Let $\mathcal{B}$ be a 0-family. Then*

(8.1) $\quad \mathrm{cl}_{\xi^\pm}\mathcal{B} = \mathcal{B} + \mathbb{R}.$

Moreover,

(8.2) *if $\mathcal{P}$ is a $\xi^{\pm}$-closed family, then $\mathcal{B} \cap \mathcal{P}$ is a 0-family;*

(8.3) *if $\mathcal{P}$ is a 0-admissible family such that $\mathcal{P} \supset \mathcal{B}$, then $\mathcal{B} = \mathcal{P} \cap \mathrm{cl}_{\xi^\pm}\mathcal{B}$.*

Proof. Let T be a niveloid such that $\mathcal{B} = \{T = 0\}$. Since $\mathrm{cl}_{\xi^\pm}\{T = 0\} = \{T = 0\} + \mathbb{R}$, (8.1) holds. Now, observe that

$$\{((T \vee \Delta_{\mathrm{cl}_{\xi^-}\mathcal{P}}) \wedge \nabla_{\mathrm{cl}_{\xi^+}\mathcal{P}}) = 0\} = \{T = 0\} \cap \mathrm{cl}_{\xi^-}\mathcal{P} \cap \mathrm{cl}_{\xi^+}\mathcal{P}.$$

Hence, if $\mathcal{P}$ is a $\xi^{\pm}$-closed family (i.e. $\mathcal{P} = \mathrm{cl}_{\xi^-}\mathcal{P} \cap \mathrm{cl}_{\xi^+}\mathcal{P}$), then $\mathcal{B} \cap \mathcal{P}$ is a 0-family, which proves (8.2). Now, to prove (8.3) apply Lemma 8.1 and (8.1). □

Theorem 8.3. *A family $\mathcal{B}$ is a 0-family if and only if one of the following equivalent properties is satisfied:*

(8.4) *$\mathcal{B}$ is 0-admissible and $\mathrm{cl}_{\xi^\pm}\mathcal{B} = \mathcal{B} + \mathbb{R}$,*

(8.5) *$\mathcal{B}$ is the intersection of a $\xi^{\pm}$-closed family with a maximal 0-admissible family.*

Proof. In virtue of (8.1), if $\mathcal{B}$ is a 0-family, then (8.4) holds. To show that (8.4) implies that $\mathcal{B}$ is a 0-family, observe that, by the 0-admissibility of $\mathcal{B}$, one has $\{\nabla_{\mathcal{B}} = 0\} \supset \mathcal{B}$; hence from Lemma 8.1 it follows that $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap (\mathcal{B}+\mathbb{R})$. Since $\mathcal{B} + \mathbb{R} = \mathrm{cl}_{\xi^\pm}\mathcal{B}$ is $\xi^{\pm}$-closed, $\mathcal{B}$ is the intersection of the 0-family $\{\nabla_{\mathcal{B}} = 0\}$ and of the $\xi^{\pm}$-closed family $\mathcal{B}+\mathbb{R}$. Thus, from (8.2) it follows that $\mathcal{B}$ is a 0-family. Now, by (8.2), the property (8.5) implies that $\mathcal{B}$ is a 0-family. Conversely, if $\mathcal{B}$ is a 0-family, then (8.5) follows from (8.3). □

In terms of coincidence conditions we have

Theorem 8.4. *A family $\mathcal{B}$ is a 0-family if and only if it is $\tau^{\pm}$-closed and $\nabla_{\mathcal{B}} = \Delta_{\mathcal{B}}$ on $\mathrm{cl}_{\xi^\pm}\mathcal{B}$.*

PROOF. Suppose $\mathcal{B}$ is a 0-family. In virtue of (6.2), by the 0-admissibility of $\mathcal{B}$, we have $\nabla_{\mathcal{B}} = \Delta_{\mathcal{B}}$ on $\mathcal{B} + \mathbb{R}$. Hence, from (8.1), $\nabla_{\mathcal{B}}$ and $\Delta_{\mathcal{B}}$ coincide on $\mathrm{cl}_{\xi^{\pm}}\mathcal{B}$. Conversely, from Theorem 7.1 we have $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathrm{cl}_{\xi^{\pm}}\mathcal{B}$, hence by (8.2), $\mathcal{B}$ is a 0-family. □

In the following corollary we collect some characterizations of the 0-families, implied by the Coincidence Theorem 7.1.

COROLLARY 8.5. *A $\tau^{\pm}$-closed family $\mathcal{B}$ is a* 0*-family if and only if one of the following equivalent properties holds:*

(8.6) $\mathcal{B} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathrm{cl}_{\xi^-}\mathcal{B}$,

(8.7) $\mathcal{B} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathrm{cl}_{\xi^+}\mathcal{B}$,

(8.8) *$\mathcal{B}$ is maximal among the* 0*-families included in* $\mathrm{cl}_{\xi^{\pm}}\mathcal{B}$.

Let T be a niveloid. If S is a niveloid such that $\{S = 0\} = \{T = 0\}$, then $\{-\infty < S < +\infty\} = \{-\infty < T < +\infty\}$ and $S = T$ on $\{-\infty < T < +\infty\}$; therefore, among all the niveloids whose 0-family is equal to $\{T = 0\}$, there are both the greatest and the least element. How to use these extrema to describe T?

For every $\mathcal{B} \subset \overline{\mathbb{R}}^X$, set

$$\mathcal{B}^+ = \mathcal{B} \cup (\mathrm{cl}_{\xi^+}\mathcal{B} \setminus \mathrm{cl}_{\xi^-}\mathcal{B}) \quad \text{and} \quad \mathcal{B}^- = \mathcal{B} \cup (\mathrm{cl}_{\xi^-}\mathcal{B} \setminus \mathrm{cl}_{\xi^+}\mathcal{B}). \tag{8.9}$$

One can prove that, if T is a niveloid, then $\mathrm{cl}_{\xi^+}\{T = 0\} \setminus \mathrm{cl}_{\xi^-}\{T = 0\} \subset \{T = +\infty\}$ and $\mathrm{cl}_{\xi^-}\{T = 0\} \setminus \mathrm{cl}_{\xi^+}\{T = 0\} \subset \{T = -\infty\}$; consequently, one checks easily the following inequalities:

$$\nabla_{\{T=0\}^+} \le T \le \Delta_{\{T=0\}^-}. \tag{8.10}$$

For a given functional T, define

$$\mathcal{K}^+(T) = \{T = +\infty\} \setminus \mathrm{cl}_{\xi^+}\{T = 0\}$$

and

$$\mathcal{K}^-(T) = \{T = -\infty\} \setminus \mathrm{cl}_{\xi^-}\{T = 0\}.$$

THEOREM 8.6 (Third Representation Theorem). *Let T be a niveloid. Then $\nabla_{\{T=0\}^+}$ (resp. $\Delta_{\{T=0\}^-}$) is the least (resp. greatest) niveloid S for which $\{S = 0\} = \{T = 0\}$. Moreover,*

$$\nabla_{\{T=0\}^+} \vee \nabla_{\mathcal{K}^+(T)} = T = \Delta_{\{T=0\}^-} \wedge \Delta_{\mathcal{K}^-(T)}. \tag{8.11}$$

PROOF. Observe that for every $\mathcal{B} \subset \overline{\mathbb{R}}^X$, the 0-families of the niveloids $\nabla_{\mathcal{B}^+}$ and $\Delta_{\mathcal{B}^-}$ are, respectively,

$$\{\nabla_{\mathcal{B}^+} = 0\} = \{\nabla_{\mathcal{B}} = 0\} \cap \mathrm{cl}_{\xi^-}\mathcal{B}, \tag{8.12}$$

$$\{\Delta_{\mathcal{B}^-} = 0\} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathrm{cl}_{\xi^+}\mathcal{B}. \tag{8.13}$$

Taking $\mathcal{B} = \{T = 0\}$, from Corollary 8.5 and properties (8.12), (8.13), we deduce that the 0-families of $\nabla_{\{T=0\}^+}$ and $\Delta_{\{T=0\}^-}$ are equal to $\{T = 0\}$. Hence, from (8.10) it follows that $\nabla_{\{T=0\}^+}$ (resp. $\Delta_{\{T=0\}^-}$) is the least (resp. greatest) niveloid S for which $\{S=0\}=\{T=0\}$.

Since $\{T=+\infty\} = (\mathrm{cl}_{\xi^+}\{T=0\} \setminus \mathrm{cl}_{\xi^-}\{T=0\}) \cup \mathcal{K}^+(T)$, by (8.9) we have $\{T=0\}\cup\{T=+\infty\} = \{T=0\}^+ \cup \mathcal{K}^+(T)$.

Therefore, the first equality of (8.11) follows from the Second Representation Theorem 2.3. Dually one proves the second equality. □

9. Sup-convolutive and inf-convolutive niveloids

A niveloid T is called *inf-convolutive* (resp. *sup-convolutive*) if $T=\nabla_{\{T=0\}}$ (resp. $T=\Delta_{\{T=0\}}$). A family $\mathcal{B}\subset\overline{\mathbb{R}}^X$ is said to be *inf-convolutive* (resp. *sup-convolutive*) if it is the 0-family of an inf-convolutive (resp. sup-convolutive) niveloid. It is clear that inf- and sup-convolutive families are 0-families; hence they are proper, 0-admissible and $\tau^{\pm}$-closed.

Lemma 9.1. *If $\mathcal{B}$ is a 0-admissible family, then $\nabla_{\mathcal{B}} = \nabla_{\{\nabla_{\mathcal{B}}=0\}}$, hence $\nabla_{\mathcal{B}}$ is inf-convolutive.*

Proof. Since $\mathcal{B}$ is 0-admissible, $\mathcal{B}\subset\{\nabla_{\mathcal{B}}=0\}$; hence

$$\mathrm{cl}_{\tau^+}\mathcal{B}\subset \mathrm{cl}_{\tau^+}\{\nabla_{\mathcal{B}}=0\}.$$

On the other hand, by definition,

$$\mathrm{cl}_{\tau^+}\mathcal{B} = \{\nabla_{\mathcal{B}}\geq 0\};\qquad \text{hence}\qquad \mathrm{cl}_{\tau^+}\mathcal{B}\supset \mathrm{cl}_{\tau^+}\{\nabla_{\mathcal{B}}=0\}.$$

Therefore, as $\mathrm{cl}_{\tau^+}\mathcal{B} = \mathrm{cl}_{\tau^+}\{\nabla_{\mathcal{B}}=0\}$, (4.1) implies $\nabla_{\mathcal{B}} = \nabla_{\{\nabla_{\mathcal{B}}=0\}}$. □

We infer that the sets of the form $\{\nabla_{\mathcal{B}}=0\}$, where $\mathcal{B}$ is a 0-admissible family, are all the inf-convolutive families. Moreover, Lemma 9.1 entails

(9.1) $$\mathcal{B} \text{ is inf-convolutive} \Leftrightarrow \mathcal{B} = \{\nabla_{\mathcal{B}}=0\}.$$

Inf-convolutive families may be characterized by a *coincidence condition* for the operators ∇ and Δ.

Theorem 9.2. *A family $\mathcal{B}$ is inf-convolutive if and only if it is $\tau^{\pm}$-closed and $\nabla_{\mathcal{B}}=\Delta_{\mathcal{B}}$ on $\mathrm{cl}_{\xi^+}\mathcal{B}$.*

Proof. From (9.1) it follows that for $\mathcal{P} := \mathrm{cl}_{\xi^+}\mathcal{B}$, the family $\mathcal{B}$ satisfies (7.11). Hence the statement follows from the Coincidence Theorem 7.1. □

The Coincidence Theorem 7.1 entails the following characterizations of inf-convolutive families in terms of either 0-admissible families or 0-families.

COROLLARY 9.3. *If $\mathcal{B}$ is a $\tau^{\pm}$-closed 0-admissible family, then the following properties are equivalent:*

(9.2) *$\mathcal{B}$ is inf-convolutive,*

(9.3) *$\mathcal{B}$ is maximal among the 0-admissible families included in $\mathrm{cl}_{\xi^+}\mathcal{B}$,*

(9.4) $\{\nabla_{\mathcal{B}} = 0\} = \{\Delta_{\mathcal{B}} = 0\} \cap \mathrm{cl}_{\xi^+}\mathcal{B}$,

(9.5) *$\mathcal{B}$ is a 0-family and $\{\nabla_{\mathcal{B}} = 0\} \subset \mathrm{cl}_{\xi^-}\mathcal{B}$.*

In order to characterize inf-convolutive niveloids we need the following lemma, stated without proof.

LEMMA 9.4. *If $\mathcal{B}$ is a 0-admissible family and T is a niveloid, then*

(9.6) *T is inf-convolutive $\Leftrightarrow \nabla_{\{T=0\}} = \Delta_{\{T=0\}}$ on $\{T > -\infty\}$,*

(9.7) $\nabla_{\{T=0\}}(f) = \sup\{T(g) : g \leq f,\ T(g) \in \mathbb{R}\}$*, for every* $f \in \overline{\mathbb{R}}^X$,

(9.8) $\nabla_{\{T=0\}} \geq \nabla_{\{T=+\infty\}} \Leftrightarrow$ *for each $f \in \{T = +\infty\}$, there exists $\{h_n\}_n \subset \{T = 0\}$ such that $f \geq h_n + n$,*

(9.9) $\nabla_{\{T=0\}} \geq \nabla_{\mathcal{K}^+(T)} \Leftrightarrow \{T = +\infty\} \subset \mathrm{cl}_{\xi^+}\{T = 0\}$,

(9.10) $\nabla_{\mathcal{B}} = \nabla_{\mathcal{B}^+} \Leftrightarrow \{\nabla_{\mathcal{B}} = 0\} \subset \mathrm{cl}_{\xi^-}\mathcal{B}$.

THEOREM 9.5. *Let T be a niveloid. The following properties are equivalent:*

(9.11) *T is inf-convolutive,*

(9.12) *there exists a 0-admissible family $\mathcal{B}$ such that $T = \nabla_{\mathcal{B}}$,*

(9.13) $T(f) = \sup\{T(g) : g \leq f,\ T(g) \in \mathbb{R}\}$*, for every* $f \in \overline{\mathbb{R}}^X$,

(9.14) *for each $f \in \{T = +\infty\}$, there exists $\{g_n\}_n$ such that $g_n \leq f$, $T(g_n) \in \mathbb{R}$ and $\sup_n T(g_n) = +\infty$,*

(9.15) *$\{T = 0\}$ is inf-convolutive and, for each $f \in \{T = +\infty\}$, there exists g such that $g \leq f$, $T(g) \in \mathbb{R}$,*

(9.16) *$\{T = 0\}$ is maximal among the 0-admissible families included in $\{T > -\infty\}$ and for every improper function f in $\{T = +\infty\}$, there exists g such that $g \leq f$, $T(g) \in \mathbb{R}$.*

PROOF. From Lemma 9.1 (resp. property (9.7)) it follows that (9.11)⇔(9.12) (resp. (9.11)⇔(9.13)). To prove (9.11)⇔(9.14), use the Second Representation Theorem 2.3 and (9.8). To verify (9.11)⇔(9.15), use the Third Representation Theorem 8.6, the properties (9.9), (9.10) and the characterization (9.5) of the inf-convolutive families. Finally, combine (9.6) and Corollary 7.4 to obtain (9.11)⇔(9.16). □

A niveloid which is both inf- and sup-convolutive niveloid is said to be *convolutive.*

COROLLARY 9.6. *A niveloid T is convolutive if and only if $\{T = 0\}$ is autodual.*

Proof. The "only if" part follows from the definition of a convolutive niveloid. If T is a niveloid, the $\nabla_{\{T=0\}} \leq T \leq \Delta_{\{T=0\}}$; hence, if $\{T = 0\}$ is autodual, then $\nabla_{\{T=0\}} = T = \Delta_{\{T=0\}}$; that is, T is convolutive. $\square$

From (9.15) it follows that a niveloid T is convolutive if and only if $\{T = 0\}$ is both inf- and sup-convolutive and for every function f, there exists a function g such that $T(g) \in \mathbb{R}$ and either $g \leq f$ or $g \geq f$. Note that a maximal 0-admissible family is both inf- and sup-convolutive, but it is not necessarily autodual (see Examples 7.6). Hence there are families which are both inf- and sup-convolutive but not 0-families of convolutive niveloids. Moreover, note that an inf-convolutive family is not necessarily sup-convolutive (see Examples 7.6).

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