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INDICES OF FIXED POINTS NOT ACCUMULATED BY PERIODIC POINTS

Luis Hernandez-Corbato

Abstract. We prove that for every integer sequence I satisfying Dold relations there exists a map $f\colon \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, such that $\mathrm{Per}(f) = \mathrm{Fix}(f) = \{o\}$, where o denotes the origin, and $(i(f^n, o))_n = I$.

1. Introduction

Given a map f defined in a Euclidean space onto itself and p a fixed point of f, the fixed point index or Lefschetz index of f at p, denoted by $i(f,p)$, is an integer which measures the multiplicity of p as a fixed point of f. The definition requires the point p to be isolated in the set of fixed points of f, which will be denoted by $\mathrm{Fix}(f)$. The index is a topological invariant of the local dynamics around p. Since a fixed point of a map is also fixed by any of its iterates f^n, $n \geq 1$, the integer $i(f^n,p)$ is defined as long as p remains isolated in $\mathrm{Fix}(f^n)$. The integer sequence $(i(f^n,p))_{n=1}^{\infty}$ will be a denominated fixed point index sequence throughout this article. In general, it is very difficult to find constraints for these invariants. In fact, the unique global rule satisfied by fixed point index sequences is encompassed in the so-called Dold relations, see [4], which are described in

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Section 2. One of the most complete references on fixed point index theory is [10].

In dimension 1, the only possible values of the index of a fixed point are $-1, 0$ and 1. From dimension 2 and on any integer sequence satisfying Dold relations may appear as a fixed point index sequence of some map. Some restrictions appear as we impose extra conditions over the map f. For instance, Shub and Sullivan proved in [20] that the sequence is periodic when f is C^1. Recently, in [5], Graff, Jezierski and Nowak–Przygodzki have given a complete description in the C^1 case. Further, if f is a homeomorphism of a surface $(i(f^n, p))_n$ follows a very restrictive periodic pattern, see for example [3], [13], [14], [18], [2]. Periodicity of the sequence has been found to be true in dimension 3 just for homeomorphisms and locally maximal fixed points (i.e. points p which have a neighbourhood V such that $\{p\}$ is the maximal invariant set in V), see [12], [8].

Any of the previously described constraints disappear when the hypotheses are substantially weakened. For instance, in the planar case if f is no longer invertible then Dold relations are the only constraints remaining. In [6], Graff and Nowak–Przygodzki showed how to define a map in the plane fixing the origin and such that the fixed point index sequence is a given integer sequence satisfying Dold relations. Their map is constructed by gluing pieces made up of small radial sectors carrying a prescribed dynamics. This operation produces lots of periodic points which accumulate in the fixed one. Incidentally, notice that, in contrast, if the map is a homeomorphism and the fixed point is accumulated by $\mathrm{Per}(f)$ but not by $\mathrm{Fix}(f^n)$ then $i(f^n, p) = 1$ (see [17] and also [11, p. 145]).

It is somehow surprising that if the fixed point p is locally maximal (in the sense previously described) and f is a continuous map in the plane, the fixed point index sequence satisfies the following three constraints (see [9], [7]): $i(f, p)$ is bounded from above by 1, the sequence is periodic and every a_k is non-positive for $k \geq 2$ (see Section 2 for a definition of a_k). It is not known to what extent these constraints remain valid. In this work we consider the hypothesis of isolation as a periodic point, which is halfway between the locally maximal hypothesis and the unrestricted case. In the case of homeomorphisms the behavior of the fixed point index under this hypothesis is very well understood and similar to the locally maximal case (see [3], [11], [15], [19]). However, we prove that for continuous maps it turns out that this weakening is enough to dissipate all three constraints:

THEOREM 1.1. *For every $d \geq 2$ and every integer sequence I satisfying Dold relations there exists a map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ fixing a point p such that*:

(a) *$I = (i(f^n, p))_n$ and*
(b) *p is not accumulated by other periodic orbits of f.*

The most interesting case included in this result is $d = 2$. For larger dimensions, it suffices to fix the map in a plane and then retract the ambient space to that plane.

The paper is organized as follows. First, we introduce several definitions and comments along with some examples of dynamics which are the basic pieces of our work. In Section 3, we carry out the construction of the map which proves Theorem 1.1. Our work relies on a definition whose discussion involves symbolic dynamics and is postponed to the last section.

2. Fixed point index

Before we start with the basic definitions, let us go quickly through the notations and conventions used in the text. The origin of the plane $\mathbb{R}^d$ is denoted by o and we shall often use polar coordinates in $S^1 \times \mathbb{R}$ to represent the punctured plane $\mathbb{R}^2 \setminus \{o\}$. Notice a small annoyance, for later convenience the radial coordinate takes values in $\mathbb{R}$ instead of $\mathbb{R}^+$ and the origin corresponds to the end $r = -\infty$. We will use two conventions for the angular coordinate: it will take values either in $(-\pi, \pi]$ or in $[0, 1)$. Angles (points in S^1) will be denoted by Greek letters. Closed arcs in S^1 are named intervals and each interval J determines a (radial) sector in the punctured plane which contains all the points (θ, r) such that $\theta \in J$. The (forward) orbit of a point x under a map F is the set $\{x, F(x), F^2(x), \ldots\}$. Sequences appear ubiquitously and are always indexed by the positive integers.

Let U be an open subset of $\mathbb{R}^d$ and $f\colon U \to \mathbb{R}^d$ a continuous map. Suppose that p is an isolated fixed point of f. The fixed point index of f at p, denoted by $i(f, p)$, is defined as the degree of the map $\mathrm{id} - f\colon U \to \mathbb{R}^d$ at p. In other words, if B is a closed ball centered at p and such that $\mathrm{Fix}(f) \cap B = \{p\}$ then $i(f, p)$ is the degree of the map

$$\phi\colon \partial B \to S^{d-1}, \qquad x \mapsto \frac{x - f(x)}{\|x - f(x)\|}.$$

As our considerations will soon be limited to dimension 2, we include a more geometrical approach to the fixed point index in the plane. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and $\gamma\colon S^1 \to \mathbb{R}^2$ be a Jordan curve disjoint from $\mathrm{Fix}(f)$. One can define another curve $\eta\colon S^1 \to \mathbb{R}^2$ by $\eta(t) = \gamma(t) - f(\gamma(t))$. The index of f along γ is defined as the winding number of η around the origin. The definition does not depend on the parametrization of the curve γ and we will often use γ to refer to the image of the curve as well.

Let p be an isolated fixed point of f and V be an open neighbourhood of p which does not contain any other fixed point. The index of f along any Jordan curve γ contained in V which winds around p is equal to the fixed point index of f at p. The closed topological disk bounded by γ plays the role of B in the definition. It is not necessary that γ is confined to V: if D is a Jordan domain in the plane and p is the only fixed point contained in D (more precisely, in the interior of D) the index of f along the curve ∂D is again $i(f,p)$.

Henceforth, we assume $d=2$ unless otherwise stated.

LEMMA 2.1. *Let p be an isolated fixed point of the map f and γ be a Jordan curve which winds (in the positive sense) around p and does not enclose any other fixed point. Suppose the vectors $\overrightarrow{px}$ and $\overrightarrow{xf(x)}$ point in different directions at every point x in γ. Then, $i(f,p)=1$.*

PROOF. The vector $\overrightarrow{px}$ makes one positive turn as x moves along γ and comes back to the starting point. Since $\overrightarrow{xf(x)}$ does not overlap with $\overrightarrow{px}$ it must also make one positive turn as x moves along γ so, by definition, the index is 1. $\square$

In view of Lemma 2.1 the most critical parts are those around the places where the vectors $\overrightarrow{px}$ and $\overrightarrow{xf(x)}$ point in the same direction. Describing the dynamics in such pieces will be enough to compute the fixed point index.

COROLLARY 2.2. *Let γ be the boundary of a closed disk D centered at p which does not contain any other point fixed by f. If $f(\gamma)\subset D$ then $i(f,p)=1$.*

EXAMPLE 2.3. Consider the planar map which fixes the origin and is given in polar coordinates by $(\theta,r)\mapsto(\theta,r-1)$, $\theta\in S^1$, $r\in\mathbb{R}$. Note that for the sake of clarity we choose the radial coordinate r to range between $-\infty$ and $+\infty$, the origin corresponding to $r=-\infty$. Let γ be the curve $r=0$. By Corollary 2.2, the index of the origin is 1. Let us modify the map in a radial sector so that the dynamics becomes more interesting. Assume now that θ takes values in $(-\pi,\pi]$ and define $S=\{(\theta,r):\theta\in[-1,1]\}$ and a map in S by $f_-(\theta,r)=(\theta\cdot c_-(\theta),r+1-2\theta^2)$, where $c_-\colon[-1,1]\to[1/2,1]$ satisfies $c_-^{-1}(1)=\{-1,0,1\}$ and also that $\theta\cdot c_-(\theta)$ is strictly increasing in $\theta\in[-1,1]$. See Figure 1.

Using the original map, f_- extends to a map in the whole plane. As x moves along the arc $\gamma\cap S$ the vector $\overrightarrow{xf(x)}$ makes one turn in the negative sense so the index must be adjusted by subtracting 1, $i(f_-,o)=1-1=0$.

Note that the vectors $\overrightarrow{ox}$ and $\overrightarrow{xf_-(x)}$ point in the same direction only at $x=(\theta=0,r=0)$ and the angular dynamics around $\theta=0$ forces the contribution to the index to be negative, as $\theta=0$ is attracting for $\theta\mapsto\theta^3$.

Since any positive iterate f_-^n, $n\geq 1$, is topologically conjugate to f_- we conclude that $i(f_-^n,o)=i(f_-,o)=0$.

A minor modification in the angular behavior of f_- at $\theta=0$ reverses the sign of its extra contribution to the index. Choose $c_+\colon[-1,1]\to[1,2]$ such that $c_+^{-1}(1)=\{-1,0,1\}$ and $\theta\cdot c_+(\theta)$ is strictly increasing in $\theta\in[-1,1]$. Define $f_+(\theta,r)=(\theta\cdot c_+(\theta),r+1-2\theta^2)$ in S and $f_+(\theta,r)=(\theta,r-1)$ otherwise. Now

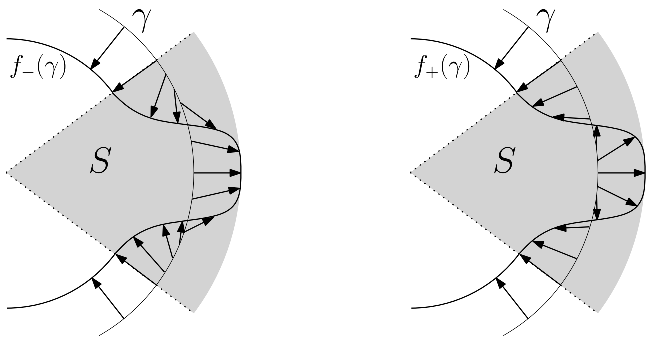

FIGURE 1. Qualitative description of maps f_- (left) and f_+ (right) within the sector S.

the vector $\overrightarrow{xf_+(x)}$ makes one turn in the positive sense as x moves along $\gamma \cap S$ so $i(f_+, o) = 1 + 1 = 2$. Notice that the modification in the angular coordinate has reversed the angular dynamics around $\theta = 0$, it is now repelling. Again, since f_+^n is conjugate to f_+ we obtain that $i(f_+^n, p) = i(f_+, p) = 2$ for every $n \geq 1$.

EXAMPLE 2.4. Let us show a way to modify arbitrarily the index using the previous construction. Fix an integer $m \geq 1$ and start with the map $(\theta, r) \mapsto (\theta, r - 1)$ as before. Take any radial sector S_m and divide it into m equal sectors $S_m = T_1 \cup \ldots \cup T_m$. Set the map in each sector T_i to be equal to $f_- \colon S \to S$, after a suitable affine transformation in the angular coordinate, and denote by $f_{m,-} \colon S_m \to S_m$ the resulting map.

It is not difficult to check that the origin has now index $i(f_{m,-}, o) = 1 - m$ as the vector $\overrightarrow{xf_{m,-}(x)}$ winds m times in the negative sense as x moves along $\gamma \cap S_m$.

In an analogous fashion one can define $f_{m,+} \colon S_m \to S_m$ and show that $i(f_{m,+}, o) = 1 + m$.

We finish this section with some general considerations about the fixed point index sequence $I = (i(f^n, p))_n$. Albrecht Dold [4] found the unique constraint satisfied by any fixed point index sequence. Define the normalized sequences $\sigma^k = (\sigma^k_n)_n$ by

$$\sigma^k_n = \begin{cases} k & \text{if } n \in k\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Dold relations, also called Dold congruences, state that if I is any fixed point index sequence then I is a (formal) integer combination of normalized sequences, i.e. there exist integers a_k, $k \geq 1$, such that

$$I = \sum_{k \geq 1} a_k \sigma^k. \tag{2.1}$$

As already commented in the introduction, Dold relations are the only general constraints governing fixed point index sequences.

3. Theorem 1.1: construction

Let us start with the proof of Theorem 1.1. Fix from now and on a sequence of integers $(a_k)_k$ which in view of equation (2.1) uniquely determines the target sequence of fixed point indices $(i(f^n, o))_n$. We will construct a map $\widetilde{f}\colon \mathbb{R}^d \to \mathbb{R}^d$ whose only periodic orbit is the origin, which is a fixed point, and such that

$$(i(\widetilde{f}^n, o))_n = \sum_{k\geq 1} a_k \sigma^k. \tag{3.1}$$

It is enough to prove the result for $d = 2$. Indeed, if f is a planar map satisfying Theorem 1.1 we split $\mathbb{R}^d = \mathbb{R}^2 \oplus \mathbb{R}^{d-2}$ and define $\widetilde{f} = (f, c)$, where c denotes the map that sends every point to the origin. From the multiplicativity of the fixed point index and the fact that c has index 1 at the origin we obtain that $(i(\widetilde{f}^n, o))_n = (i(f^n, o))_n$. Clearly $\widetilde{f}$ satisfies the requirements of Theorem 1.1.

Our target map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ will have the appearance of a skew-product in polar coordinates. We start with a very simple dynamics in the base, the angular coordinate, namely $e_2(\theta) = 2\theta \mod 1$ (now we assume θ ranges in $[0, 1)$). The dynamics will become richer after we replace the angles in a set of periodic orbits of e_2 by intervals and extend the map to them. The set of periodic orbits Λ in which this blow-up procedure will be carried out is defined by the following two properties:

(i) Λ is composed of exactly one orbit of each period in a set $\mathcal{P} \subset \mathbb{N}$.
(ii) $\Lambda' \cap \mathrm{Per}(e_2) = \emptyset$, i.e. no point of accumulation of Λ is periodic under e_2.

The task of showing that the definition of Λ is not vacuous is postponed to Section 4, where we produce a set satisfying both of its defining properties. We use the notation X' to refer to the derived set of a set X. It is composed of all accumulation points, also called limit points, of X. Property (i) ensures that Λ' is invariant under e_2. Note that (ii) implies that any $x \in \Lambda$ is isolated in Λ.

First, we begin with a map f_0 whose radial dynamics makes all the indices $i(f_0^n, o)$ to be equal to 1. We group in $\mathcal{P}$ the set of exponents n for which we need to modify the map to achieve the desired value for the index of f^n which, in view of (3.1), is

$$i(f^n, o) = \sum_{k|n} k a_k. \tag{3.2}$$

Thus, we set that 1 belongs to $\mathcal{P}$ if and only if $a_1 \neq 1$ and $k \in \mathcal{P}$ if and only if $a_k \neq 0$ for $k > 1$.

The map f_0 is given in polar coordinates by

$$f_0\colon (\theta, r) \mapsto (e_2(\theta), r - g_0(\theta)).$$

The stretching in the radial direction is determined by the map $g_0 \colon S^1 \to \mathbb{R}$ which we define by $g_0(\theta) = \operatorname{dist}(\theta, \Lambda')$. Note that (ii) guarantees that the orbit under f_0 of every point whose angular coordinate is periodic under e_2 tends to the origin. However, if $\Lambda' \neq \emptyset$ there are points whose ω-limit is not $\{o\}$ and they are not periodic.

An application of Corollary 2.2 shows that $i(f_0^n, o) = 1$ for any $n \geq 1$. The dynamics of f_0 is somehow complicated though. For any r_0, $\Lambda' \times \{r_0\}$ is a closed invariant set contained in the circle $\{r = r_0\}$ which does not contain any periodic point. The origin is then far from being locally maximal, since the maximal invariant set contained in the closed disk $\{r \leq r_0\}$ is $(\Lambda' \times \{r \leq r_0\}) \cup \{o\}$.

Now, we enrich the dynamics of e_2 by replacing each point $\alpha \in \Lambda$ with a non-degenerate interval J_α. Let $\pi \colon S^1 \to S^1$ be the map which realizes this operation, $\pi^{-1}(\alpha) = J_\alpha$ for every $\alpha \in \Lambda$ and π is a homeomorphism outside the intervals J_α. Then, e_2 lifts to a map $h \colon S^1 \to S^1$ which satisfies $\pi \circ h = e_2 \circ \pi$. Define accordingly the map g in $\pi^{-1}(S^1 \setminus \Lambda)$ so that $\pi \circ g = g_0 \circ \pi$ and set

$$f \colon (\theta, r) \mapsto (h(\theta), r - g(\theta)).$$

It is intentionally still left to complete the definition of g, h for angles inside the intervals J_α and thus the definition of f within the radial sectors S_α determined by them.

Let $\{\alpha_0, \dots, \alpha_{n-1}\}$ be an orbit of period n under e_2 contained in Λ. The dynamics of f in the sectors $S_{\alpha_0}, \dots S_{\alpha_{n-1}}$ will be responsible for the value of the integer coefficient a_n in (3.1). We define $f_{|S_{\alpha_i}} \colon S_{\alpha_i} \to S_{\alpha_{i+1}}$ to be equal to, after affine rescaling in the angular coordinate, $f_{m,*}$, where $* = +$ or $-$. The choice of sign and integer m is

$$(m, *) = \begin{cases} (a_1 - 1, +) & \text{if } n = 1 \text{ and } a_1 \geq 2, \\ (1 - a_1, -) & \text{if } n = 1 \text{ and } a_1 \leq 0, \\ (a_n, +) & \text{if } n \geq 2 \text{ and } a_n > 0, \\ (-a_n, -) & \text{if } n \geq 2 \text{ and } a_n < 0. \end{cases} \tag{3.3}$$

It is convenient to say a few words about the dynamics of f before focusing on the index computation. It still shares some of the properties of f_0. For instance, for every $r_0 \in \mathbb{R}$, $\pi^{-1}(\Lambda') \times \{r_0\}$ is invariant and the maximal invariant set in the closed disk $\{r \leq r_0\}$ is $(\pi^{-1}(\Lambda') \times \{r \leq r_0\}) \cup \{o\}$. Each sector S_α is periodic and contains an odd number ($2m + 1$ exactly) of periodic rays of the same period as S_α, which are alternatively attracting or repelling, the two rays in the boundary of S_α being attracting. Here attracting means that the orbit of every point in the ray tends to the origin and repelling means that it goes to infinity. The orbit of any other point in S_α goes to infinity and approaches a repelling periodic ray provided the sign in (3.3) is "$-$". Otherwise, in case the plus sign was assigned in (3.3), most of the orbits tend to the origin and approach an attracting periodic ray.

In the complement of the regions already described the behavior of the orbits is not particularly simple. Orbits of points in rays of angle θ such that $\theta \notin \pi^{-1}(\Lambda')$ and whose orbits under h never land in an interval J_α tend to the origin. However, any of the previously described asymptotics might occur if the orbit eventually lands in a point with angle in $\pi^{-1}(\Lambda')$ or in some J_α.

On top of all the previous discussion, let us emphasize that $\mathrm{Per}(f) = \{o\}$. Indeed, in any orbit the radial coordinate only stabilizes if the angular coordinate eventually lies in $\pi^{-1}(\Lambda')$. Since in $\pi^{-1}(\Lambda')$ there are no periodic points under h, the origin is an isolated periodic point.

PROPOSITION 3.1. $i(f^n, o) = \sum_{k|n} k a_k$.

PROOF. The computation of the index will be done using the curve γ, which is the boundary of a circle centered at o (anyone works fine).

Consider the collection $\mathcal{J}$ of intervals J_α which satisfy $h^n(J_\alpha) = J_\alpha$. It contains every J_α such that n is a multiple of the period of $\alpha \in \Lambda$. Consider the union of the radial sectors S_α determined by all J_α in $\mathcal{J}$ and denote by C its complement in the plane. The picture in C is easy to analyze. Indeed, as a point x moves along a component J' of $\gamma \cap C$ the vector $\overrightarrow{xf^n(x)}$ never points in the same direction as $\overrightarrow{ox}$. Moreover, it starts and ends its tour along J' pointing to the opposite direction as $\overrightarrow{ox}$ because $g(\theta)$ is strictly negative in Λ and the endpoints of J' are fixed by h^n. Incidentally, note that the arc $f^n(J')$ might wind around the origin several times but this behavior does not make any impact in the index because $f^n(J')$ lies inside the disk enclosed by γ.

Thus, the problem amounts to examine what happens in every arc $\gamma \cap S_\alpha$ where $J_\alpha \in \mathcal{J}$. Denote by $(m, *)$ the pair associated to S_α according to (3.3). As noticed in Example 2.4, all the maps $f^l_{m,*}$, $l \geq 1$, are conjugate. Up to an affine transformation in the angular coordinate, the restriction of f^n to S_α is equal to $f^l_{m,*}$, where $l = n/k$ and k is the period of α. This remark allows to conclude that, as x moves along $\gamma \cap S_\alpha$, the vector $\overrightarrow{xf^n(x)}$ turns exactly m times in the (positive or negative) sense given by $*$.

Consequently, the dynamics in sector S_α adds $a_1 - 1$ to the index if $k = 1$ and a_k if $k \geq 2$. We need to take care of k sectors for each k divisor of n contained in $\mathcal{P}$ so we finally obtain

$$i(f^n, o) = 1 + (a_1 - 1) + \sum_{k \geq 2, k|n} k a_k$$

and the result follows. $\square$

In order to finish the proof of Theorem 1.1 it suffices to notice that the result of Proposition 3.1 simply rephrases equation (3.1).

4. Symbolic dynamics: definition of Λ

This section is devoted to discussion about the definition of Λ. Difficulties arise when trying to meet property (ii) and this makes the example we present here a bit involved. Notice that if Λ did not satisfy (ii) there would exist periodic points β under h accumulated by intervals J_α. Since the absolute value of g is arbitrarily small in J_α as the period of α grows, we would have $g(\beta) = 0$ and so every point in the ray $\{\theta = \beta\}$ would be periodic, contrary to our hypothesis.

Symbolic dynamics eases the analysis of properties from subsets of our system whose dynamics can be encoded properly. We will construct Λ from a subset of a symbolic dynamical system defined using the Prouhet–Thue–Morse sequence. Several other approaches exist as well. For instance, one may use a symbolic sequence for which the relative density of each symbol is irrational.

Let Σ_2 be the set of one-sided infinite sequences in two symbols $\{0, 1\}$ and σ the shift map in Σ_2. The dynamical system (S^1, e_2) is a factor of (Σ_2, σ). The semiconjugation $\pi\colon \Sigma_2 \to S^1$ first sends an infinite sequence $s = d_1 \dots d_n \dots$ to the number $x = \sum\limits_n d_n 2^{-n}$ in $[0, 1]$ which has s as binary expansion and then projects it to S^1. For the sake of clarity, here we view S^1 as the interval $[0, 1]$ whose endpoints are identified. The following diagram is commutative:

$$\begin{tikzcd} \Sigma_2 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_2 \arrow[d, "\pi"] \\ S^1 \arrow[r, "e_2"'] & S^1. \end{tikzcd}$$

Let $(s_n)_n$ be a sequence of periodic infinite sequences whose periods tend to infinity. Suppose that $(s_n)_n$ has limit s and s is periodic, i.e. it is made up of the infinite repetition of some word w (a finite sequence of 0's and 1's). Then, the sequence s_n starts with the word w for sufficiently large n. Furthermore, given any positive integer k it must be the case that s_n starts with the word w^k, where w^k denotes the word $ww \dots w$ consisting of w repeated k times in a row. Thus, if for some $k \geq 1$ we ensure that s_n does not start with an instance of the pattern w^k, for any word w and large enough n, we prove that the limit of $(s_n)_n$ is not a periodic sequence.

Our goal is to find a set of periodic sequences A in Σ_2 satisfying the following properties:

(a) For every $n \geq 1$, A contains a sequence of period n.
(b) A is invariant under σ, that is, it contains the whole orbit of each of its periodic points.

(c) There exists an integer k such that at most a finite number of elements of A start with the pattern w^k.

Once we obtain A we may simply set $\Lambda = \pi(A)$ and check properties (i)–(ii) from the previous section. The semiconjugation guarantees that any point of Λ is periodic. We shall remove from Λ all orbits whose periods do not belong to $\mathcal{P}$. The previous considerations and condition (c) ensure that property (ii) is satisfied.

The set A will be created using the Prouhet–Thue–Morse sequence, which is defined as follows. Start with the word 0 and at each step do the replacements $0 \mapsto 01$, $1 \mapsto 10$. The words built in the first stages are

$$0,\ 01,\ 0110,\ 01101001,\ 0110100110010110,\ \ldots$$

The process continues ad infinitum and the words converge (note that their starting subwords are equal) to an infinite sequence $\mathbf{t}$ named after Prouhet, Thue and Morse, who discovered it independently,

$$\mathbf{t} = 0110100110010110100101100110100 1\ldots$$

This sequence exhibits an aperiodic yet recurrent behavior and has shown up in various fields of mathematics. One striking feature of $\mathbf{t}$ makes it interesting to us: it is cube-free, that is it does not contain any instance of the pattern w^3 (a word appearing three times in a row). This result is part of one of the foundational works on the field of combinatorics on words and was first proved by Axel Thue in 1912 [21] and then rediscovered by Marston Morse in 1921 [16]. For example the sequences

$$10\mathbf{010101}10100110,\ \ 010110\mathbf{100100100}101100110$$

are not cube-free because they contain the words $010101 = 01^3$ and $100100100 = 100^3$, respectively.

A word v is said to be conjugate to w if there are (possibly empty) words x, y such that $w = xy$ and $v = yx$. The circular word of w is the set consisting of w and all of its conjugates. There is an evident one-to-one correspondence between periodic infinite sequences whose period is a divisor of n and words of length n: any such sequence s is formed by the repetition of a word w. As a tiny trivial remark, note that the orbit of s under the action of the shift map is made up of the infinite sequences generated from each of the conjugates of the word w. For example the circular word of 100 is $\{100, 001, 010\}$ and the orbit of $s = 100100100\ldots$ under σ has period 3:

$$\sigma(s) = 001001001\ldots,\ \ \sigma^2(s) = 010010010\ldots,\ \ \sigma^3(s) = 100100100\ldots = s.$$

Consider again the Prouhet–Thue–Morse infinite sequence $\mathbf{t} = d_1 \ldots d_n \ldots$ and, for every $n \geq 1$, define the word $s_n = d_1 \ldots d_n$ of length n. Let s_n^* be the infinite sequence generated by repetition of s_n. Set A to be the union of the orbits under σ of all the sequences s_n^*.

Proposition 4.1. *A satisfies properties* (a)–(c) *above.*

Proof. The first two properties follow from the definition so we shall concentrate on (c). We will prove that no conjugate of s_n^* starts with an instance of the pattern w^6 if n is large enough. In other words, there does not exist any word w such that a conjugate of s_n starts with w^6. Suppose on the contrary that some conjugate yx of $s_n = xy$ starts like that and $n > 6$ length(w). It follows that either x or y contains the word w^3. This fact leads to contradiction because both x and y are subwords of the Prouhet–Thue–Morse sequence which is known to be cube-free. □

The previous method generates a set of circular words of every length which are 6-power free, i.e. do not contain an instance of the patter w^6. The construction is very elementary and very far from being optimal. For a more up-to-date account on this topic together with an optimal result we refer the reader to [1].

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LUIS HERNANDEZ-CORBATO
IMPA
Estrada dona Castorina 110
Rio de Janeiro, BRAZIL
E-mail address: luishcorbato@mat.ucm.es, lcorbato@impa.br