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REMARKS ON THE PAPER "COVERING MANIFOLDS FOR ANALYTIC FAMILIES OF LEAVES OF FOLIATIONS BY ANALYTIC CURVES" (TOPOL. METHODS NONLINEAR ANAL. 11 (1998), 361–373)

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The main result of paper [3] is the theorem

Theorem 1 [3, Theorem 1.2]). *A covering manifold corresponding to a codimension one Stein cross-section for an analytic foliation by curves with singularities of $\mathbb{C}^n$, is a Stein manifold itself.*

Below we improve it:

Theorem 2. *Theorem* 1 *holds with* $\mathbb{C}^n$ *replaced by any Stein manifold.*

Moreover, we add some missing details in the proof of Theorem 1, and simplify some other arguments.

Definition of a covering manifold of family of leaves is given in paper [3]. We remind the notations.

Suppose $\mathcal{F}$ is a foliation with singularities of $\mathbb{C}^n$ by analytic curves, B a Stein hypersurface transversal to the leaves, φ_p the leaf through the point $p \in B$, M the union of the leaves φ_p, $p \in B$, $\widetilde{M}$ the corresponding covering manifold over

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the leaves of M, and $\widetilde{\pi}\colon \widetilde{M} \to M$ the natural projection. The universal covering of the leaf φ_p is denoted $\widetilde{\varphi}_p$. There is also the projection $\pi\colon \widetilde{M} \to B$ along the leaves, which gives to $\widetilde{M}$ the structure of a skew cylinder with the leaves $\widetilde{\varphi}_p$ (see Definition 1 of [3]).

Lemma 1 below claims that $\widetilde{M}$ is a Riemann domain. Here we accept the concept of a Riemann domain in sense of [2, 5.4], it means that the manifold must have a locally biholomrphic projection to a domain in $\mathbb{C}^n$ and must be holomorphically separable.

The manifold $\widetilde{M}$ has a holomorphic envelope S, and this holomorphic envelope is a Stein manifold itself (see [1, 1G]). The set $\widetilde{M}$ is mapped into S by some mapping i, which is locally biholomorphic and is an open embedding.

This used in the proof of Theorem 1, as soon as the following lemma is proved.

Lemma 3 ([3, Lemma 2.1]). *The manifold $\widetilde{M}$ is a Riemann domain over M.*

Proof. We pay attention here mostly to the details different from the proof of Lemma 2.1 in [3].

Let b_1, b_2 be nonseparable points on $\widetilde{M}$. They must have the same image under the projection $\widetilde{\pi}$ and belong to the same fiber of the skew cylinder $\widetilde{M}$; otherwise they be separated by the coordinate functions on $\mathbb{C}^n$ or the coordinate functions on the base lifted to $\widetilde{M}$. Let $b = \widetilde{\pi}b_1 = \widetilde{\pi}b_2$, $p = \pi b_1 = \pi b_2$. Let φ_p be the corresponding leaf and $\pi_1(\varphi_p, p)$ its fundamental group. Suppose that the points b_j are represented by the curves $\gamma_j \subset \varphi_p$ beginning at p and landing at b. These curves may be so chosen that their lifts to $\widetilde{M}$ will be simple and smooth; but the curves themselves may have selfintersections. Denote $\mu = \gamma_1^{-1}\gamma_2$.

In paper [3] four cases were considered:

(a) The curve μ is simple nonclosed.
(b) The curve μ is simple and closed.
(c) The curve μ is closed but not necessary simple.
(d) The curve μ is nonclosed and selfintersected.

Really there is no need to consider cases (a) and (d), because in these cases the points b_1, b_2 have different projections on M. In the paper [3] at the consideration of case (b) were neglected the situation when the leaf φ_p can be extended as an analytic set to some singular point a. That is, the leaf is a separatrix. Now we consider this case.

We will construct a holomorphic function I on $\widetilde{M}$ that separates points b_1 and b_2 thus bringing the assumption above to a contradiction and proving Lemma 3 in case (b).

Let ω be one-form on $\mathbb{C}^n$ with rational coefficients, such that its restriction to the leaves of the foliation is a holomorphic one-form: $\omega_p = \omega|_{\varphi_p}$, see (4) below for the main example.

Denote by ω_p^* the pullback of ω_p: $\omega_p^* = \widetilde{\pi}^*\omega$. For any point $q \in \widetilde{M}$ let $p = \pi(q)$ and γ_q be a curve on the fiber $\widetilde{\varphi}_p = \pi^{-1}p$ that begins at p and ends at q. The function

$$I_\omega \colon q \mapsto \int_{\gamma_q} \omega *_p \tag{1}$$

is well-defined on $\widetilde{M}$ because the fibers are simply connected.

The function I_ω is holomorphic on $\widetilde{M}$ because of the analytic dependence on the fibers on the initial condition.

We will now construct the form ω such that I_ω would separate b_1 and b_2. This is equivalent to the inequality

$$\int_\mu \omega \neq 0. \tag{2}$$

So we will find a polynomial 1-form ω that satisfies (2). In case (b), μ is a simple loop on the fiber φ_p; this fiber may be a separatrix of some singular points; by definition, they are not included in φ_p. Without loss of generality, μ is positively oriented. Hence, μ is homological on φ_p to a sum of simple loops: $\mu = \mu_1 + \ldots + \mu_s$, m_j is a simple loop of φ_p around a singular point a_j. Recall that the union $\varphi_p \cup a_j$ is an analytic curve near a_j plausibly with a singularity at a_j. The integral in the left hand side of (2) equals to the sum of residues of $\omega|_{\varphi_p}$ at the points a_j times $2\pi i$. We will construct ω so that

$$\operatorname{res}_{a_1}\omega|_{\varphi_p} \neq 0 = \operatorname{res}_{a_j}\omega|_{\varphi_p}, \quad j = 2, \ldots, s. \tag{3}$$

Choose the coordinates $z_1, \ldots, z_n$ on the affine space $\mathbb{C}^n$ such that $z_j(a_1) = 0$ and $z_j|\varphi_p \neq 0$ identically for all j. The foliation on $\mathbb{C}^n$ can be determined by a global holomorphic vector field V on $\mathbb{C}^n$:

$$V = P_1 \frac{\partial}{\partial z_1} + \ldots + P_n \frac{\partial}{\partial z_n}.$$

The form

$$dt = \frac{dz_1}{P_1} = \ldots = \frac{dz_n}{P_n} \tag{4}$$

is holomorphic on φ_p and the nearby fibers.

Proposition 4. *Let $\bar{\varphi}_p$ be $\varphi_p \cup a$. Then form* (2) *can be extended meromorphically to $\bar{\varphi}_p$ and has a pole at the point a.*

Proof. Let us uniformize $\bar{\varphi}_p$ at the point a, let $f \colon D \to \overline{\varphi}_p$ be the uniformizing map; suppose D is the disk centered at zero. Let $\widetilde{V} = f^*V$ be the pullback of the field V on D, $\widetilde{V}(0) = 0$ since q is a singular point. As $(dt)^*\widetilde{V} = 1$ we obtain that the form $(dt)^*$ has a pole at zero and the proposition holds. $\square$

Let $q \in \widetilde{M}$ be a point on the universal cover $\widetilde{\varphi}_{p'}$, for some $p' \in B$, γ_q the path on $\varphi_{p'}$ corresponding to this point. Since the form dt is holomorphic on the leaves and holomorphically depends on a leaf, the integral of type (1):

$$t = \int_{\gamma_q} dt \tag{5}$$

is well-defined on the universal cover $\varphi_{p'}$ and defines a holomorphic function on $\widetilde{M}$.

Let w be the uniformizing chart on the leaf φ_p near a_1. Then $(z_j|\varphi_p) = C_j w^{k_j} + \dots$ for some $C_j \neq 0$, $k_j \in \mathbb{N}$, here and below dots replace the higher order terms. The restriction to φ_p of any polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ is a polynomial in $w^{k_1}, \dots, w^{k_n}$. Hence, $P_1|_{\varphi_p} = Cw^d + \dots$, $C \neq 0$,

$$d = \sum l_i k_{ij} \tag{6}$$

for some $l_i \in \mathbb{Z}^+$, $d > 0$. We may assume $l_1 \neq 0$.

Consider now a polynomial $f = gh$ such that

$$g = z_1^{l_1 - 1} \Pi_2^n z_j^{l_j}; \quad h(0) = 1, \quad h(a_j) = 0, \quad j = 2, \dots, n.$$

Moreover, the order of zero of h at a_j is high, as specified below.

Let now $\omega = f\,dt$. Note that

$$g\,dt|\varphi_p = g \frac{dz_1}{P_1}\bigg|_{\varphi_p} = \frac{C_1\,dw}{Cw + \dots}, \quad C_1 C \neq 0.$$

Hence, $\mathrm{res}_{a_1} g\,dt|_{\varphi_p} \neq 0$. The same holds for ω.

Let us now choose the order of h at a_j, $j \neq 1$ so high that $res_{a_j}\omega|_{\varphi_p} = 0$. This implies (3), hence Lemma 3 in case (b).

In (b) the curve μ is closed but not necessary simple and may be homological but nonhomotopical to zero. This case is considered as in the proof of Lemma 2.1 in [3], only case (b) in the reference is treated as above. Thus we have proved Lemma 3. $\square$

Now we complete the proof of the Theorem 1 as in paper [3] using the Closure Lemma and the Connectedness Lemma.

The proof of Theorem 2 is the same as of Theorem 1, only we should use the existence of a holomorphic envelope for a manifold that is mapped locally byholomorphically to a Stein manifold (see [4]). After that we can embed the Stein manifold of Theorem 2 into $\mathbb{C}^N$ as closed analytic subset and continue as for Theorem 1.

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