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ON SINGULAR NONPOSITONE SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. We prove the existence of a large positive solution for the boundary value problems

$$\begin{aligned} -\Delta u &= \lambda(-h(u) + g(x,u)) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$, λ is a positive parameter, $g(x,\,\cdot\,)$ is sublinear at ∞, and h is allowed to become ∞ at $u=0$. Uniqueness is also considered.

1. Introduction

Consider the boundary value problems

$$\begin{aligned} -\Delta u &= \lambda(-h(u) + g(x,u)) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $h\colon(0,\infty)\to[0,\infty)$, $g\colon\overline{\Omega}\times[0,\infty)\to\mathbb{R}$, and λ is a positive parameter.

The existence and uniqueness of positive (1.1) when $f(x,u)\equiv -h(u)+g(x,u)$ is nonnegative and sublinear at ∞ have been studied extensively (see [2], [3], [5]–[9] and the references therein). We are interested here in studying positive solutions of (1.1) in the challenging case when $f(x,u)$ becomes $-\infty$ at $u=0$,

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which does not appear to have been considered in the literature. Our main result, in particular, gives the existence of a large positive solution for the problem

$$
\begin{aligned}
-\Delta u &= \lambda\left(\frac{-1}{u^{\alpha}\ln^{\beta}(1+u)} + u^{\gamma} + k(x)\right) && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

for λ large, where $\alpha, \beta \geq 0$, $\alpha + \beta < \min(1, 2/N)$, $0 < \gamma < 1$ and $k \in C(\overline{\Omega})$. Uniqueness in a class of large positive solutions is also obtained. Our approach is based on the Schauder Fixed Point Theorem.

2. Main results

We make the following assumptions:

(A.1) $h\colon (0,\infty) \to [0,\infty)$ is of class C^1, nonincreasing, $h(u) \to 0$ as $u \to \infty$, and there exists $p > \max(1, N/2)$ such that $h \in L^p(0,T)$ for all $T > 0$.

(A.2) $g\colon \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is continuous and nondecreasing in u.

(A.3) There exist positive numbers L, L_1 such that $g(x,u) > 2L$ for $x \in \Omega$, $u > L_1$, and

$$\lim_{u\to\infty} \frac{g(x,u)}{u} = 0$$

uniformly for $x \in \Omega$.

(A.4) There exists $q \in (0,1)$ such that $g(x,u)/u^q$ is nonincreasing for each $x \in \Omega$.

(A.5) There exists a positive number m such that

$$\sup_{x\in\Omega} g(x,u) \leq m \inf_{x\in\Omega} g(x,u) \quad \text{for all } u > 0.$$

Let ϕ be the solution of

$$-\Delta\phi = 1 \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

By a solution of (1.1), we mean a function $u \in C^1(\overline{\Omega})$ which satisfies (1.1) in the weak sense. Our main result is

THEOREM 2.1. *Let* (A.1)–(A.3) *hold. Then there exists a positive number* λ_0 *such that for* $\lambda > \lambda_0$, *problem* (1.1) *has a solution* u *with* $u \geq \lambda L\phi$ *in* Ω. *If, in addition,* (A.4) *and* (A.5) *hold, then the solution is unique in this class.*

LEMMA 2.2. *Let* (A.1) *hold. Then* $h(c\phi) \in L^p(\Omega)$ *for all* $c > 0$.

PROOF. By the maximum principle, there exists a constant $k_1 > 0$ such that $\phi \geq k_1 d$ in Ω, where $d(x) = d(x,\partial\Omega)$. Hence it suffices to prove the result with ϕ replaced by d. This is now obvious because near a point of $\partial\Omega$, we can choose local coordinates for Ω where $a(x)$ is one of the co-ordinates (and the other co-ordinates are co-ordinates of $\partial\Omega$). Note that there will be a Jacobian in the change of variables but this will be bounded. $\square$

Lemma 2.3. *Let* (A.1)–(A.5) *hold. Then there exist positive numbers* λ^*, c_1, c_2 *such that, if* u *is a solution of* (1.1) *with* $\lambda > \lambda^*$ *and*

$$u \geq \lambda L\phi \quad \text{in } \Omega$$

then

$$c_1 G^{-1}(\lambda)\phi \leq u \leq c_2 G^{-1}(\lambda)\phi \quad \text{in } \Omega,$$

where

$$G(z) = \frac{z}{\widetilde{g}(z)}, \quad \widetilde{g}(z) = \inf_{x\in\Omega} g(x,z), \quad z > L_1.$$

Proof. Note that G is increasing on $(0,\infty)$ and $G(z) \to \infty$ as $z \to \infty$, by (A.4). Let u be a solution of (1.1) satisfying $u \geq \lambda L\phi$ in Ω with $\lambda > \lambda^*$, where $\lambda^* > 0$ is to be chosen later. Define $\delta = \sup\{c > 0 : u \geq c\phi \text{ in } \Omega\}$. Then $\delta \geq \lambda L$ and $u \geq \delta\phi$ in Ω. Let v_λ satisfy

$$-\Delta v_\lambda = h(\delta\phi) \quad \text{in } \Omega, \qquad v_\lambda = 0 \quad \text{on } \partial\Omega.$$

Since $h(\lambda L\phi) \to 0$ pointwise in Ω as $\lambda \to \infty$ and

$$h(\lambda L\phi) \leq h(\lambda^* L\phi) \in L^p(\Omega),$$

by Lemma 2.2, it follows from the Lebesgue dominated convergence theorem that $||h(\lambda L\phi)||_{L^p} \to 0$ as $\lambda \to \infty$. Since $p > \max(1, N/2)$, we have that $v_\lambda \in C^1(\overline{\Omega})$ and

$$|v_\lambda|_{C^1} \leq M||v_\lambda||_{W^{2,p}} \leq M_1||h(\delta\phi)||_{L^p} \leq M_1||h(\lambda L\phi)||_{L^p} \tag{2.1}$$

(see [1], [4]), and hence $|v_\lambda|_{C^1} \to 0$ as $\lambda \to \infty$.

Let $K > 0$ be such that

$$g(x,u) \geq -K \tag{2.2}$$

for all $x \in \Omega$, $u > 0$. Then we have, for $\lambda^* > L_1/L|\phi|_\infty$,

$$\begin{aligned}
-\Delta(u + \lambda v_\lambda) &= \lambda(-h(u) + g(x,u)) + \lambda h(\delta\phi) \geq \lambda\widetilde{g}(\delta\phi) \\
&= \lambda(\widetilde{g}(\delta\phi)\chi_{\{x:\phi(x)>L_1/\lambda L\}} - K\chi_{\{x:\phi(x)\leq L_1/\lambda L\}}) \\
&= \lambda\bigg(\frac{\delta\phi}{G(\delta\phi)}\chi_{\{x:\phi(x)>L_1/\lambda L\}} - K\chi_{\{x:\phi(x)\leq L_1/\lambda L\}}\bigg) \\
&\geq \lambda\bigg[\frac{\delta\phi}{G(\delta|\phi|_\infty)} - \bigg(\frac{\delta|\phi|_\infty}{G(\delta|\phi|_\infty)} + K\bigg)\chi_{\{x:\phi(x)\leq L_1/\lambda L\}}\bigg],
\end{aligned}$$

where χ_B denotes the characteristic function of B, i.e. $\chi_B\ (x) = 1$ if $x \in B$, 0 if $x \notin B$. This implies

$$u + \lambda v_\lambda \geq \lambda\bigg(\frac{\delta}{G(\delta|\phi|_\infty)}\psi - \bigg(\frac{\delta|\phi|_\infty}{G(\delta|\phi|_\infty)} + K\bigg)w_\lambda\bigg),$$

in Ω, where ψ and w_λ satisfy

$$-\Delta\psi = \phi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega,$$

and

$$-\Delta w_\lambda = \chi_{\{x:\phi(x)\le L_1/\lambda L\}} \quad \text{in } \Omega, \qquad w_\lambda = 0 \quad \text{on } \partial\Omega,$$

respectively. Note that $|w_\lambda|_{C^1} \to 0$ as $\lambda \to \infty$. Consequently, for λ^* large,

$$\begin{aligned}
u &\ge \lambda\biggl(\frac{\delta}{G(\delta|\phi|_\infty)}\psi - \biggl(\frac{\delta|\phi|_\infty}{G(\delta|\phi|_\infty)} + K\biggr)w_\lambda - v_\lambda\biggr)\\
&\ge \lambda\biggl(\frac{\delta}{G(\delta|\phi|_\infty)}\psi - \biggl(\frac{\delta|\phi|_\infty}{G(\delta|\phi|_\infty)} + K\biggr)|w_\lambda|_{C^1}d - |v_\lambda|_{C^1}d\biggr)\\
&\ge \lambda\biggl(\frac{\delta k_0}{G(\delta|\phi|_\infty)} - \biggl(\frac{\delta|\phi|_\infty}{G(\delta|\phi|_\infty)} + K\biggr)k|w_\lambda|_{C^1} - |v_\lambda|_{C^1}k\biggr)\phi \ge \frac{\lambda\delta k_0}{2G(\delta|\phi|_\infty)}\phi,
\end{aligned}$$

where k and k_0 are positive numbers such that $\psi \ge k_0\phi$, $d \le k\phi$ in Ω. Here we have used the fact that

$$\frac{\delta k_0}{G(\delta|\phi|_\infty)} = \frac{k_0}{|\phi|_\infty}\widetilde{g}(\delta|\phi|_\infty) \ge \frac{k_0}{|\phi|_\infty}\widetilde{g}(L_1) > 0.$$

By the maximality of δ, we obtain $\lambda\delta k_0/(2G(\delta|\phi|_\infty)) \le \delta$ or

$$\delta \ge \frac{1}{|\phi|_\infty}G^{-1}\biggl(\frac{\lambda k_0}{2}\biggr).$$

Using (A.4), it can be verified that for $C > 0$ and $\lambda > G(L_1)\max(1, C)$,

$$\frac{1}{\max(1, C^{-1/(1-q)})}G^{-1}(\lambda) \le G^{-1}(\lambda C) \le \max(1, C^{1/(1-q)})G^{-1}(\lambda),$$

and hence $\delta \ge c_1 G^{-1}(\lambda)\phi$ in Ω, where c_1 is a positive constant depending only on k_0, $|\phi|_\infty$.

Next, using (A.5), we obtain

$$-\Delta u \le \lambda g(x,u) \le \lambda m\widetilde{g}(|u|_\infty), \tag{2.3}$$

which implies

$$u \le \lambda m\widetilde{g}(|u|_\infty)\phi$$

in Ω. Hence

$$G(|u|_\infty) \le \lambda m|\phi|_\infty,$$

or, equivalently,

$$|u|_\infty \le G^{-1}(\lambda m|\phi|_\infty).$$

Using this in (2.3), we infer that

$$-\Delta u \le \lambda m\widetilde{g}(G^{-1}(\lambda m|\phi|_\infty)),$$

and so

$$u \le \lambda m\widetilde{g}(G^{-1}(\lambda m|\phi|_\infty))\phi = \frac{G^{-1}(\lambda m|\phi|_\infty)}{|\phi|_\infty}\phi \le c_2 G^{-1}(\lambda)\phi,$$

where c_2 is a positive constant depending only on m and $|\phi|_\infty$. This completes the proof of Lemma 2.3. $\square$

Proof of Theorem 2.1. Let $\lambda > \lambda_0 > 0$ and define $\mathbf{K} = \{u \in C(\overline{\Omega}) : \lambda L\phi \le u \le c_\lambda \text{ in } \Omega\}$, where c_λ and λ_0 are large numbers to be chosen later. For $u \in \mathbf{K}$, we have $h(u) \le h(\lambda L\phi)$ since h is nonincreasing. By Lemma 2.2, $h(u) \in L^p(\Omega)$ and so the problem

$$\begin{aligned} -\Delta v &= \lambda(-h(u) + g(x,u)) \quad && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega \end{aligned}$$

has a unique solution $v = Au \in W_0^{2,p}(\Omega) \cap C^1(\overline{\Omega})$. We shall verify that $A\colon \mathbf{K} \to \mathbf{K}$ if λ_0 is large enough. Let $v = Au$ for some $u \in \mathbf{K}$. Let z_λ satisfy

$$-\Delta z_\lambda = h(\lambda L\phi) \quad \text{in } \Omega, \qquad z_\lambda = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

and note that $|z_\lambda|_{C^1} \to 0$ as $\lambda \to \infty$. Then we have

$$-\Delta(v + \lambda z_\lambda) = -\lambda h(u) + \lambda g(x,u) + \lambda h(\lambda L\phi) \ge \lambda g(x,u) \tag{2.5}$$

in Ω. By (A.2),

$$g(x,u) \ge 2L\chi_{\{x:u(x)>L_1\}} - K\chi_{\{x:u(x)\le L_1\}} = 2L - (K+2L)\chi_{\{x:u(x)\le L_1\}}, \tag{2.6}$$

where K is defined in (2.2). Let ψ_λ satisfy

$$-\Delta\psi_\lambda = \chi_{\{x:u(x)<L_1\}} \quad \text{in } \Omega, \qquad \psi_\lambda = 0 \quad \text{on } \partial\Omega,$$

and note that

$$\{x \in \Omega : u(x) \le L_1\} \subseteq \{x \in \Omega : \phi(x) < L_1/(\lambda L)\}$$

and the Lebesgue measure of the latter set goes to 0 as λ goes to ∞. Hence

$$||\chi_{\{x:u(x)<L_1\}}||_{L^p} \le ||\chi_{\{x:\phi(x)\le L_1/\lambda L\}}||_{L^p} \to 0 \quad \text{as } \lambda \to \infty,$$

and therefore $|\psi_\lambda|_{C^1} \to 0$ as $\lambda \to \infty$. From (2.5), (2.6) and the comparison principle, we obtain

$$v + \lambda z_\lambda \ge 2\lambda L\phi - \lambda(K+2L)\psi_\lambda,$$

in Ω, which implies

$$\begin{aligned} v &\ge 2\lambda L\phi - \lambda z_\lambda - \lambda(K+2L)\psi_\lambda \\ &\ge 2\lambda L\phi - \lambda|z_\lambda|_{C^1} d - \lambda(K+2L)|\psi_\lambda|_{C^1} d \\ &\ge \lambda[2L - |z_\lambda|_{C^1} k - (K+2L)k|\psi_\lambda|_{C^1}]\phi \ge \lambda L\phi \end{aligned}$$

for λ large enough so that $|z_\lambda|_{C^1} k + (K+2L)k|\psi_\lambda|_{C^1} < L$, which is achieved if λ_0 is chosen so that

$$kM_1||h(\lambda_0 L\phi)||_{L^p} + k(K+2L)M_1||\chi_{\{x:\phi(x)<L_1/\lambda_0 L\}}||_{L^p} < L,$$

where M_1 is defined in (2.1).

Next, we have

$$-\Delta v \leq \lambda g(x, u) \leq \lambda m\widetilde{g}(|u|_\infty) \leq \lambda m\widetilde{g}(c_\lambda),$$

where $\widetilde{g}$ is defined in Lemma 2.2. By the comparison principle and the fact that $\lim\limits_{z\to\infty} \widetilde{g}(z)/z = 0$, we infer that

$$v \leq \lambda m\widetilde{g}(c_\lambda)\phi \ \leq \lambda m\widetilde{g}(c_\lambda)|\phi|_\infty \leq c_\lambda$$

in Ω if c_λ is large enough. Thus, for $\lambda > \lambda_0$, A maps $\mathbf{K}$ into itself and the Schauder fixed point theorem gives the existence of a solution u of (1.1) in $\mathbf{K}$.

Next, suppose that (A.4), (A.5) hold and let u, u_1 be solutions of (1.1) satisfying $u \geq \lambda L\phi$ in Ω. By increasing λ_0 if necessary, we assume that $\lambda_0 > \lambda^*$, where λ^* is given by Lemma 2.2. By Lemma 2.2, $u \geq (c_1/c_2)u_1$in Ω. Let τ be the maximum number such that $u \geq \tau u_1$ in Ω . Then $\tau \geq c_1/c_2 \equiv c_0$. Suppose that $\tau < 1$. We shall show that this leads to a contradiction. By (A.2) and (A.4),

$$\begin{aligned}-\Delta u &= \lambda(-h(u) + g(x, u)) \\ &\geq \lambda(-h(\tau u_1) + g(x, \tau u_1) \geq -\lambda h(\tau u_1) + \lambda\tau^q g(x, u_1),\end{aligned}$$

which implies

$$-\Delta(u - \tau^q u_1) \geq \lambda(\tau^q h(u_1) - h(\tau u_1)). \tag{2.7}$$

By the mean value theorem,

$$|(\tau^q h(u_1) - h(\tau u_1))| = (1-\tau)|(u_1 h'(cu_1) - qc^{q-1}h(u_1))|,$$

where c is between τ and 1. Since $th'(t) \leq h(t)$ for $t > 0$,

$$|u_1 h'(cu_1) - qc^{q-1}h(u_1)| \leq (c^{-1} + qc_0^{q-1})h(c_0 u_1) \leq (c_0^{-1} + qc_0^{q-1})h(\lambda Lc_0\phi),$$

it follows from (2.7) that

$$u - \tau^q u_1 \geq -\lambda(1-\tau)c_3\widetilde{z}_\lambda \tag{2.8}$$

in Ω, where $c_3 = c_0^{-1} + qc_0^{q-1}$ and $\widetilde{z}_\lambda$ satisfies

$$-\Delta\widetilde{z}_\lambda = h(\lambda Lc_0\phi) \quad \text{in } \Omega, \qquad \widetilde{z}_\lambda = 0 \quad \text{on } \partial\Omega.$$

Since $\tau^q - \tau \geq \tau^q(1-q)(1-\tau)$ and $|\widetilde{z}_\lambda|_{C^1} \to 0$ as $\lambda \to \infty$, it follows from (2.8) that

$$\begin{aligned}u - \tau u_1 &= u - \tau^q u_1 + (\tau^q - \tau)u_1 \geq (\tau^q - \tau)u_1 - \lambda(1-\tau)c_3\widetilde{z}_\lambda \\ &\geq \lambda(1-\tau)[c_0^q L(1-q) - c_3|\widetilde{z}_\lambda|_{C^1}k]\phi \geq \frac{\lambda(1-\tau)c_0^q L(1-q)}{2}\phi\end{aligned}$$

for λ large enough, a contradiction with the maximality of τ. Thus $\tau \geq 1$, which completes the proof of Theorem 2.1. □

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