Topological **M**ethods in **N**onlinear **A**nalysis
Volume 41, No. 1, 2013, 163–190

A GENERAL DEGREE FOR FUNCTION TRIPLES

Martin Väth

Abstract. Consider a fixed class of maps F for which there is a degree theory for the coincidence problem $F(x) = \varphi(x)$ with compact φ. It is proved that under very natural assumptions this degree extends to a degree for function triples which in particular provides a degree for coincidence inclusions $F(x) \in \Phi(x)$.

1. Introduction

In the paper [15], we proposed a general procedure which allows to extend any degree theory (satisfying certain axioms) for coincidences of function pairs $F(x) = \varphi(x)$ with

$$Y \xleftarrow{F} X \xrightarrow{\varphi} Y \tag{1.1}$$

to a related degree for coincidences of function triples $F(x) \in q(p^{-1}(x))$ with

$$Y \xleftarrow{F} X \xleftarrow{p} \Gamma \xrightarrow{q} Y \tag{1.2}$$

Here, roughly speaking, admissibility of homotopies, additivity of the degree and other related notions are meant with respect to the space X (and therefore, despite one could interpret the above diagram also differently, the triple degree

2010 *Mathematics Subject Classification.* Primary 47H11, 54H25, 55M20, 55M25; Secondary 47H04, 47H10, 54C60.

Key words and phrases. Fixed point index, degree theory, coincidence index, coincidence degree, multivalued map, nonlinear Fredholm map.

The paper was written in the framework of a DFG project (Az. VA 206/2-1). Financial support by the DFG is gratefully acknowledged.

is in some sense a measure of the number of solutions of the inclusion $F(x) \in q(p^{-1}(x))$ which in general is rather different than the number of solutions of the coincidence equation $F(p(z)) = q(z)$, although the existence of solutions of both equations are equivalent).

The motivation for this extension to triples was explained in [15] and is not repeated here in detail. One motivation is that this extension allows a unified treatment of the single- and multivalued case, and one obtains as a special case a degree theory for inclusions $F(x) \in \Phi(x)$ (identifying Φ with a map $q \circ p^{-1}$ where in the simplest case p and q are obtained from Φ as the projections of the graph of Φ). We just point out that the case that F is the identity map is well-studied: In this case our theory is essentially contained in the fixed point theory of so-called admissible pairs (p, q) for which a Lefschetz number was introduced in [5] and also a corresponding fixed point index was developed, see e.g. [6, Section 47 or 52].

Unfortunately, the assumptions concerning the map p (and thus the map Φ) in the extension procedure of [15] are rather technical. Essentially, the only situation where these assumptions can be verified in current practice is the case when p is a Vietoris map (i.e. Φ is acyclic or – with a more complicated choice of p and q – a composition of acyclic maps) and the image space Y is a finite-dimensional vector space.

It is the aim of this paper to drop the assumption that Y has finite dimension. We do this essentially by approximating compact maps by maps in finite-dimensional subspaces in the spirit of Leray–Schauder's construction of the degree from the Brouwer degree. However, the situation is not so simple, because this approximation procedure does not necessarily lead to "close" maps for the given original degree, and so our main difficulty is to prove that our new degree is well-defined anyway.

Of course, in any case we have to pay a price for our approach. We have to require that the original degree for (1.1) allows a finite-dimensional reduction at all: It is not known whether all of the examples of the degree theories proposed in [15] satisfy the corresponding reduction axiom which we require later. So, roughly speaking, the classes of maps F for which the theory in this paper is applicable is by current knowledge restricted to the following cases:

(i) Linear Fredholm operators of index 0 or of positive index: If F has such a form, one can use for function pairs (1.1) the Mawhin degree [10] (see also [3], [13]) or the Nirenberg degree [4], [11], [12], respectively. The reduction property for finite-dimensional subspaces for this degree is well-known (see e.g. [9]).

(ii) Nonlinear Fredholm operators of index 0: A corresponding degree for (1.1) with the reduction property for finite-dimensional subspaces has been obtained in [1].

It seems likely that also the degree of [17] for nonlinear Fredholm operators of positive index satisfies the reduction property for finite-dimensional subspaces, but this is far from being obvious and requires further investigations. On the other hand, it seems unlikely that the Skrypnik-Browder degree for monotone maps F satisfies the reduction property for finite-dimensional subspaces.

In this paper, we restrict our attention to *compact* function triples (F, p, q) (i.e. to the case when q is a compact map). This is completely sufficient, because any degree theory for such compact triples automatically extends to a degree theory for noncompact triples by the procedure developed in [16].

Roughly speaking, this paper is one part of a trilogy which in a sense resembles the development of the Leray–Schauder degree: While in [15] a degree for finite-dimensional Y was developed and in [16] the extension from compact triples to noncompact triples was studied, it is the main purpose of this paper to study the missing "link" from the finite-dimensional case to compact maps. Although this is an easy step in the definition of the Leray–Schauder degree (when $F = p$ is the identity map), this is rather nontrivial in the general case, because it seems that in contrast to the Leray–Schauder degree one cannot simply use an approximation argument.

We point out that for the case that F is a *linear* Fredholm operator of nonnegative index, a corresponding degree for function triples (1.2) was already developed in [2], so for this particular class of maps F our degree is not new. Nevertheless, even in this case our *approach* is new: In contrast to [2], we do not have to invoke deep results from infinite-dimensional cohomotopy theory to define the degree. As a minor extension, even in this linear case, we can often avoid to work in Banach spaces so that our theory applies also e.g. to maps in locally convex spaces (or, more general, in so-called admissible spaces).

The plan of this paper is as follows: The announced result about the extension of a degree from function pairs to function triples will be presented in two variants in Theorems 4.10 and 4.11 in Section 4. The crucial step for the proof is the introduction of a new notion of a *relative* degree theory for which we establish some technical preliminaries in Sections 2 and 3. Somewhat surprisingly, this relative degree theory, which is a bit technical to formulate but for a fixed class of maps easier to establish than an "ordinary" (we will call it for clarity "absolute") degree theory, can be used directly to obtain such an absolute degree theory and the desired extension for triples. Hence, actually our main result is that a relative degree theory for pairs induces (uniquely) an absolute degree for triples. This result is presented in two variants in Theorems 4.7 and 4.8 (the announced Theorems 4.10 and 4.11 are easy consequences). In all these theorems only a mild form of the homotopy invariance is proved and needed. Therefore, we discuss in the last Section 5 a variant of the homotopy invariance which one will probably use in practice if one wants to work with the new theory.

2. Absolute and relative degree theories for function pairs

As explained in the introduction, we want to extend a given coincidence degree for function pairs to function triples. Let us first explain what we mean by such a degree for function pairs. Roughly speaking, this is a map which associates to a certain class of function pairs (F,φ) a "number" such that certain axioms described below are fulfilled. These axioms ensure that this "number" is some "topological measure" for the number of coincidence points of F and φ. Typical examples are the Leray–Schauder degree of $F-\varphi$ or the Brouwer coincidence degree for maps between (finite-dimensional) manifolds.

Unfortunately, in order to formulate the following reduction property, which will be crucial in our considerations, we have to be precise and observe that the degree depends also on the underlying spaces. This makes the following definition (and our notation) even more technical than in [15].

Throughout this paper, let X and Y be fixed topological spaces (not necessarily Hausdorff), and G be a commutative semigroup with neutral element 0. G is the set of values of the degree.

Let $\mathcal{O}$ be a family of open subsets $\Omega\subseteq X$, and $\mathcal{F}$ be a nonempty family of pairs (F,Ω) with $\Omega\in\mathcal{O}$ and $F\colon\overline{\Omega}\to Y$. We require that for each $(F,\Omega)\in\mathcal{F}$ and each $\Omega_0\subseteq\Omega$ with $\Omega_0\in\mathcal{O}$ also $(F|_{\overline{\Omega}_0},\Omega_0)\in\mathcal{F}$.

The "canonical" situation one should have in mind is that X and Y are normed spaces (or X even a Banach manifold) $\mathcal{O}$ the system of all open (or all open and bounded) subsets of X, and the functions F are from a certain class like e.g. (linear or nonlinear oriented) Fredholm operators. We point out that we do not require that F is continuous (typically, our later requirements are already satisfied if F is sequentially demicontinuous).

In view of the crucial reduction property, we consider a fixed family $\mathcal{Y}$ of nonempty closed subsets $Y_0\subseteq Y$. In the most important examples, Y is a Banach space (or a topological vector space) and $\mathcal{Y}$ the family of all finite-dimensional subspaces of Y.

By a *compact* map we always mean a map whose range is contained in a compact subset of Y (recall that we do not assume that Y is Hausdorff, so that this is a weaker requirement than to assume that the closure of the range is compact). We use the notation

$$\operatorname{coin}_A(F,\varphi):=\{x\in A: F(x)=\varphi(x)\}. \tag{2.1}$$

It will be convenient to use this notation also (with the obvious meaning) if A is larger than the domain of definition of F or φ.

Definition 2.1. By $\mathcal{F}_0^{\mathrm{abs}}$, we denote the system of all (F,φ,Ω) where (F,Ω) in $\mathcal{F}$, and $\varphi\colon\overline{\Omega}\to Y$ is continuous and compact and $\mathrm{coin}_{\partial\Omega}(F,\varphi)=\emptyset$.

We say that $\mathcal{F}$ *provides an absolute degree* $\deg\colon\mathcal{F}_0^{\mathrm{abs}}\to G$ for $\mathcal{Y}$ if deg has the following properties:

(a) (Existence) $\deg(F,\varphi,\Omega)\neq 0$ implies $\mathrm{coin}_{\Omega}(F,\varphi)\neq\emptyset$.

(b) (Homotopy Invariance) If $(F,\Omega)\in\mathcal{F}$, $Y_0\in\mathcal{Y}$, and $h\colon[0,1]\times\overline{\Omega}\to Y_0$ is continuous and compact and such that $(F,h(t,\,\cdot\,),\Omega)\in\mathcal{F}_0^{\mathrm{abs}}$ for each $t\in[0,1]$, then

$$\deg(F,h(0,\,\cdot\,),\Omega)=\deg(F,h(1,\,\cdot\,),\Omega).$$

(c) (Reduction) If $\varphi(\overline{\Omega})\subseteq Y_0\in\mathcal{Y}$ and $\psi\colon\overline{\Omega}\to Y_0$ is continuous and compact with $\varphi(x)=\psi(x)$ on $\overline{\Omega}\cap F^{-1}(Y_0)$ then

$$\deg(F,\varphi,\Omega)=\deg(F,\psi,\Omega).$$

An absolute degree might or might not possess the following properties.

(d) (Restriction) If $(F,\varphi,\Omega)\in\mathcal{F}_0^{\mathrm{abs}}$ and $\Omega_0\in\mathcal{O}$ is contained in Ω with $\mathrm{coin}_{\Omega}(F,\varphi)\subseteq\Omega_0$, then

$$\deg(F,\varphi,\Omega)\neq 0\implies \deg(F,\varphi,\Omega_0)=\deg(F,\varphi,\Omega). \tag{2.2}$$

(e) (Excision) As the restriction property, but with (2.2) replaced by

$$\deg(F,\varphi,\Omega_0)=\deg(F,\varphi,\Omega).$$

(f) (Additivity) If $(F,\varphi,\Omega)\in\mathcal{F}_0^{\mathrm{abs}}$ and $\Omega_1,\Omega_2\in\mathcal{O}$ are disjoint with $\Omega=\Omega_1\cup\Omega_2$, then

$$\deg(F,\varphi,\Omega)=\deg(F,\varphi,\Omega_1)+\deg(F,\varphi,\Omega_2).$$

Examples of absolute degrees with $\mathcal{Y}=\{Y\}$ have been given in [15]. However, if Y is a topological vector space and $\mathcal{Y}$ a family of finite-dimensional subspaces, the reduction property is a rather nontrivial requirement. This property is satisfied e.g. by the Leray–Schauder degree. It is also satisfied for the degree for linear Fredholm maps in Banach spaces, if $\mathcal{F}$ consists of a single map F (and its restrictions), and if $\mathcal{Y}$ denotes the family of all finite-dimensional linear subspaces which are transversal to F, see e.g. [9, Theorem 5.25(iv)].

If F is a fixed nonlinear Fredholm map of index 0, deg denotes the degree from [1], then the condition of the reduction property holds for those $Y_0\in\mathcal{Y}$ with the additional property that they are transversal to $F'(x)$ for all x in a neighbourhood of $\mathrm{coin}_{\Omega}(F,\varphi)$.

Unfortunately, if F' is not constant, it may thus happen that Y_0 can depend on φ which is not admissible. Therefore, in order to apply our theory in that case, one needs an additional consideration which we sketch here only; details will be given elsewhere: Assume for simplicity that $\mathcal{F}$ consists of only one (continuous extension of a) nonlinear Fredholm map $F\colon \overline{\Omega} \to Y$ of index 0 and its restrictions. One can show that each compact subset of Ω has an open neighbourhood Ω_0 such that there is a finite-dimensional subspace $Y_0 \subseteq Y$ which is transversal to $F'(x)$ for all $x \in \Omega_0$. Then the reduction property holds if $\mathcal{Y}$ denotes the family of all finite-dimensional subspaces containing Y_0 and $\mathcal{O}$ the family of all open sets $\Omega_0 \subseteq \Omega$ with the property that Y_0 is transversal to $F'(x)$ for all $x \in \Omega_0$. A similar assertion holds if $\mathcal{F}$ constists of all maps from a fixed Fredholm homotopy.

The theory of this paper will then provide a triple degree theory with this restricted family $\mathcal{O}$ (and $\mathcal{F}$). *Afterwards*, one can use the excision property of the obtained degree to obtain a more satisfactory theory for *all* open subsets of Ω. Since this can be done for any fixed Fredholm map/homotopy, one obtains finally a degree theory for function triples involving the class of all nonlinear Fredholm maps of index 0. However, details for the latter will be published elsewhere.

In contrast, it is unknown whether the reduction property (or at least some sufficient variant of it) holds for the homotopically defined degree of 0-epi maps or for the Skrypnik-Browder degree for monotone maps with a reasonable large class $\mathcal{Y}$.

Actually, we will not need an absolute degree but only a relative degree. The latter is a slightly less restrictive requirement as we will see.

Definition 2.2. By $\mathcal{F}_0^{\mathrm{rel}}$, we denote the system of all $(F, \varphi, \Omega, Y_0)$ where $(F, \Omega) \in \mathcal{F}$, $Y_0 \in \mathcal{Y}$, and $\varphi\colon \overline{\Omega} \cap F^{-1}(Y_0) \to Y_0$ is continuous and compact and $\mathrm{coin}_{\partial\Omega}(F, \varphi) = \emptyset$.

We say that $\mathcal{F}$ *provides a relative degree* $\deg\colon \mathcal{F}_0^{\mathrm{rel}} \to G$ for $\mathcal{Y}$ if the following properties hold (we use the more suggestive notation $\deg_{Y_0}(F, \varphi, \Omega)$ for the degree of $(F, \varphi, \Omega, Y_0) \in \mathcal{F}_0^{\mathrm{rel}}$).

(a) (Existence) $\deg_{Y_0}(F, \varphi, \Omega) \neq 0$ implies $\mathrm{coin}_{\Omega}(F, \varphi) \neq \emptyset$.

(b) (Homotopy Invariance) If $(F, \Omega) \in \mathcal{F}$, $Y_0 \in \mathcal{Y}$, and if the homotopy $h\colon [0,1] \times (\overline{\Omega} \cap F^{-1}(Y_0)) \to Y_0$ is continuous and compact and such that $(F, h(t, \cdot\,), \Omega, Y_0) \in \mathcal{F}_0^{\mathrm{rel}}$ for each $t \in [0,1]$, then

$$\deg_{Y_0}(F, h(0, \cdot\,), \Omega) = \deg_{Y_0}(F, h(1, \cdot\,), \Omega).$$

(c) (Reduction) If $(F, \varphi, \Omega, Y_1) \in \mathcal{F}_0^{\mathrm{rel}}$ and $Y_0 \in \mathcal{Y}$ satisfies $\varphi(F^{-1}(Y_1) \cap \overline{\Omega}) \subseteq Y_0 \subseteq Y_1$ then

$$\deg_{Y_0}(F, \varphi, \Omega) = \deg_{Y_1}(F, \varphi, \Omega).$$

A relative degree might or might not possess the following properties.

(d) (Restriction) If $(F,\varphi,\Omega,Y_0)\in\mathcal{F}_0^{\mathrm{rel}}$ and $\Omega_0\in\mathcal{O}$ is contained in Ω with $\mathrm{coin}_\Omega(F,\varphi)\subseteq\Omega_0$, then

$$\deg_{Y_0}(F,\varphi,\Omega)\neq 0 \implies \deg_{Y_0}(F,\varphi,\Omega_0)=\deg_{Y_0}(F,\varphi,\Omega).$$

(e) (Excision) Under the same assumptions as above:

$$\deg_{Y_0}(F,\varphi,\Omega_0)=\deg_{Y_0}(F,\varphi,\Omega).$$

(f) (Additivity) If $(F,\varphi,\Omega,Y_0)\in\mathcal{F}_0^{\mathrm{rel}}$ and $\Omega_1,\Omega_2\in\mathcal{O}$ are disjoint with $\Omega=\Omega_1\cup\Omega_2$, then

$$\deg_{Y_0}(F,\varphi,\Omega)=\deg_{Y_0}(F,\varphi,\Omega_1)+\deg_{Y_0}(F,\varphi,\Omega_2).$$

In order to motivate the above definitions, one should think of the case that $X=Y$ is a Banach space, F the identity map, and $\mathcal{Y}$ the system of all finite-dimensional subspaces of Y. Then the Leray–Schauder degree (of $F-\varphi$) is an example of an absolute degree while the Brouwer degree (on the finite-dimensional subspaces) is an example of a relative degree in the above sense.

In this example, the Brouwer degree is of course only a simple special case of the Leray–Schauder degree. We will verify soon that this is not accidental: In practically all natural situations an absolute degree induces in a canonical way a relative degree.

Although we are mainly interested in extending an absolute degree to function triples, it appears not possible to do this without first passing to the (simpler) relative degree: We will extend the latter to a relative degree for function triples. This degree theory in turn gives rise to an absolute degree for function triples which is then the desired extension of the original absolute degree. The reason why we have to proceed this way is explained after the proof of Theorem 4.10.

So in a sense, the relative degree is only an artificial intermediate step, needed to prove our main result. However, since we thus actually base our considerations on a relative degree, this has a positive side effect: If one wants to develop a new degree theory and to apply our results to this new degree, one does not have to fully develop the corresponding (absolute) degree theory: It suffices that one has elaborated the corresponding relative degree. By our results one then obtains *automatically* the required degree theory, even for function triples. For example, as a (very simple) special case of our main result we obtain the existence of the Leray–Schauder degree only from the existence of the Brouwer degree.

Let us show now that each absolute degree indeed gives rise to a relative degree under a very natural assumption on $\mathcal{Y}$.

Definition 2.3. We call a subset $K \subseteq Y$ an *extensor set* for a space M if, for each closed $A \subseteq M$, each continuous compact map $f\colon A \to Y$ with $f(A) \subseteq K$ has an extension to a continuous compact map $f\colon M \to Y$ with $f(M) \subseteq K$.

Roughly speaking, all sets which are homeomorphic to retracts of convex subsets of locally convex or normed spaces are extensor sets under mild additional assumptions; for details, see [7] and [15]. In our applications the following observation usually suffices which is a straightforward consequence of the Tietze–Urysohn extension theorem and the fact that all closed convex subsets of $\mathbb{R}^n$ are retracts of $\mathbb{R}^n$.

Proposition 2.4. *Assume that $K \subseteq Y$ is closed and an* ER*, i.e. homeomorphic to a retract of $\mathbb{R}^n$ (or, equivalently, to a retract of a closed convex subset of $\mathbb{R}^n$). Then K is an extensor set for each T_4 space M.*

Now we can prove in a straightforward manner the first of the results which we had announced before.

Theorem 2.5 (Absolute Degree Induces Relative Degree). *Let $\mathcal{F}$ provide an absolute degree* deg *for $\mathcal{Y}$ where each $Y_0 \in \mathcal{Y}$ has the following two properties for each $(F,\Omega) \in \mathcal{F}$.*

(a) *Y_0 is an extensor set for $[0,1] \times \overline{\Omega}$.*
(b) *$F^{-1}(Y_0) \cap \overline{\Omega}$ is closed.*

Then $\mathcal{F}$ provides a unique relative degree for $\mathcal{Y}$ with the following normalization property.

- (Normalization) *If $(F,\varphi,\Omega) \in \mathcal{F}_0^{\mathrm{abs}}$ and $\varphi(\overline{\Omega}) \subseteq Y_0 \in \mathcal{Y}$ then*
$$\deg_{Y_0}(F,\varphi,\Omega) = \deg(F,\varphi,\Omega).$$

If deg *has the restriction, excision or addititivity property, then also the relative degree has the corresponding property.*

Proof. Given $(F,\varphi,\Omega,Y_0) \in \mathcal{Y}$, we extend φ to a continuous compact map $\varphi\colon \overline{\Omega} \to Y_0$. Since φ assumes its values only in Y_0, this extension automatically satisfies $\mathrm{coin}_{\partial\Omega}(F,\varphi) = \emptyset$, and so $(F,\varphi,\Omega) \in \mathcal{F}_0^{\mathrm{rel}}$. By the normalization property, the only way to define the relative degree is by putting
$$\deg_{Y_0}(F,\varphi,\Omega) := \deg(F,\varphi,\Omega).$$
Note that, since deg has the reduction property, this definition is actually independent of the particular choice of the extension of φ. The proof of the other properties is now straightforward and therefore left to the reader. For the proof of the homotopy invariance note that we can extend the given homotopy, because each $Y_0 \in \mathcal{Y}$ is also an extensor set for $[0,1] \times \overline{\Omega}$. □

3. Relative degree theories for function triples

Let us now extend the main result of [15] to the relative degree. Essentially, this result tells us that the relative degree extends in a unique way to a "triple-degree" which is defined for function triples (F,p,q). Here, $(F,\Omega)\in\mathcal{F}$, and $p\colon\Gamma\to X$ and $q\colon\Gamma\to Y$ are continuous (with some topological space Γ). We point out that the space Γ is *not* fixed (and so we actually speak about *classes* of function triples). For such a triple and $A\subseteq X$ we use a notation similar to (2.1):

$$\begin{aligned}\mathrm{COIN}_A(F,p,q)&=\{x\in A\mid F(x)\in q(p^{-1}(x))\}\\&=\{x\in A\mid \exists z\in\Gamma : x=p(z),\ F(p(z))=q(z)\}.\end{aligned}$$

Unfortunately, we need for p the following technical assumption whose discussion we postpone until Theorem 3. Let in the following definition be $F\colon\overline{M}\to Y$ (with some $M\subseteq X$) and $Y_0,Y_1\subseteq Y$.

Definition 3.1. We call the map p an (F,M,Y_0,Y_1)*-compact homotopy surjection on* $A\subseteq M$ if $p(\Gamma)\supseteq F^{-1}(Y_1)$ and the following holds.

For each continuous compact $q\colon p^{-1}(F^{-1}(Y_1))\to Y_0$ with $\mathrm{COIN}_A(F,p,q)=\emptyset$ there is a continuous map $\varphi\colon F^{-1}(Y_1)\to Y_0$ and a continuous compact map $h\colon[0,1]\times p^{-1}(F^{-1}(Y_1))\to Y_0$ satisfying $h(0,\,\cdot\,)=q$ and $h(1,\,\cdot\,)=\varphi\circ p$ (on $p^{-1}(F^{-1}(Y_1))$) such that

$$\mathrm{COIN}_A(F,p,h(t,\,\cdot\,))=\emptyset\quad(0\le t\le 1).$$

Note that φ is automatically compact, and $\mathrm{coin}_A(F,\varphi)=\emptyset$.

Definition 3.2. We call the map p an (F,M,Y_0,Y_1)*-compact homotopy injection on* $A\subseteq M$ if each two continuous compact maps $\varphi,\widetilde{\varphi}\colon F^{-1}(Y_1)\to Y_0$ with

$$\mathrm{coin}_A(F,\varphi)=\mathrm{coin}_A(F,\widetilde{\varphi})=\emptyset$$

which are as in Definition 3.1 (with the same map q) are homotopic in the following sense.

There is a continuous compact map $H\colon[0,1]\times F^{-1}(Y_1)\to Y_0$ with $H(0,\,\cdot\,)=\varphi$ and $H(1,\,\cdot\,)=\widetilde{\varphi}$ such that

$$\mathrm{coin}_A(F,H(t,\,\cdot\,))=\emptyset\quad(0\le t\le 1).$$

If p is also a (F,M,Y_0,Y_1)-compact homotopy surjection on A, we call p an (F,M,Y_0,Y_1)*-compact homotopy bijection on A*.

Definition 3.3. By $\mathcal{T}^{\mathrm{rel}}$, we denote the class of all (F,p,Ω,Y_0) where $(F,\Omega)\in\mathcal{F}$, $Y_0\in\mathcal{Y}$, $F^{-1}(Y_0)$ is closed in X, and on each closed $A\subseteq\overline{\Omega}$ with $A\supseteq\partial\Omega$, the map p is an $(F,\overline{\Omega},Y_0,Y_0)$-compact homotopy bijection.

By $\mathcal{T}_0^{\mathrm{rel}}$, we denote the class of all (F,p,q,Ω,Y_0) where $(F,p,\Omega,Y_0)\in\mathcal{T}^{\mathrm{rel}}$ and q is a continuous compact function $q\colon p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))\to Y_0$ (q might also be defined on a larger space Γ), and

$$\mathrm{COIN}_{\partial\Omega}(F,p,q)=\emptyset.$$

Theorem 3.4 (Relative Degree Induces Relative Triple-Degree). *Let $\mathcal{F}$ provide a relative degree* $\deg\colon\mathcal{F}_0^{\mathrm{rel}}\to G$ *for $\mathcal{Y}$. Then there is a unique triple-degree* DEG *which associates to each $(F,p,q,\Omega,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$ an element of G which depends only on $(F,p|_D,q|_D,\Omega,Y_0)$ with $D:=p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))$, such that the following properties hold for each $(F,p,q,\Omega,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$.*

(a) (Normalization) *If $(F,\varphi,\Omega,Y_0)\in\mathcal{F}_0^{\mathrm{rel}}$ and $\varphi\circ p=q$, then*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega)=\deg_{Y_0}(F,\varphi,\Omega).$$

(b) (Homotopy Invariance in the Third Argument) *If h is (an extension of) a continuous compact function $h\colon[0,1]\times p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))\to Y$ and $(F,p,h(t,\,\cdot\,),\Omega,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$ for each $t\in[0,1]$, then*

$$\mathrm{DEG}_{Y_0}(F,p,h(t,\,\cdot\,),\Omega)\ \textit{is independent of}\ t\in[0,1].$$

(c) (Existence) $\mathrm{DEG}_{Y_0}(F,p,q,\Omega)\neq 0$ *implies* $\mathrm{COIN}_{\Omega}(F,p,q)\neq\emptyset$.

(d) (Reduction) *Let $(F,p,q,\Omega,Y_1)\in\mathcal{T}_0^{\mathrm{rel}}$ satisfy $q(p^{-1}(\overline{\Omega}\cap F^{-1}(Y_1)))\subseteq Y_0\subseteq Y_1$. If p is even an (F,Ω,Y_0,Y_1)-compact homotopy surjection on $\partial\Omega$ then*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega)=\mathrm{DEG}_{Y_1}(F,p,q,\Omega).$$

(e) (Strong Independence from Γ) *If also $(F,\widetilde{p},\widetilde{q},\Omega,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$ and there is a continuous map $J\colon p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0)))\to\widetilde{p}^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))$ such that $p(z)=\widetilde{p}(J(z))$ and $q(z)=\widetilde{q}(J(z))$ for all $z\in p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))$, then*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega)=\mathrm{DEG}_{Y_0}(F,\widetilde{p},\widetilde{q},\Omega).$$

The uniqueness of DEG *already follows from the first two of these properties. If* deg *satisfies in addition the restriction, excision, resp. additivity property, then* DEG *automatically satisfies the corresponding property*:

(f) (Restriction) *If $(F,p,q,\Omega,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$ and $\Omega_0\in\mathcal{O}$ is contained in Ω with $\mathrm{COIN}_{\Omega}(F,p,q)\subseteq\Omega_0$, then $(F,p,q,\Omega_0,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$, and*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega)\neq 0\implies\mathrm{DEG}_{Y_0}(F,p,q,\Omega_0)=\mathrm{DEG}_{Y_0}(F,p,q,\Omega).$$

(g) (Excision) *Under the same assumptions as above, $(F,p,q,\Omega_0,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$, and*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega_0)=\mathrm{DEG}_{Y_0}(F,p,q,\Omega).$$

(h) (Additivity) *If* $(F,p,q,\Omega,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$ *and* $\Omega_1,\Omega_2\in\mathcal{O}$ *are disjoint with* $\Omega=\Omega_1\cup\Omega_2$, *then* $(F,p,q,\Omega_i,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$, *and*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega)=\mathrm{DEG}_{Y_0}(F,p,q,\Omega_1)+\mathrm{DEG}_{Y_0}(F,p,q,\Omega_2.$$

It appears that the additional requirement on p which we pose in the above formulation of the reduction property cannot be dropped. This is the reason why we cannot immediately prove an analogous theorem for the absolute degree but have to pass to a relative degree theory first.

Proof. Except for the reduction property, we may assume that we have $X=F^{-1}(Y_0)$. To see this, replace X and Ω by their respective intersection with $F^{-1}(Y_0)$ (here we use that $F^{-1}(Y_0)$ is closed and Ω is open so that we implicitly also replace $\overline{\Omega}$ and $\partial\Omega$ by their intersection with $F^{-1}(Y_0)$). Hence, except for the reduction property, we may assume that $Y_0=Y$, and so all assertions (except of the reduction property) follow from the main result of [15].

It remains to show that the (uniquely determined!) degree actually satisfies the reduction property. We apply the assumption that p is an $(F,\overline{\Omega},Y_0,Y_1)$-compact homotopy surjection on $\partial\Omega$ for the map q: Note that indeed $q\colon p^{-1}(\overline{\Omega}\cap F^{-1}(Y_1)))\to Y_0$ and $\mathrm{COIN}_{\partial\Omega}(F,p,q)=\emptyset$. Let $\varphi\colon\overline{\Omega}\cap F^{-1}(Y_1)\to Y_0$ and $h\colon[0,1]\times p^{-1}(\overline{\Omega}\cap F^{-1}(Y_1))\to Y_0$ be the corresponding maps of Definition 3.1. By the homotopy invariance in the third argument and the normalization property, we obtain for $i=0,1$ that

$$\begin{aligned}\mathrm{DEG}_{Y_i}(F,p,q,\Omega)&=\mathrm{DEG}_{Y_i}(F,p,h(0,\,\cdot\,),\Omega)=\mathrm{DEG}_{Y_i}(F,p,h(1,\,\cdot\,),\Omega)\\&=\mathrm{DEG}_{Y_i}(F,p,\varphi\circ p,\Omega)=\deg_{Y_i}(F,\varphi,\Omega).\end{aligned}$$

Since φ assumes its values only in $Y_0\subseteq Y_1$, the reduction property of the (relative) degree implies that the right-hand side is independent of $i\in\{0,1\}$. $\square$

Corollary 3.5 (Relative Triple-Degree with the Excision Property). *If the relative degree in Theorem* 3.4 *has the excision property, then an analogous result holds where we replace* $\mathcal{T}_0^{\mathrm{rel}}$ *by the family of all* (F,p,q,Ω,Y_0) *with* $q\colon p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))\to Y_0$ *and the property that there is some* $\Omega_0\in\mathcal{O}$ *with*

$$\mathrm{COIN}_{\overline{\Omega}}(F,p,q)\subseteq\Omega_0\subseteq\Omega \tag{3.1}$$

such that $(F,p,q,\Omega_0,Y_0)\in\mathcal{T}_0^{\mathrm{rel}}$ *if we require that in such a case*

$$\mathrm{DEG}_{Y_0}(F,p,q,\Omega)=\mathrm{DEG}_{Y_0}(F,p,q,\Omega_0). \tag{3.2}$$

In particular, the reduction property holds under the requirement that p *is an* (F,Ω_0,Y_0,Y_1)*-compact homotopy surjection on* $\partial\Omega_0$ *for some set* $\Omega_0\in\mathcal{O}$ *with property* (3.1).

Proof. We use the relation (3.2) to define the left-hand side. By the excision property, this definition is independent of the particular choice of Y_0. The verification of all other properties is straightforward. $\square$

So far, all elements of $\mathcal{Y}$ could have been arbitrary closed subsets of Y: We did not pose any assumptions like finite dimensions or a group structure. However, we have simply hidden all topological difficulties in the technical definition of (F, M, Y_0, Y_1)-compact homotopy bijections. In order to verify this property, we use a special case of a result of [15] which we formulate now.

We recall that a map p is called *proper* if preimages of compact sets are compact. In metric spaces it is equivalent to require that p maps closed sets onto closed sets and has compact fibres $p^{-1}(x)$.

A topological space Y_0 is called a (metric) AR if Y_0 is homeomorphic to a retract of a normed space. A metric space is called R_δ if it is the intersection of a decreasing sequence of compact AR spaces. A metric space is an R_δ if and only if it is the intersection of a decreasing sequence of compact contractible nonempty metric spaces (or, equivalently, if it has the shape of a point). The continuity of the Čech cohomology implies that each R_δ space is acyclic with respect to Čech cohomology. (The converse is not true.)

Theorem 3.6. *Let $Y_0, Y_1 \subseteq Y$. Let $M \subseteq X$ be closed and $F\colon M \to Y$. Let $p\colon \Gamma \to X$ be proper with $p(\Gamma) \supseteq F^{-1}(Y_1)$ $(\subseteq M)$. Suppose that the following holds*:

(a) *$F^{-1}(Y_1)$ and $p^{-1}(F^{-1}(Y_1))$ are both metrizable.*
(b) *Y_0 is homeomorphic to a topological group.*
(c) *Y_0 is an extensor set for $F^{-1}(Y_1)$ and for $[0,1] \times p^{-1}(F^{-1}(Y_1))$.*
(d) *$F^{-1}(Y_0 \cap Y_1)$ is closed in $F^{-1}(Y_1)$, and the (restricted) map $F\colon F^{-1}(Y_0 \cap Y_1) \to Y_0$ is continuous.*
(e) *$F^{-1}(Y_0 \cap Y_1)$ is compact.*
(f) *At least one of the following two requirements holds:*
   (f$_1$) *All fibres $p^{-1}(x)$ $(x \in F^{-1}(Y_1))$ are R_δ, and the large inductive dimension of $F^{-1}(Y_1)$ is finite.*
   (f$_2$) *All fibres $p^{-1}(x)$ $(x \in F^{-1}(Y_1))$ are acyclic with respect to Čech cohomology with coefficients in $\mathbb{Z}$. In addition, there is some $y_0 \in Y$ such that $Y \setminus \{y_0\}$ is homotopically n-simple for each $n \geq 1$. Moreover, $\dim(F^{-1}(Y_1)) < \infty$ where $\dim$ denotes the covering dimension, and*

$$\sup_{x \in F^{-1}(Y_1)} \dim p^{-1}(x) < \infty.$$

Then p is an (F, M, Y_0, Y_1)-compact homotopy bijection on each closed $A \subseteq M$.

Proof. In case $F^{-1}(Y_1) = X$, the result follows from [15] with $\mathcal{K} := \{Y_0\}$. The general case reduces to the special case: To see this, replace X and M by their intersection with $F^{-1}(Y_1)$. □

We point out that the assumption $\dim(F^{-1}(Y_1)) < \infty$ is rather natural if F has finite-dimensional fibres and $\dim Y_1 < \infty$, see [14].

Moreover, if F is a nonlinear Fredholm map and $Y_1 \subseteq Y$ is a finite-dimensional submanifold, then $F^{-1}(Y_1)$ is contained in a finite-dimensional submanifold in each sufficiently small neighbourhood of a compact set (since by increasing Y_1, one can assume that Y_1 is transverse to F in a neighbourhood of a compact set). For this reason, Corollary 3.5 is rather important, because it implies that in case of compact $\mathrm{COIN}_{\overline{\Omega}}(F, p, q)$ it suffices to restrict attention to such neighbourhoods, and so the assumption on finite (inductive or covering) dimension of F^{-1} is "automatically" satisfied.

4. Absolute degree theories for function triples

In contrast to the previous section, we assume in this section that Y is a topological Hausdorff vector space which possesses a family $\mathcal{Y}$ of subspaces with the following property.

Definition 4.1. We call a system $\mathcal{Y}$ of nonempty closed linear subspaces $Y_0 \subseteq Y$ *admissible* for Y if the following holds:

(a) $\mathcal{Y}$ is directed upwards with respect to inclusion, i.e. for each $Y_0, Y_1 \in \mathcal{Y}$ there is some $Y_2 \in \mathcal{Y}$ with $Y_2 \supseteq Y_0 \cup Y_1$.
(b) For each compact $K \subseteq Y$ and each neighbourhood $U \subseteq Y$ of 0 there is some $Y_0 \in \mathcal{Y}$ and a continuous map $\pi: K \to Y_0$ with $\pi(y) \in y + U$.

The well-known Schauder projections imply that for each normed space Y the family $\mathcal{Y}$ of all finite-dimensional subspaces is admissible for Y. It is a folklore result that this also holds for locally convex spaces, but we could not find a proof in literature. So we provide a proof for the reader's convenience.

Proposition 4.2. *If Y is a locally convex space, then the family $\mathcal{Y}$ of all finite-dimensional subspaces is admissible for Y.*

Proof. Let a neighbourhood $U \subseteq Y$ of 0 be given. Choose a neighbourhood V of 0 with $\operatorname{conv} V \subseteq U$. Since Y is completely regular (see e.g. [8, §15, 2(3)]), there is a continuous function $\varphi: Y \to \mathbb{R}$ such that $W := \{y \in Y : \varphi(y) \neq 0\}$ is an open neighbourhood of 0 with $W \subseteq V$. For each compact set $K \subseteq Y$ there are finitey many points $y_1, \ldots, y_n \in K$ with $K \subseteq (y_1 + W) \cup \ldots \cup (y_n + W)$.

Then the function

$$\pi(y) := \frac{\sum_{k=1}^{n} \varphi(y_k - y) y_k}{\sum_{k=1}^{n} \varphi(y_k - y)}$$

is defined an continuous on K and assumes its values in $\operatorname{span}\{y_1, \dots, y_n\} \in \mathcal{Y}$. Moreover, for each $y \in Y$, since $y_k - y \in W$ for each k with $\varphi(y_k - y) \neq 0$, we obtain

$$\pi(y) - y = \frac{\sum_{k=1}^{n} \varphi(y_k - y)(y_k - y)}{\sum_{k=1}^{n} \varphi(y_k - y)} \in \operatorname{conv} W \subseteq \operatorname{conv} V \subseteq U. \qquad \square$$

We do not explicitly require that all spaces in $\mathcal{Y}$ have finite dimension. However, the following requirements are usually only satisfied if this is the case.

Definition 4.3. Let $(F, \Omega) \in \mathcal{F}$, $p\colon \Gamma \to X$, and $C \subseteq \Omega$. Then we write $(F, p, \Omega) \in \mathcal{T}^{\mathrm{abs}}(C)$, resp. $(F, p, \Omega) \in \mathcal{T}^{\mathrm{abs}}$ if for each $Y_0 \in \mathcal{Y}$ we find some $\Omega_0 \in \mathcal{O}$ with $C \subseteq \Omega_0 \subseteq \Omega$, resp. $\Omega_0 := \Omega$ such that the following holds:

(a) p is continuous with $p(\Gamma) \supseteq \overline{\Omega}_0 \cap F^{-1}(Y_0)$.
(b) $F\colon \overline{\Omega} \to Y$ is proper and has a closed graph.
(c) $F(\overline{\Omega}_0) \cap Y_0$ is relatively compact.
(d) Y is metrizable, or for each continuous compact map $h\colon [0,1] \times p^{-1}(\overline{\Omega}) \to Y$ and each closed $A \subseteq \overline{\Omega}$ the set

$$\begin{aligned} \bigcup \{F(x) - h(t, p^{-1}(x)) : t \in [0,1],\ x \in A\} \\ = \{F(p(z)) - h(t, z) : t \in [0,1],\ z \in p^{-1}(A)\} \end{aligned}$$

is closed.
(e) $\overline{\Omega}_0 \cap F^{-1}(Y_0)$ is metrizable and closed in X.
(f) $p^{-1}(\overline{\Omega}_0 \cap F^{-1}(Y_0))$ is metrizable and compact.
(g) Y_0 is an extensor set for $\overline{\Omega}_0$ and for $[0,1] \times p^{-1}(\overline{\Omega}_0)$.
(h) At least one of the following two requirements holds:
(h_1) All fibres $p^{-1}(x)$ ($x \in \overline{\Omega}_0 \cap F^{-1}(Y_0)$) are R_δ, and the inductive dimension of $\overline{\Omega}_0 \cap F^{-1}(Y_0)$ is finite.
(h_2) All fibres $p^{-1}(x)$ ($x \in \overline{\Omega}_0 \cap F^{-1}(Y_0)$) are acyclic with respect to Čech cohomology with coefficients in $\mathbb{Z}$. Moreover, the covering dimension $\dim(\overline{\Omega}_0 \cap F^{-1}(Y_0))$ is finite and

$$\sup_{x \in \overline{\Omega}_0 \cap F^{-1}(Y_0)} \dim p^{-1}(x) < \infty.$$

Definition 4.4. Let with the above notation $q\colon\Gamma\to Y$ be such that the restriction of q to $p^{-1}(\overline{\Omega})$ is continuous and compact and such that

$$C := \mathrm{COIN}_{\overline{\Omega}}(F,p,q)\subseteq\Omega.$$

If $(F,p,\Omega)\in\mathcal{T}^{\mathrm{abs}}$, resp. $(F,p,\Omega)\in\mathcal{T}^{\mathrm{abs}}(C)$, then we write $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$, resp. $(F,p,q,\Omega)\in\mathcal{T}_c^{\mathrm{abs}}$.

Clearly, $\mathcal{T}_0^{\mathrm{abs}}\subseteq\mathcal{T}_c^{\mathrm{abs}}$.

The following lemmas will be proved in a more general setting in the next section.

Lemma 4.5. *Let (F,p,Ω) belong to $\mathcal{T}^{\mathrm{abs}}$ or to $\mathcal{T}^{\mathrm{abs}}(C)$. Then for each $Y_0\in\mathcal{Y}$ the restriction of F to $F^{-1}(Y_0)$ is continuous.*

Lemma 4.6. *Let (F,p,Ω) belong to $\mathcal{T}^{\mathrm{abs}}$ or to $\mathcal{T}^{\mathrm{abs}}(C)$. Then for each continuous compact map $h\colon[0,1]\times p^{-1}(\overline{\Omega})\to Y$ and each closed $A\subseteq\overline{\Omega}$ with*

$$\mathrm{COIN}_A(F,p,h(t,\,\cdot\,))=\emptyset\quad(0\le t\le 1)$$

there is a neighbourhood $U\subseteq Y$ of 0 with $F(p(z))-h(t,z)\notin U$ for each $z\in p^{-1}(A)$ and each $t\in[0,1]$.

The main result of this paper can now be formulated as follows.

Theorem 4.7 (Relative Degree Induces an Absolute Triple-Degree). *Let $\mathcal{Y}$ be admissible for Y and let $\mathcal{F}$ provide a relative degree $\deg\colon\mathcal{F}_0^{\mathrm{rel}}\to G$ for $\mathcal{Y}$. Then there is a unique degree DEG which associates to each $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ an element of G which depends only on (F,Ω) and the restriction of p and q to $p^{-1}(\overline{\Omega})$ such that the following properties hold for each $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$.*

(a) (Normalization) *If the range of q is contained in $Y_0\in\mathcal{Y}$ and there is some φ with $(F,\varphi,\Omega,Y_0)\in\mathcal{F}_0^{\mathrm{rel}}$ and $\varphi\circ p=q$ (on $p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))$) then*

$$\mathrm{DEG}(F,p,q,\Omega)=\deg_{Y_0}(F,\varphi,\Omega).$$

(b) (Homotopy Invariance in the Third Argument) *Let $h\colon[0,1]\times p^{-1}(\overline{\Omega})\to Y$ be continuous and compact with $\mathrm{COIN}_{\partial\Omega}(F,p,h(t,\,\cdot\,))=\emptyset$ $(0\le t\le 1)$. Then $(F,p,h(t,\,\cdot\,),\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ for each $t\in[0,1]$ and*

$$\mathrm{DEG}(F,p,h(t,\,\cdot\,),\Omega)\ \text{is independent of}\ t\in[0,1].$$

(c) (Existence) $\mathrm{DEG}(F,p,q,\Omega)\neq 0$ *implies* $\mathrm{COIN}_\Omega(F,p,q)\neq\emptyset$.

(d) (Reduction) *If the range of q is contained in $Y_0\in\mathcal{Y}$ and $\widetilde{q}\colon p^{-1}(\overline{\Omega})\to Y_0$ is continuous and compact with $\widetilde{q}(z)=q(z)$ on $p^{-1}(\overline{\Omega}\cap F^{-1}(Y_0))$, then $(F,p,\widetilde{q},\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ and*

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,p,\widetilde{q},\Omega).$$

(e) (Strong Independence from Γ) *If also* $(F,\widetilde{p},\widetilde{q},\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ *and there is a continuous map* $J\colon p^{-1}(\overline{\Omega})\to\widetilde{p}^{-1}(\overline{\Omega})$ *such that* $p(z)=\widetilde{p}(J(z))$ *and* $q(z)=\widetilde{q}(J(z))$ *for all* $z\in p^{-1}(\overline{\Omega})$, *then*

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,\widetilde{p},\widetilde{q},\Omega).$$

The uniqueness already follows from the normalization and homotopy invariance. If deg *satisfies in addition the restriction, excision, resp. additivity property, then* DEG *automatically satisfies the corresponding property*:

(f) (Restriction) *If* $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{rel}}$ *and* $\Omega_0\in\mathcal{O}$ *is contained in* Ω *with* $\mathrm{COIN}_\Omega(F,p,q)\subseteq\Omega_0$, *then* $(F,p,q,\Omega_0)\in\mathcal{T}_0^{\mathrm{rel}}$, *and*

$$\mathrm{DEG}(F,p,q,\Omega)\neq 0\implies\mathrm{DEG}(F,p,q,\Omega_0)=\mathrm{DEG}(F,p,q,\Omega).$$

(g) (Excision) *Under the same assumptions as above,* $(F,p,q,\Omega_0)\in\mathcal{T}_0^{\mathrm{rel}}$, *and*

$$\mathrm{DEG}(F,p,q,\Omega_0)=\mathrm{DEG}(F,p,q,\Omega).$$

(h) (Additivity) *If* $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{rel}}$ *and* $\Omega_1,\Omega_2\in\mathcal{O}$ *are disjoint with* $\Omega=\Omega_1\cup\Omega_2$, *then* $(F,p,q,\Omega_i)\in\mathcal{T}_0^{\mathrm{rel}}$, *and*

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,p,q,\Omega_1)+\mathrm{DEG}(F,p,q,\Omega_2).$$

Theorem 4.8 (Relative Degree with Excision Induces Strong Absolute Triple-Degree). *Assume in Theorem* 4.7 *that* deg *has the excision property. Then an analogous result holds with* $\mathcal{T}_c^{\mathrm{abs}}$ *instead of* $\mathcal{T}_0^{\mathrm{abs}}$ *and the following strengthening of the homotopy invariance*:

(b) (Homotopy Invariance in the Third Argument) *Let* $h\colon[0,1]\times p^{-1}(\overline{\Omega})\to Y$ *be continuous and compact and* $\mathrm{COIN}_{\overline{\Omega}}(F,p,h(t,\,\cdot\,))\subseteq C$ $(0\le t\le 1)$ *for some* $C\subseteq\Omega$ *such that* $(F,p,\Omega)\in\mathcal{T}^{\mathrm{abs}}(C)$. *Then* $(F,p,h(t,\,\cdot\,),\Omega)\in\mathcal{T}_c^{\mathrm{abs}}$ *for each* $t\in[0,1]$ *and*

$$\mathrm{DEG}(F,p,h(t,\,\cdot\,),\Omega)\ \textit{is independent of}\ t\in[0,1].$$

The uniqueness of DEG *already follows from the normalization, homotopy invariance, and excision property.*

Remark 4.9. If deg has the excision property, then the degree of Theorem 4.7 is the restriction of the degree of Theorem 4.8. This is an immediate consequence of the uniqueness claim of Theorem 4.7.

We now prove Theorems 4.7 and 4.8 simultaneously.

Proof. We observe first that for each $(F,p,\Omega)\in\mathcal{T}^{\mathrm{abs}}$ (resp. $(F,p,\Omega)\in\mathcal{T}^{\mathrm{abs}}(C)$) and each $Y_0,Y_1\in\mathcal{Y}$ with $Y_0\subseteq Y_1$ we have for $\Omega_0:=\Omega$ (resp. some $\Omega_0\in\mathcal{O}$ with $C\subseteq\Omega_0\subseteq\Omega$) that the map p is an (F,Ω_0,Y_0,Y_1)-compact homotopy bijection on each closed set $A\subseteq\overline{\Omega}_0$. This follows from Theorem 3.6 and

Lemma 4.5. In particular, if $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ (resp. $(F,p,q,\Omega)\in\mathcal{T}_c^{\mathrm{abs}}$) and q assumes its values only in Y_0, then the degree $\mathrm{DEG}_{Y_0}(F,p,q,\Omega)$ of Theorem 3.4 (resp. of Corollary 3.5) is defined. Moreover, this value is independent of the particular choice of Y_0. Indeed, if $Y_1\in\mathcal{Y}$ is another space which contains the range of q, choose some $Y_2\in\mathcal{Y}$ with $Y_2\supseteq Y_0\cup Y_1$. Then the reduction property in Theorem 3.4 (resp. in Corollary 3.5) implies

$$\mathrm{DEG}_{Y_i}(F,p,q,\Omega)=\mathrm{DEG}_{Y_2}(F,p,q,\Omega)\quad (i=0,1).$$

This proves the independence of the particular choice of Y_0. We claim that *if* a degree exists which satisfies the normalization, homotopy invariance (and exicison property), then we must in the above situation even have

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}_{Y_0}(F,p,q,\Omega). \tag{4.1}$$

In fact, let $Y_1:=Y_0$ (and choose $\Omega_0\in\mathcal{O}$ correspondingly; however, in view of the excision property, we may replace Ω by Ω_0 on both sides of (4.1) and thus assume $\Omega=\Omega_0$). Now choose h and φ as in Definition 3.1 with $Y_1:=Y_0$ and $A:=\partial\Omega$. Since Y_0 is an extensor set, we can extend φ to a continuous compact map $\varphi\colon\overline{\Omega}\to Y_0$. Since φ assumes its values only in Y_0, we have $\mathrm{coin}_{\partial\Omega}(F,\varphi)=\emptyset$ and so $(F,\varphi,\Omega,Y_0)\in\mathcal{F}_0^{\mathrm{rel}}$. Next, we extend h to a continuous compact map $h\colon[0,1]\times p^{-1}(\overline{\Omega})\to Y$ by putting first $h(0,\,\cdot\,):=q$ and $h(1,\,\cdot\,):=\varphi\circ p$ and then using that Y_0 is an extensor set. Since h assumes its values only in Y_0, we have also for the extended map $\mathrm{COIN}_{\partial\Omega}(F,p,h(t,\,\cdot\,),\Omega)=\emptyset$. Using the homotopy invariance of DEG, resp. of DEG_{Y_0}, we see that the equality (4.1) is actually equivalent to

$$\mathrm{DEG}(F,p,\varphi\circ p,\Omega)=\mathrm{DEG}_{Y_0}(F,p,\varphi\circ p,\Omega).$$

By the normalization of DEG, resp. DEG_{Y_0} the two sides of this equality must have the same value $\deg_{Y_0}(F,\varphi,\Omega)$.

We thus have proved that for any $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ for which q assumes its values only in $Y_0\in\mathcal{Y}$, we must define $\mathrm{DEG}(F,p,q,\Omega)$ by (4.1); and we have seen simultaneously that this definition is independent of the particular choice of Y_0.

Now let $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ (resp. $(F,p,q,\Omega)\in\mathcal{T}_c^{\mathrm{abs}}$) be arbitrary. For $A:=\partial\Omega$ (resp. for $A:=\overline{\Omega}\setminus\Omega_0$ with C and Ω_0 as in Definitions 4.4 and 4.3), we find by Lemma 4.6 some neighbourhood $U\subseteq Y$ of 0 with $F(p(z))-q(z)\notin U$ for $z\in A$. Choose some neighbourhood $V\subseteq Y$ of 0 with $tV+sV\subseteq U$ for $t,s\in[0,1]$. Let $Y_0\in\mathcal{Y}$ and π be as in Definition 4.1 (with the neighbourhood V and K a compact set containing the range of q). For $h_\pi(t,z):=t\pi(q(z))+(1-t)q(z)$ $(0\le t\le 1)$ we have $\mathrm{COIN}_A(F,p,h(t,\,\cdot\,))=\emptyset$. Indeed, otherwise we would find some $z\in A$ with $F(p(z))=h(t,z)$ which would imply $F(p(z))-q(z)=t(\pi(q(z))-q(z))\in tV\subseteq U$, contradicting our choice of U. By the homotopy invariance we conclude that we must have

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}(F,p,\pi\circ q,\Omega). \tag{4.2}$$

Since $\pi\circ q$ assumes its values only in Y_0, (4.1) thus implies that we must have

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}_{Y_0}(F,p,\pi\circ q,\Omega). \tag{4.3}$$

We have already seen that the right-hand side is actually independent of the particular choice of Y_0, and we show now that it is independent of the choice of U, V, K, π (and Ω_0) either. Thus, let $\widetilde{U}$, $\widetilde{V}$, $\widetilde{K}$, $\widetilde{\pi}$ (and $\widetilde{\Omega}_0$) be different choices; enlarging Y_0 if necessary ($\mathcal{Y}$ is directed upwards), we can assume that the corresponding space Y_0 is the same for both choices. The set $V_0:=\widetilde{V}\cap V$ is a neighbourhood of 0, and we find a corresponding π_0 as in Definition 4.1. Enlarging Y_0 once more if necessary, we may assume that π_0 assumes its values in Y_0. Consider now the homotopy $h(t,\,\cdot\,):=t(\pi_0\circ q)+(1-t)(\pi\circ q)$. If there is some $z\in A$ with $F(p(z))=h(t,z)$, then

$$F(p(z))-q(z)=t(\pi_0(q(z))-q(z))+(1-t)(\pi(q(z))-q(z))\in tV_0+(1-t)V\subseteq U$$

which is not possible. Hence, the homotopy invariance implies

$$\mathrm{DEG}_{Y_0}(F,p,\pi\circ q,\Omega)=\mathrm{DEG}_{Y_0}(F,p,\pi_0\circ q,\Omega).$$

For symmetry reasons, an analogous equality must also hold for the alternative choices, i.e.

$$\mathrm{DEG}_{Y_0}(F,p,\widetilde{\pi}\circ q,\Omega)=\mathrm{DEG}_{Y_0}(F,p,\pi_0\circ q,\Omega).$$

Combining these equalities, we find that the right-hand side of (4.3) is indeed independent of our particular choices of the involved sets and functions. This proves the uniqueness of DEG and that we actually can define DEG by (4.3).

Let us now show that the such defined degree DEG has all required properties. The normalization property is clear from the very definition; this in turn immediately implies the reduction property. To see the homotopy invariance, note that by Lemma 4.6 the set U in our definition of $\mathrm{DEG}(F,p,h(t,\,\cdot\,),\Omega)$ can be chosen independently of $t\in[0,1]$. Moreover, the range of h is contained in a compact set K, and so also π and Y_0 can be chosen independently of $t\in[0,1]$. We thus are only to show that

$$\mathrm{DEG}_{Y_0}(F,p,\pi\circ h(t,\,\cdot\,)) \text{ is independent of } t\in[0,1].$$

However, this follows immediately from the homotopy invariance of DEG_{Y_0} in Theorem 3.4 (resp. Corollary 3.5).

To see the strong independence from Γ, we observe similarly that it can be arranged that our definition of $\mathrm{DEG}(F,p,q,\Omega)$ and $\mathrm{DEG}(F,\widetilde{p},\widetilde{q},\Omega)$ works with the same maps π and the same $Y_0\in\mathcal{Y}$. Hence, the required property follows from the corresponding property of Theorem 3.4.

Concerning the existence property, assume that $\mathrm{COIN}_\Omega(F,p,q)=\emptyset$. Applying Lemma 4.6 with $A:=\overline{\Omega}$, we find some neighbourhood $U\subseteq Y$ of 0 with $F(p(z))-q(z)\notin U$ for all $z\in p^{-1}(A)$. Proceeding with this U as in our definition of the degree, we find that the corresponding map π even satisfies $\mathrm{COIN}_A\,\Omega(F,p,\pi\circ q)=\emptyset$, and so $\mathrm{DEG}_{Y_0}(F,p,q,\Omega)=\emptyset$ by the existence property of Theorem 3.4.

The proof of the restriction and exhaustion properties reduce with $A:=\overline{\Omega}\backslash\Omega_0$ in a similar manner to the corresponding properties of Theorem 3.4. Finally, the proof of the additivity reduces immediately to the additivity of DEG_{Y_0}. □

Our original intentention was to extend an *absolute* degree to function triples. The corresponding result reads as follows.

Theorem 4.10 (Absolute Degree Induces Absolute Triple-Degree). *Let $\mathcal{Y}$ be admissible for Y, and let $\mathcal{F}$ provide an absolute degree* $\deg\colon\mathcal{F}_0^{\mathrm{abs}}\to G$ *for $\mathcal{Y}$. In addition, suppose that each $Y_0\in\mathcal{Y}$ is even an extensor set for $[0,1]\times\overline{\Omega}$ for each $\Omega\in\mathcal{O}$.*

Then there is a unique degree DEG *with all properties of Theorem* 4.7 *when one replaces the normalization property by the following property*:

(a) (Normalization) *If the range of q is contained in $Y_0\in\mathcal{Y}$ and there is some φ with $(F,\varphi,\Omega)\in\mathcal{F}_0^{\mathrm{abs}}$ and $\varphi\circ p=q$, then*

$$\mathrm{DEG}(F,p,q,\Omega)=\deg(F,\varphi,\Omega).$$

The uniqueness of DEG *already follows from the normalization and homotopy invariance.*

Theorem 4.11 (Absolute Degree with Excision Induces Strong Absolute Triple-Degree). *If in Theorem* 4.10 *the degree has the excision property, then an analogous result to Theorem* 4.10 *holds when we replace $\mathcal{T}_0^{\mathrm{abs}}$ by $\mathcal{T}_c^{\mathrm{abs}}$ and the homotopy property by the corresponding stronger property of Theorem* 4.9*. The uniqueness of* DEG *already follows from the normalization, homotopy invariance, and excision property.*

Remark 4.12. If deg has the excision property, then the degree of Theorem 4.10 is the restriction of the degree of Theorem 4.11. This is an immediate consequence of the uniqueness claim of Theorem 4.9.

We now prove Theorems 4.10 and 4.11 simultaneously.

Proof. In view of Theorem 2.5, the existence is an immediate consequence of Theorem 4.7 (resp. of Theorem 4.8). To see the uniqueness, let $(F,p,q,\Omega)\in\mathcal{T}_0^{\mathrm{abs}}$ and choose U, V, π, Y_0, and Ω_0 as in the proof of Theorem 4.7/4.8. Then, as in that proof, the homotopy invariance implies (4.2). This implies that $\mathrm{DEG}(F,p,\pi\circ q,\Omega)$ is independent of the particular choice of U, V, π, Y_0, and Ω_0. Hence, to show that $\mathrm{DEG}(F,p,q,\Omega)$ is uniquely determined, we may assume that q assumes its values in some $Y_0 \in \mathcal{Y}$. Now we construct h and φ as in the first part of the proof of Theorem 4.7/4.8 and obtain by the homotopy invariance the equality

$$\mathrm{DEG}(F,p,q,\Omega) = \mathrm{DEG}(F,p,\varphi \circ p,\Omega).$$

This shows on the one hand, that this value is independent of the particular choice of φ. Since on the other hand the right-hand side of this equality is $\deg(F,\varphi,\Omega)$ by the normalization property, the uniqueness of $\mathrm{DEG}(F,p,q,\Omega)$ is established. □

We point out once more that it appears impossible to prove Theorem 4.10 without using Theorem 4.7 as an intermediate step (at least implicitly). In fact, an analysis of our proof shows that we have

$$\mathrm{DEG}(F,p,q,\Omega) = \deg(F,\varphi,\Omega)$$

where $\varphi\colon \overline{\Omega} \to Y_0$ is obtained by extension of a homotopy which starts from the restricted map $\pi \circ q\colon \overline{\Omega} \cap F^{-1}(Y_0) \to Y_0$. Thus, one would expect that the natural way to prove Theorem 4.10 directly is to verify all properties for this definition. However, it appears impossible to show directly that this definition is independent of the particular choice of Y_0: For a different choice $\widetilde{Y}_0$ of Y_0 the corresponding map $\widetilde{\varphi}$ can have a rather different form than φ and need not be close to it. In particular, the homotopy $h(t,x) = t\varphi + (1-t)\widetilde{\varphi}$ might have coincidence points with F on $\partial\Omega$ and so at least the "classical" argument in the construction of the Leray–Schauder degree breaks down completely. In fact, it appears even that there need not be an admissible homotopy connecting φ and $\widetilde{\varphi}$ at all. In our actual proof we only showed rather implicitly that $\deg(F,\varphi,\Omega) = \deg(F,\widetilde{\varphi},\Omega)$ by using several homotopies in different subsets, using the reduction property.

Since all degrees in this paper are uniquely determined, it is not surprising that they are compatible in the following sense.

Corollary 4.13 (Compatibility of the Absolute Triple-Degrees). *Let the assumptions of Theorem* 4.10 *be satisfied. Then the corresponding degree* DEG *is exactly that of Theorem* 4.7 *if one starts from the induced relative degree of Theorem* 2.5. *In the analogous sense, the degrees of Theorems* 4.11 *and* 4.8 *coincide.*

Proof. By the uniqueness, we only have to verify that the degree DEG of Theorem 4.10 and 4.11 satisfies also the normalization property of Theorem 4.7. Thus, let $(F,p,q,\Omega) \in \mathcal{T}_0^{\mathrm{abs}}$ and $(F,\varphi,\Omega,Y_0) \in \mathcal{F}_0^{\mathrm{rel}}$ and $\varphi \circ p = q$ on $p^{-1}(\overline{\Omega} \cap F^{-1}(Y_0))$.

We can extend φ to a continuous compact map $\varphi\colon p^{-1}(\overline{\Omega}) \to Y_0$ and define $\widetilde{q} := \varphi \circ p$. Since φ assumes only values in Y_0, we have $\mathrm{coin}_{\partial\Omega}(F, \varphi)$. Hence, $(F, \varphi, \Omega) \in \mathcal{F}_0^{\mathrm{abs}}$, and by the reduction and normalization property (of Theorem 4.10), we obtain

$$\mathrm{DEG}(F, p, q, \Omega) = \mathrm{DEG}(F, p, \widetilde{q}, \Omega) = \deg(F, \varphi, \Omega).$$

By the normalization property of Theorem 2.5, this is $\deg_{Y_0}(F, \varphi, \Omega)$, and so the normalization property of Theorem 4.7 is established. □

We could also have used the above argument to prove the uniqueness of the degree of Theorem 4.10 from the normalization, homotopy invariance and reduction property. Note that our earlier uniqueness proof did not need the latter.

Corollary 4.14 (Compatibility of Relative and Absolute Triple-Degrees). *Let the assumptions of Theorem* 4.7 *resp.* 4.10 *be satisfied. Let* $(F, p, q, \Omega) \in \mathcal{T}_0^{\mathrm{abs}}$ *be such that* q *assumes its values only in* $Y_0 \in \mathcal{Y}$. *Then*

$$\mathrm{DEG}(F, p, q, \Omega) = \mathrm{DEG}_{Y_0}(F, p, q, \Omega)$$

where the right-hand side is the degree of Theorem 3.4*; in the situation of Theorem* 4.10*, the relative degree used in Theorem* 3.4 *is that of Theorem* 2.5*.*

In the analogous sense, the degree of Theorem 4.8*, resp.* 4.11 *coincides with that of Corollary* 3.5*.*

Proof. Concerning the degree of Theorem 4.7, the claim has been shown in the proof of (4.1). The claim concerning the degree of Theorem 4.10 follows from this in view of Corollary 4.13. The proof concerning the degrees of Theorem 4.8 and 4.11 is analogous. □

5. Homotopy Invariance

For most applications, the homotopy invariance with respect to the third argument is not general enough: One would like to have invariance of the degree $\mathrm{DEG}(F, p, q, \Omega)$ under simultaneous homotopies of all three involved functions (F, p, q). In our axioms on the degree (Definition 2.1, resp. 2.2) the function F is always fixed, and it is in general not true that this definition is stable under *arbitrary* homotopies. However, in all known examples of degree theories, there exists a certain class of homotopies of F under which deg, resp. $\deg_{Y_0}$ is invariant. We will restrict our consideration to the class of such "admissible" homotopies in the following sense.

Definition 5.1. Let deg be an absolute degree. Then we denote by $\mathcal{A}^{\mathrm{abs}}$ the system of all (H,Ω) where $\Omega \in \mathcal{O}$ and $H\colon [0,1]\times\overline{\Omega} \to Y$ (not necessarily continuous) are such that the following holds:

- $(H(t,\,\cdot\,),\Omega) \in \mathcal{F}$ $(0 \le t \le 1)$ and for each continuous compact map $h\colon [0,1]\times\overline{\Omega} \to Y$ with $\operatorname{coin}_{[0,1]\times\partial\Omega}(H,h) = \emptyset$ the value
$$\deg(H(t,\,\cdot\,),h(t,\,\cdot\,),\Omega)$$
is independent of $t \in [0,1]$.

We need also a corresponding notion for the relative degree.

Definition 5.2. Let deg be a relative degree for $\mathcal{Y}$. Then we denote by $\mathcal{A}^{\mathrm{rel}}$ the system of all (H,Ω,Y_0) where $\Omega \in \mathcal{O}$, $Y_0 \in \mathcal{Y}$ and $H\colon [0,1]\times\overline{\Omega} \to Y$ (not necessarily continuous) are such that the following holds:

- $(H(t,\,\cdot\,),\Omega) \in \mathcal{F}$ $(0 \le t \le 1)$ and for each continuous compact map $h\colon ([0,1]\times\overline{\Omega}) \cap H^{-1}(Y_0) \to Y_0$ with $\operatorname{coin}_{[0,1]\times\partial\Omega}(H,h) = \emptyset$ the value
$$\deg_{Y_0}(H(t,\,\cdot\,),h(t,\,\cdot\,),\Omega)$$
is independent of $t \in [0,1]$.

We automatically have included the case of constant $H(t,\,\cdot\,) = F$:

Proposition 5.3. *If $(F,\Omega) \in \mathcal{F}$ and $H(t,\,\cdot\,) := F$ $(0 \le t \le 1)$ then $(H,\Omega) \in \mathcal{A}^{\mathrm{abs}}$ and $(H,\Omega,Y_0) \in \mathcal{A}^{\mathrm{rel}}$ for each $Y_0 \in \mathcal{Y}$.*

Proof. The claim is a reformulation of the homotopy invariance axiom. □

One might expect that (at least in natural situations) if a homotopy H is admissible for the absolute degree then it is also admissible for the induced relative degree. The following result states that this is indeed the case.

Proposition 5.4. *Let the assumptions of Theorem 2.5 be satisfied, and let $(H,\Omega) \in \mathcal{A}^{\mathrm{abs}}$ and $Y_0 \in \mathcal{Y}$ be such that $H^{-1}(Y_0)$ is closed. Then we have for the induced relative degree of Theorem 2.5 that $(H,\Omega,Y_0) \in \mathcal{A}^{\mathrm{rel}}$.*

Proof. Since Y_0 is an extensor set and $H^{-1}(Y_0)$ is closed, we can extend the map h of Definition 11 to a continuous compact map $h\colon [0,1]\times\overline{\Omega} \to Y_0$. Since h assumes its values only in Y_0, we have $\operatorname{coin}_{[0,1]\times\partial\Omega}(H,h) = \emptyset$. Since $(H,\Omega) \in \mathcal{A}^{\mathrm{abs}}$, it follows that $\deg(H(t,\,\cdot\,),h(t,\,\cdot\,),\Omega)$ is independent of t. By the normalization property of Theorem 2.5, we have
$$\deg_{Y_0}(H(t,\,\cdot\,),h(t,\,\cdot\,),\Omega) = \deg(H(t,\,\cdot\,),h(t,\,\cdot\,),\Omega),$$
and so also $\deg_{Y_0}(H(t,\,\cdot\,),h(t,\,\cdot\,),\Omega)$ is independent of t. □

For our aim to formulate a homotopy invariance of $\mathrm{DEG}(F,p,q,\Omega)$ with respect to all three functions (F,p,q), Definition 10 apparently is an appropriate requirement with respect to F. Concerning p and q, it is not natural at all to consider homotopies in the usual sense, because in the applications one wants to interpret $q\circ p^{-1}$ as a multivalued map and so one wants to consider multivalued homotopies. The appropriate setting is therefore the diagram

$$Y \xleftarrow{\ H\ } [0,1]\times\overline{\Omega} \xleftarrow{\ P\ } \Gamma \xrightarrow{\ Q\ } Y$$

In most application, Γ will be (a subset of) a space of the form $[0,1]\times\widetilde{\Gamma}$ (so that P and Q can indeed be interpreted as homotopies) but we do not require this. We therefore have to define in a slightly more technical manner how to interpret the corresponding maps P_t and Q_t at the time $t\in[0,1]$.

Definition 5.5. Let $(H,\Omega,Y_0)\in\mathcal{A}^{\mathrm{rel}}$. Then we write $(H,P,Q,\Omega,Y_0)\in\mathcal{H}_0^{\mathrm{rel}}$ if $P\colon\Gamma\to[0,1]\times X$ is continuous with some topological space Γ and $Q\colon D\to Y_0$ with $D:=P^{-1}(H^{-1}(Y_0))$ are such that the following holds.

(a) $H^{-1}(Y_0)$ is closed in $[0,1]\times\overline{\Omega}$.
(b) $P(\Gamma)\supseteq([0,1]\times\overline{\Omega})\cap H^{-1}(Y_0)$.
(c) There is a continuous map $\varphi\colon H^{-1}(Y_0)\to Y_0$ and a continuous compact map $h\colon[0,1]\times D\to Y_0$ with $h(0,Z)=Q$ and $h(1,\,\cdot\,)=\varphi\circ P$ such that

$$\mathrm{COIN}_{[0,1]\times\partial\Omega}(H,P,h(t,\,\cdot\,))=\emptyset\quad(0\le t\le 1).$$

(d) $(H(t,\,\cdot\,),P_t,Q_t,\Omega,Y_0)\in\mathcal{F}_0^{\mathrm{rel}}$ for each $t\in[0,1]$ where Q_t is the restriction of Q to

$$\Gamma_t:=P^{-1}(\{t\}\times\overline{\Omega})\cap H^{-1}(Y_0),$$

and $P_t\colon\Gamma_t\to X$ is defined by $P(z)=(t,P_t(z))$.

Theorem 5.6 (Homotopy Invariance of the Relative Triple-Degree). *With the above notations, if $(H,P,Q,\Omega,Y_0)\in\mathcal{H}_0^{\mathrm{rel}}$, then*

$$\mathrm{DEG}_{Y_0}(H(t,\,\cdot\,),P_t,Q_t,\Omega)\ \textit{is independent of}\ t\in[0,1].$$

Proof. In case $Y_0=Y$ the claim has been proved in [15]. The general case reduces to this special case similarly as in the proof of Theorem 3.4. $\square$

In order to verify that $(H,P,Q,\Omega,Y_0)\in\mathcal{H}_0^{\mathrm{rel}}$, the following test is convenient:

Theorem 5.7. *Let $(H,\Omega,Y_0)\in\mathcal{A}^{\mathrm{rel}}$, $P\colon\Gamma\to[0,1]\times X$, and $Q\colon\Gamma\to Y_0$ where Γ is some topological space. Suppose that the following holds*:

(a) $P(\Gamma)\supseteq([0,1]\times\overline{\Omega})\cap H^{-1}(Y_0)$.
(b) *$H^{-1}(Y_0)$ and $P^{-1}(H^{-1}(Y_0))$ are closed, compact, and metrizable.*
(c) *The restriction of H to $H^{-1}(Y_0)$ is continuous.*

(d) *The restrictions of P and Q to $P^{-1}(H^{-1}(Y_0))$ are continuous.*
(e) *Y_0 is homeomorphic to a topological group.*
(f) *Y_0 is an extensor set for $H^{-1}(Y_0)$ and for $[0,1] \times P^{-1}(H^{-1}(Y_0))$.*
(g) *At least one of the following two requirements holds:*
  (g_1) *All fibres $P^{-1}(x)$ $(x \in H^{-1}(Y_0))$ are R_δ, and the inductive dimension of $H^{-1}(Y_0)$ is finite.*
  (g_2) *All fibres $p^{-1}(x)$ $(x \in H^{-1}(Y_0))$ are acyclic with respect to Čech cohomology with coefficients in $\mathbb{Z}$. In addition, there is some $y_0 \in Y$ such that $Y \setminus \{y_0\}$ is homotopically n-simple for each $n \geq 1$. Moreover, $\dim(H^{-1}(Y_0)) < \infty$ where $\dim$ denotes the covering dimension, and*

$$\sup_{x \in H^{-1}(Y_0)} \dim p^{-1}(x) < \infty.$$

(h) $\mathrm{COIN}_{[0,1]\times\partial\Omega}(H,P,Q) = \emptyset$.

Then $(H,P,Q,\Omega,Y_0) \in \mathcal{H}_0^{\mathrm{rel}}$.

Proof. We use the notation of Definition 5.5. The assertion that we have $(H(t,\,\cdot\,),P_t,Q_t,\Omega,Y_0) \in \mathcal{F}_0^{\mathrm{rel}}$ is a straightforward consequence of Theorem 3.6 (with $Y_1 := Y_0$ and $M := \overline{\Omega}$). We also apply Theorem 3.6 with $M := H^{-1}(Y_0)$ and find that P is an $(H,[0,1]\times\overline{\Omega},Y_0,Y_0)$-compact homotopy bijection on each closed $A \subseteq [0,1]\times\overline{\Omega}$. In particular, P is an $(H,[0,1]\times\overline{\Omega},Y_0,Y_0)$-compact homotopy surjection on $\partial\Omega$. This implies that maps φ and h as required in Definition 5.5 do exist. □

In order to formulate a corresponding result for the absolute triple-degree of Theorem 4.7 or 4.10, we assume as in that theorems that Y is a topological Hausdorff vector space with an admissible family $\mathcal{Y}$ of subspaces.

We treat the situations of Theorem 4.7 (and 4.8) or of Theorem 4.10 (and 4.11) simultaneously, and so we assume that we have given either a relative or an absolute degree. Our following hypotheses in these two situations differ only in this obvious respect:

Definition 5.8. Let $H\colon [0,1]\times\overline{\Omega} \to Y$ and $P\colon \Gamma \to [0,1]\times X$ with a topological space Γ. Then we write $(H,P,\Omega) \in \mathcal{H}^{\mathrm{rel}\to\mathrm{abs}}$, resp. $(H,P,\Omega) \in \mathcal{H}^{\mathrm{abs}}$ if the following holds for each $Y_0 \in \mathcal{Y}$:

(a) $(H,\Omega,Y_0) \in \mathcal{A}^{\mathrm{rel}}$, resp. $(H,\Omega) \in \mathcal{A}^{\mathrm{abs}}$.
(b) P is continuous and proper and $P(\Gamma) \supseteq ([0,1]\times\overline{\Omega}) \cap H^{-1}(Y_0)$.
(c) $H\colon [0,1]\times\overline{\Omega} \to Y$ is proper and has a closed graph.
(d) $H([0,1]\times\overline{\Omega}) \cap Y_0$ is relatively compact.

(e) Y is metrizable, or for each continuous compact $h\colon [0,1] \times P^{-1}([0,1] \times \overline{\Omega}) \to Y$ and each closed $A \subseteq \overline{\Omega}$ the set

$$\begin{aligned}\bigcup\{H(t,x) - h(s,P^{-1}(t,x)) : s,t \in [0,1],\, x \in A\}\\ = \{H(P(z)) - h(t,z) : t \in [0,1],\, z \in P^{-1}([0,1] \times A)\}\end{aligned}$$

is closed in Y.

(f) $H^{-1}(Y_0)$ is metrizable and closed in X.

(g) $P^{-1}(H^{-1}(Y_0))$ is metrizable.

(h) Y_0 is an extensor set for $[0,1] \times \overline{\Omega}$ and for $[0,1] \times P^{-1}([0,1] \times \overline{\Omega})$.

(i) At least one of the following two requirements holds:

(i$_1$) All fibres $P^{-1}(t,x)$ ($(t,x) \in H^{-1}(Y_0)$) are R_δ, and the inductive dimension of $H^{-1}(Y_0)$ is finite.

(i$_2$) All fibres $p^{-1}(t,x)$ ($(t,x) \in H^{-1}(Y_0)$) are acyclic with respect to Čech cohomology with coefficients in $\mathbb{Z}$. Moreover, the covering dimension $\dim(H^{-1}(Y_0))$ is finite and

$$\sup_{(t,x)\in H^{-1}(Y_0)} \dim P^{-1}(t,x) < \infty.$$

If in addition $Q\colon \Gamma \to Y$ is such that the restriction of Q to $P^{-1}([0,1] \times \overline{\Omega})$ is continuous and compact and

$$\mathrm{COIN}_{[0,1]\times\partial\Omega}(H,P,Q) = \emptyset$$

then we write $(H,P,Q,\Omega) \in \mathcal{H}_0^{\mathrm{rel}\to\mathrm{abs}}$, resp. $(H,P,Q,\Omega) \in \mathcal{H}_0^{\mathrm{abs}}$.

The following lemmas contain Lemma 4.5 and 4.6 as special cases.

Lemma 5.9. *Let (H,P,Ω) belong to $\mathcal{H}^{\mathrm{rel}\to\mathrm{abs}}$ or $\mathcal{H}^{\mathrm{abs}}$. Then for each $Y_0 \in \mathcal{Y}$ the restriction of H to $H^{-1}(Y_0)$ is continuous.*

Proof. By hypothesis, the spaces $M := H^{-1}(Y_0)$ and Y_0 are metrizable. It thus suffices to show that H is sequentially continuous. If this is not the case, we find a sequence $a_n \in M$ with $a_n \to a$ such that $H(a_n)$ is outside some fixed neighbourhood of $H(a)$. Since $H(M)$ is contained in a compact subset of Y_0 (by hypothesis and since Y_0 is closed), we may assume that $H(a_n) \to y$ for some $y \in Y_0$, and so $y \neq H(a)$. This contradicts the fact that H has a closed graph.□

Lemma 5.10. *Let (H,P,Ω) belong to $\mathcal{H}^{\mathrm{rel}\to\mathrm{abs}}$ or to $\mathcal{H}^{\mathrm{abs}}$. Then for each continuous compact map $h\colon [0,1] \times P^{-1}([0,1] \times \overline{\Omega}) \to Y$ and each closed $A \subseteq \overline{\Omega}$ the set*

$$R_A := \bigcup\{H(t,x) - h(s,P^{-1}(t,x)) : s,t \in [0,1],\, x \in A\}$$

is closed in Y.

Proof. We have to prove this only for the case that Y is metrizable, since in the other case, the assertion is assumed in Definition 5.8. Let $A \subseteq \overline{\Omega}$ be closed and $y_n \in R_A$ converge to some $y \in Y$. There are $t_n, s_n \in [0,1]$, $x_n \in A$, and $z_n \in P^{-1}(t_n, x_n)$ such that $y_n = H(t_n, x_n) - h(s_n, z_n)$. By the compactness, we may assume that $h(s_n, z_n) \to y^*$ converges; similarly, we may assume that $s_n \to s$ converges. It follows that $H(t_n, x_n) = y_n + h(s_n, z_n) \to y + y^*$ converges and so, since H is proper, the elements (t_n, x_n) are contained in a compact set $K \subseteq [0,1] \times A$. Hence, the sequence $(z_n)_n$ is contained in the compact set $P^{-1}(K)$ and thus has some cluster point $z \in P^{-1}(A)$. By continuity of P, it follows that $(t,x) := P(z)$ is a cluster point of $((t_n, x_n))_n$, and so since H has a closed graph and $H(t_n, x_n) \to y^*$, we must have $H(t,x) = y^*$. Since h has also a closed graph and $h(s_n, z_n) = H(t_n, x_n) - y_n \to y^* - y$, we obtain similarly $h(s,z) = y^* - y$, and so $y = y^* - h(s,z) \in H(t,x) - h(s, p^{-1}(t,x)) \in R_A$. □

Lemma 5.11. *Let (H, P, Ω) belong to $\mathcal{H}^{\mathrm{rel}\to\mathrm{abs}}$ or to $\mathcal{H}^{\mathrm{abs}}$. Then for each continuous compact map $h\colon [0,1] \times P^{-1}([0,1] \times \overline{\Omega}) \to Y$ and each closed $A \subseteq \overline{\Omega}$ with*

$$\mathrm{COIN}_{[0,1]\times A}(H, P, h(t,\,\cdot\,)) = \emptyset \quad (0 \le t \le 1)$$

there is a neigbhorhood $U \subseteq Y$ of 0 with $H(P(z)) - h(t,z) \notin U$ for each $z \in P^{-1}([0,1] \times A)$ and each $t \in [0,1]$.

Proof. By Lemma 5.10, the set

$$R_A = \{H(P(z)) - h(t,z) : t \in [0,1],\, z \in P^{-1}([0,1] \times A)\}$$

is closed. The assumption of the lemma means that $0 \notin R_A$, and so there is a neigbhorhood $U \subseteq Y$ of 0 which is disjoint from R_A. □

Theorem 5.12 (Homotopy Invariance of the Absolute Triple-Degree). *Let the assumptions of Theorem 4.7, resp. 4.10 be satisfied and let $(H, P, Q, \Omega) \in \mathcal{H}_0^{\mathrm{rel}\to\mathrm{abs}}$, resp. $(H, P, Q, \Omega) \in \mathcal{H}_0^{\mathrm{abs}}$. For $t \in [0,1]$, let Q_t be the restriction of Q to*

$$\Gamma_t := P^{-1}(\{t\} \times \overline{\Omega}),$$

and $P_t\colon \Gamma_t \to X$ be defined by $P(z) = (t, P_t(z))$.

Then $(H(t,\,\cdot\,), P_t, Q_t, \Omega) \in \mathcal{T}_0^{\mathrm{abs}}$ for each $t \in [0,1]$ and

$$\mathrm{DEG}(H(t,\,\cdot\,), P_t, Q_t, \Omega) \textit{ is independent of } t \in [0,1],$$

where DEG *is the degree of Theorem* 4.7/4.8*, resp.* 4.10/4.11*.*

Proof. In view of Remark 4.9 (resp. Remark 4.12), it suffices to consider the degree of Theorem 4.7 (resp. of Theorem 4.10). By Corollary 4.13, and since each $(H, P, Q, \Omega) \in \mathcal{H}_0^{\mathrm{abs}}$ belongs to $\mathcal{H}_0^{\mathrm{rel}\to\mathrm{abs}}$ (in view of Proposition 5.4), it suffices to consider the situation of Theorem 4.7. We omit the straightforward proof which shows that indeed $(H(t,\,\cdot\,), P_t, Q_t, \Omega) \in \mathcal{T}_0^{\mathrm{abs}}$.

By Lemma 5.11, we find some neighbourhood $U \subseteq Y$ of 0 with $H(P(z)) - Q(z) \notin U$ for each $z \in A := P^{-1}([0,1] \times \partial\Omega)$. Choose a neighbourhood $V \subseteq Y$

of 0 with $tV + sV \subseteq U$ for each $t, s \in [0,1]$, and let $Y_0 \in \mathcal{Y}$ and π be correspondingly as in Definition 4.1. From the proof of Theorem 4.7 (more precisely from (4.3)), we know that

$$\operatorname{DEG}(H(t,\,\cdot\,), P_t, Q_t, \Omega) = \operatorname{DEG}_{Y_0}(H(t,\,\cdot\,), P_t, \pi \circ Q_t, \Omega) \quad (t \in [0,1]). \tag{5.1}$$

By Theorem 5.7, we have $(H, P, \pi \circ Q, \Omega, Y_0) \in \mathcal{H}_0^{\mathrm{rel}}$, and so Theorem 5.6 implies that the right-hand side of (5.1) is indeed independent of $t \in [0,1]$. $\square$

We point out that the homotopy invariance in the third argument is a special case of Theorem 5.12 (and similarly for the relative case). To see this, put $H(t,\,\cdot\,) := F$ (Proposition 5.3), $\widetilde{\Gamma} := [0,1] \times \Gamma$, and define $P: \widetilde{\Gamma} \to [0,1] \times X$ and $Q: \widetilde{\Gamma} \to Y$ by $P(t,z) := p(z)$ and $Q(t,z) := h(t,z)$. Now identify $\{t\} \times \Gamma$ with Γ, using the fact that the degree of a function triple is invariant under homeomorphisms of the intermediate space Γ (by the strong independence of DEG from Γ).

The reader might wonder why we did not formulate a corresponding stronger homotopy invariance for Theorems 4.8 resp. 4.11, i.e. whether the requirement on the homotopies can be relaxed in the presence of the excision property. Such a result could indeed be formulated, but its proof would be a trivial application of the excision property (of the extended triple-degree) and of Theorem 5.12. In this sense, Theorem 5.12 already implicitly includes this apparently more general case.

References

[1] P. Benevieri and M. Furi, *A Degree for Locally Compact Perturbations of Fredholm Maps in Banach Spaces*, Abstr. Appl. Anal. **2006** (2006), 1–20.

[2] D. Gabor and W. Kryszewski, *A coincidence theory involving Fredholm operators of nonnegative index*, Topol. Methods Nonlinear Anal. **15** (2000), no. 1, 43–59.

[3] R.E. Gaines and J.L. Mawhin, *Coincidence degree, and nonlinear differential equations*, Lect. Notes in Math., vol. 568, Springer, Berlin, Heidelberg, New York, 1977.

[4] K. Gęba, I. Massabo, and A. Vignoli, *Generalized topological degree and bifurcation*, Nonlinear Analysis and its Applications, Proc. NATO Adv. Study Inst., (Maratea/Italy 1985 (Dordrecht)) (S.P. Singh, ed.), D. Reidel, 1986, pp. 55–73.

[5] L. Górniewicz, *Homological methods in fixed-point theory of multi-valued maps*, Dissertationes Math. (Rozprawy Mat.), no. 129, Polish Scientific Publ., Warszawa, 1976.

[6] ______, *Topological fixed point theory of multivalued mappings*, 2nd ed., Springer, Dordrecht, 2006.

[7] I.-S. Kim and M. Väth, *Some results on the extension of single- and multivalued maps*, Topol. Methods Nonlinear Anal. **28** (2006), 133–153.

[8] G. Köthe, *Topologische lineare Räume I*, 2nd ed., Springer, Berlin, Heidelberg, New York, 1966.

[9] W. Kryszewski, *Homotopy properties of set-valued mappings*, Univ. N. Copernicus Publishing, Toruń, 1997.

[10] J. L. MAWHIN, *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations **12** (1972), 610–636.

[11] L. NIRENBERG, *An application of generalized degree to a class of nonlinear problems*, Contributions to Nonlinear Analysis. Proc. Symp. Univ. Wisconsin, (Madison, 1971 (New York, London)) (E. H. Zarantonello, ed.), Academic Press, 1971, pp. 57–74.

[12] ______, *Generalized degree and nonlinear problems*, 3ieme Coll. sur l'Analyse fonction. (Liege 1970 (Louvain, Belgique)), Centre Belge de Recherches Matheématiques, Vander éditeur, 1971, pp. 1–9.

[13] J. PEJSACHOWICZ AND A. VIGNOLI, *On the topological coincidence degree for perturbations of Fredholm operators*, Boll Un. Mat. Ital. B(5) **17** (1980), 1457–1466.

[14] E. G. SKLYARENKO, *Theorem on dimension-lowering maps*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **10** (1962), 423–432.

[15] M. VÄTH, *Merging of degree and index theory*, Article ID 36361, Fixed Point Theory Appl. **2006** (2006), 30 pages.

[16] ______, *Degree and index theories for noncompact function triples*, Topol. Methods Nonlinear Anal. **29** (2007), no. 1, 79–118.

[17] V.G. ZVYAGIN AND N.M. RATINER, *Oriented degree of Fredholm maps of non-negative index and its applications to global bifurcation of solutions*, Global Analysis—Studies and Applications V (Yu.G. Borisovich and Yu.E. Gliklikh, eds.), Lect. Notes in Math., vol. 1520, Springer, Berlin, Heidelberg, New York, 1992, pp. 111–137.

Manuscript received October 5, 2011

MARTIN VÄTH
Free University of Berlin
Department of Mathematics (WE1)
Arnimallee 3
D-14195 Berlin, GERMANY

E-mail address: vaeth@mathematik.uni-wuerzburg.de