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ON EXISTENCE OF GLOBAL IN TIME SOLUTIONS TO THERMOELASTICITY WITH A QUADRATIC NONLINEARITY FOR SMALL DATA

Leszek Bartczak

Abstract. In this paper we study a simplified model of thermoviscoplasticity. We prove local in time existence and uniqueness of solution. Moreover, for sufficiently small data, global in time existence is proved.

1. Introduction

In this paper we consider the problem of thermoelasticity with a nonlinear term in the equation of heat conduction. The considered body occupies initially a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. Let $(t,x) \in (0,T) \times \Omega$, where x represents the material point while t denotes time.

$$\begin{aligned} -\mathrm{div}_x \mathcal{D}(\varepsilon(u(t,x))) &= -\nabla_x \theta(t,x) + f(t,x), \\ \partial_t \theta(t,x) - \Delta\theta(t,x) &= -\mathrm{div}_x \partial_t u(t,x) + |\mathcal{D}(\varepsilon(u(t,x)))|^2. \end{aligned} \tag{1.1}$$

The first equation results from the balance of momentum in the quasistatic case. The second equation follows from the balance of energy and describes the heat conduction in the considered body. The function $u\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ is the displacement, $\varepsilon(u) := \frac{1}{2}(\nabla_x u + \nabla_x^T u)$ represents the linear strain tensor. We can observe that $\varepsilon(u) \in \mathcal{S}^3$, where $\mathcal{S}^3$ is the set of 3×3 symmetric matrices

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with real entries. We assume that the operator $\mathcal{D}: \mathcal{S}^3 \to \mathcal{S}^3$ is linear, symmetric and positive definite. The temperature of the body is described by the function $\theta: \Omega \times \mathbb{R}_+ \to \mathbb{R}$. The vector-valued function $f: \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ represents volume forces.

We complete the problem (1.1) with Dirichlet boundary condition:

$$\begin{aligned} u(t,x)|_{\partial\Omega} &= u_D(t,x), \\ \theta(t,x)|_{\partial\Omega} &= \theta_D(t,x). \end{aligned}$$

and impose the initial data:

$$\theta(0,x) = \theta_0(x).$$

Furthermore we have to add the compatibility condition in the following form

$$\theta_0(x)|_{\partial\Omega} = \theta_D(0,x).$$

Our aim is to find a solution (u,θ) and prove its uniqueness. First we show the existence of a local in time solution to (1.1) using Banach fixed point theorem. Then we prove that for some sort of the given data we can solve our problem for any $t \in \mathbb{R}_+$.

Our motivation to investigate such a system of equations comes from the study of mathematical models of thermoviscoplasticity. Mathematical analysis of models in thermoviscoplasticity is not sufficiently developed. The question of existence, uniqueness and the behaviour in time of the solutions is still open in general, only some special cases are resolved. Recently author in [2] has proved the existence and uniqueness of the solution to a problem of termoviscoplasticity in the case of quasistatic model with Bodner–Partom constitutive equation on plastic part of the strain tensor. Nevertheless in the most of models in thermoviscoplasticity the crucial therm – the stress times the time derivative of a plastic part of the strain is only integrable over Ω. Problem (1.1) is a simplification of this case.

Remark 1.1. Let us denote

$$\mathcal{E}(\varphi) := \frac{1}{2} \int_\Omega \mathcal{D}(\varepsilon(\varphi))\varepsilon(\varphi)\,dx \quad \text{for } \varphi \in H^1(\Omega).$$

One easily observes that there exist positive constants $c_1, c_2 \in \mathbb{R}_+$ such that

$$c_1\mathcal{E}(\varphi) \le \|\nabla_x \varphi\|^2_{L^2(\Omega)} \le c_2(\mathcal{E}(\varphi) + \|\varphi|_{\partial\Omega}\|^2_{H^{1/2}(\partial\Omega)}) \tag{1.2}$$

(this immediately follows from Korn's inequality and properties of the operator $\mathcal{D}$). In particular $\mathcal{E}(\,\cdot\,)$ is equivalent to the standard norm in the space $H^1_0(\Omega)$.

2. Reduction of the boundary condition

In this section our aim is to transform the problem (1.1) into a homogeneous one. As in [2] we solve the following two linear problems. The first of the considered problems has the form

$$
\begin{aligned}
\partial_t\widetilde{\theta}(t,x) &= \Delta\widetilde{\theta}(t,x),\\
\widetilde{\theta}(t,x)|_{\partial\Omega} &= \theta_D(t,x),\\
\widetilde{\theta}(0,x) &= \widetilde{\theta}_0(x),
\end{aligned}
\tag{2.1}
$$

where $\widetilde{\theta}_0$ is some function compatible with the boundary data θ_D. This means that the following condition is satisfied:

$$
\widetilde{\theta}_0(x) - \theta_0(x) \in H^1_0(\Omega).
\tag{2.2}
$$

The second problem is to solve the linear elasticity system with the temperature from (2.1)

$$
\begin{aligned}
-\mathrm{div}_x\mathcal{D}(\varepsilon(\widetilde{u}(t,x))) &= -\nabla_x\widetilde{\theta}(t,x) + f(t,x),\\
\widetilde{u}(t,x)|_{\partial\Omega} &= u_D(t,x).
\end{aligned}
\tag{2.3}
$$

Lemma 2.1. (a) *Assume that*

$$
\theta_D \in L^2(0,T;H^{3/2}(\partial\Omega)) \cap H^{3/4}(0,T;L^2(\partial\Omega)) \cap L^\infty(0,T;H^{1/2}(\partial\Omega)).
$$

Moreover, let the initial data $\widetilde{\theta}_0$ belong to the space $H^1(\Omega)$ and satisfy the compatibility condition (2.2). *Then the problem* (2.1) *has a unique solution $\widetilde{\theta}$ satisfying $\widetilde{\theta} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ with $\partial_t\widetilde{\theta} \in L^2(0,T;L^2(\Omega))$ and*

$$
\begin{aligned}
&\|\widetilde{\theta}\|_{L^2(0,T;H^2(\Omega))} + \|\widetilde{\theta}\|_{L^\infty(0,T;H^1(\Omega))} + \|\partial_t\widetilde{\theta}\|_{L^2(0,T;L^2(\Omega))}\\
&\qquad\le C(T)(\|\widetilde{\theta}_0\|_{H^1(\Omega)} + \|\theta_D\|_{L^2(0,T;H^{3/2}(\partial\Omega))}\\
&\qquad\quad + \|\theta_D\|_{H^{3/4}(0,T;L^2(\partial\Omega))} + \|\theta_D\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))}).
\end{aligned}
$$

(b) *Let $f \in H^1(0,T;L^2(\Omega))$, $u_D \in L^\infty(0,T;H^{3/2}(\partial\Omega)) \cap H^1(0,T;H^{1/2}(\partial\Omega))$. Then the problem* (2.3) *has a unique solution $\widetilde{u}$ belonging to $L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega))$ and we have the estimates:*

$$
\begin{aligned}
\|\widetilde{u}\|_{L^\infty(0,T;H^2(\Omega))} &\le C(\|\widetilde{\theta}\|_{L^\infty(0,T;H^1(\Omega))}\\
&\qquad + \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|u_D\|_{L^\infty(0,T;H^{3/2}(\partial\Omega))}),\\
\|\partial_t\widetilde{u}\|_{L^2(0,T;H^1(\Omega))} &\le C(\|\partial_t\widetilde{\theta}\|_{L^2(0,T;L^2(\Omega))}\\
&\qquad + \|\partial_t f\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t u_D\|_{L^2(0,T;H^{1/2}(\partial\Omega))}).
\end{aligned}
$$

Remark 2.2. The first part of the above assertion follows immediately from the theory of parabolic equations (cf. [4]). The second part results from the theory of linear elasticity (cf. [6]).

Next we define $(\widehat{u},\widehat{\theta}) := (u-\widetilde{u},\theta-\widetilde{\theta})$ to obtain the problem with homogeneous boundary values.

$$
\begin{aligned}
-\operatorname{div}_x \mathcal{D}(\varepsilon(\widehat{u}(t,x))) &= -\nabla_x \widehat{\theta}(t,x),\\
\partial_t \widehat{\theta}(t,x) - \Delta\widehat{\theta}(t,x) &= -\operatorname{div}_x \partial_t \widehat{u}(t,x)\\
&\quad + |\mathcal{D}(\varepsilon(\widehat{u}(t,x)+\widetilde{u}(t,x)))|^2 + \operatorname{div}_x \partial_t \widetilde{u}(t,x),\\
\widehat{u}(t,x)|_{\partial\Omega} &= 0,\\
\widehat{\theta}(t,x)|_{\partial\Omega} &= 0,\\
\widehat{\theta}(0,x) &= \theta_0(x) - \widetilde{\theta}_0(x)\ [=:\theta_0^\star].
\end{aligned}
\tag{2.4}
$$

In the problem above and in further investigations we will treat $\widetilde{u}$ and $\widetilde{\theta}$ as known functions.

3. Existence of a local in time solution to (2.4) and (1.1)

In this section we prove the existence of a local in time solution to problem (2.4). To obtain this result we use the Banach fixed point theorem. First let us take $\theta^\star \in L^\infty(0,T;H_0^1(\Omega))$ such that $\|\theta^\star\|_{L^\infty(0,T;H_0^1(\Omega))} \le M$, where $M > 1$ will be indicated later. Additionally, we set $B(M) := \{\xi \in L^\infty(0,T;H_0^1(\Omega)) \mid \|\xi\|_{L^\infty(0,T;H_0^1(\Omega))} \le M\}$. Then we solve the following problem:

$$
\begin{aligned}
-\operatorname{div}_x \mathcal{D}(\varepsilon(v(t,x))) &= -\nabla_x \theta^\star(t,x),\\
v(t,x)|_{\partial\Omega} &= 0.
\end{aligned}
\tag{3.1}
$$

Lemma 3.1 (Existence of a solution to (3.1)). *For any $\theta^\star \in B(M)$ there exists a unique solution $v \in L^\infty(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ to the problem* (3.1)*. Moreover, the following estimates are satisfied*:

$$
\begin{aligned}
\|v\|_{L^\infty(0,T;H_0^1(\Omega))} &\le C\|\theta^\star\|_{L^\infty(0,T;L^2(\Omega))},\\
\|v\|_{L^\infty(0,T;H^2(\Omega))} &\le C\|\nabla_x\theta^\star\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
\tag{3.2}
$$

Remark 3.2. The lemma above is a standard result from the theory of linear elasticity (see [6]).

Next let us consider the second problem (with v solving (3.1)):

$$
\begin{aligned}
-\operatorname{div}_x \mathcal{D}(\varepsilon(u(t,x))) &= -\nabla_x \theta(t,x),\\
\partial_t \theta(t,x) - \Delta\theta(t,x) &= -\operatorname{div}_x \partial_t u(t,x)\\
&\quad + |\mathcal{D}(\varepsilon(v(t,x)+\widetilde{u}(t,x)))|^2 - \operatorname{div}_x \partial_t \widetilde{u}(t,x),\\
u(t,x)|_{\partial\Omega} &= 0,\\
\theta(t,x)|_{\partial\Omega} &= 0,\\
\theta(0,x) &= \theta_0^\star(x).
\end{aligned}
\tag{3.3}
$$

Lemma 3.3 (Existence of a solution to (3.3)). *There exists a unique solution* (u,θ) *for the problem* (3.3) *belonging to* $H^1(0,T;H^1_0(\Omega))$ *and* $L^\infty(0,T;H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$*, respectively. Moreover, the following estimates hold*:

$$\begin{aligned}(3.4)\qquad &\sup_{0\le t\le T}\|\nabla_x u(t)\|^2_{L^2(\Omega)}+\sup_{0\le t\le T}\|\theta(t)\|^2_{L^2(\Omega)}+\|\nabla_x\theta\|^2_{L^2(0,T;L^2(\Omega))}\\ &\qquad\le C(\|v+\widetilde{u}\|^4_{L^4(0,T;H^2(\Omega))}+\|\mathrm{div}_x\partial_t\widetilde{u}\|^2_{L^2(0,T;L^2(\Omega))}+\|\theta_0^\star\|^2_{L^2(\Omega)}),\end{aligned}$$

$$\begin{aligned}(3.5)\qquad &\|\nabla_x\partial_t u\|^2_{L^2(0,T;L^2(\Omega))}+\|\partial_t\theta\|^2_{L^2(0,T;L^2(\Omega))}+\sup_{0\le t\le T}\|\nabla_x\theta(t)\|^2_{L^2(\Omega)}\\ &\qquad\le C(\|v+\widetilde{u}\|^4_{L^4(0,T;H^2(\Omega))}+\|\mathrm{div}_x\partial_t\widetilde{u}\|^2_{L^2(0,T;L^2(\Omega))}+\|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)}).\end{aligned}$$

Remark 3.4. Lemma 3.3 follows from the more general fact, which is proven in Theorem 5.1 in the appendix and from the following estimate

$$\|\,|\mathcal{D}(\varepsilon(v(t)+\widetilde{u}(t)))|^2\,\|_{L^2(\Omega)}=\|\mathcal{D}(\varepsilon(v(t)+\widetilde{u}(t)))\|^2_{L^4(\Omega)}\le C\|v(t)+\widetilde{u}(t)\|^2_{H^2(\Omega)}.$$

The estimate above follows immediately from the Sobolev embedding theorem and properties of the operator $\mathcal{D}$.

Remark 3.5. Note that constants on right hand sides of inequalities (3.2), (3.4) and (3.5) depend only on the geometry of the domain Ω and coefficients of the operator $\mathcal{D}$ but they are independent of the length of the time interval $(0,T)$.

Remark 3.6. The operator $\mathcal{P}\colon B(M)\to L^\infty(0,T;H^1_0(\Omega))$ such that $\mathcal{P}\colon\theta^\star\mapsto\theta$ is well defined. Indeed to get θ from $\theta^\star$ we solve problems (3.1) and (3.3).

We are going to use the Banach fixed point theorem for the operator $\mathcal{P}$ so we need to prove that $\mathcal{P}$ maps $B(M)$ into $B(M)$ and that $\mathcal{P}$ is a contraction on small time interval.

Proposition 3.7. *For any given data* $\theta_0^\star$ *we can choose such a constant* $M\ge 0$ *and such a short time interval* $(0,T)$ *that the operator* $\mathcal{P}\colon B(M)\to B(M)$ *i.e.* $\|\theta\|_{L^\infty(0,T;H^1_0(\Omega))}\le M$.

Proof. To prove the proposition it is enough to observe that:

$$\|v\|_{L^4(0,T;H^2(\Omega))}\le\sqrt[4]{T}\,\|v\|_{L^\infty(0,T;H^2(\Omega))}.$$

Then immediately from Lemma 3.3 and Lemma 3.1 we can estimate as follows:

$$\begin{aligned}\sup_{0\le t\le T}\|\theta(t)\|^2_{H^1_0(\Omega)}&\le C(T\|v\|^4_{L^\infty(0,T;H^2(\Omega))}+\|\widetilde{u}\|^4_{L^4(0,T;H^2(\Omega))}\\ &\qquad+\|\mathrm{div}_x\partial_t\widetilde{u}\|^2_{L^2(0,T;H^2(\Omega))}+\|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)})\\ &\le C(TM^4+\|\widetilde{u}\|^4_{L^4(0,T;H^2(\Omega))}\\ &\qquad+\|\mathrm{div}_x\partial_t\widetilde{u}\|^2_{L^2(0,T;H^2(\Omega))}+\|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)})<M^2,\end{aligned}$$

where

$$M^2 > C(\|\widetilde{u}\|^4_{L^4(0,T;H^2(\Omega))} + \|\mathrm{div}_x \partial_t \widetilde{u}\|^2_{L^2(0,T;H^2(\Omega))} + \|\nabla_x \theta^\star_0\|^2_{L^2(\Omega)})$$

and

$$T < \frac{M^2 - C(\|\widetilde{u}\|^2_{L^2(0,T;H^2(\Omega))} + \|\mathrm{div}_x \partial_t \widetilde{u}\|^2_{L^2(0,T;H^2(\Omega))} + \|\nabla_x \theta^\star_0\|^2_{L^2(\Omega)})}{CM^4}. \quad \square$$

Now we are going to show that $\mathcal{P}$ is a contraction on $B(M)$ for a sufficiently short time interval. To prove it we take $\theta^\star_1, \theta^\star_2 \in B(M)$ and solve the problem (3.1). Next for solutions v_1, v_2 we solve the problem (3.3). Let $(\theta_1, u_1), (\theta_2, u_2)$ be the corresponding solutions. We denote by $\overline{\theta}^\star := \theta^\star_1 - \theta^\star_2$, $\overline{v} := v_1 - v_2$, $(\overline{\theta}, \overline{u}) := (\theta_1, u_1) - (\theta_2, u_2)$. Functions $(\overline{\theta}, \overline{u})$ satisfy

$$\begin{aligned}
-\mathrm{div}_x \mathcal{D}(\varepsilon(\overline{u}(t,x))) &= -\nabla_x \overline{\theta}(t,x), \\
\partial_t \overline{\theta}(t,x) - \Delta \overline{\theta}(t,x) &= -\mathrm{div}_x \partial_t \overline{u}(t,x) |\mathcal{D}(\varepsilon(v_1(t,x)) + \widetilde{u}(t,x))|^2 \\
&\quad - |\mathcal{D}(\varepsilon(v_2(t,x) + \widetilde{u}(t,x)))|^2, \\
\overline{u}(t,x)|_{\partial\Omega} &= 0, \\
\overline{\theta}(t,x)_{\partial\Omega} &= 0, \\
\overline{\theta}(0,x) &= 0.
\end{aligned} \tag{3.6}$$

Proposition 3.8 (Estimates for differences). *For almost all $t \le T$, where T was fixed in Proposition 3.7 functions $\overline{\theta}$, $\overline{u}$ satisfy the following estimates*:

$$\begin{aligned}
\sup_{0\le\tau\le t} \|\nabla_x \overline{u}(\tau)\|^2_{L^2(\Omega)} + \sup_{0\le\tau\le t} \|\overline{\theta}(\tau)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x \overline{\theta}(\tau)\|^2_{L^2\Omega)} \\
\le Ct(M^2 + 2\|\widetilde{u}\|^2_{L^\infty(0,T;H^2(\Omega))}) \sup_{0\le\tau\le t} \|\overline{\theta}^\star(\tau)\|^2_{H^1_0(\Omega)},
\end{aligned}$$

$$\begin{aligned}
2\int_0^t \|\nabla_x \partial_t \overline{u}\|^2_{L^2(\Omega)} + \int_0^t \|\partial_t \overline{\theta}\|^2_{L^2(\Omega)} + \sup_{0\le\tau\le t} \|\nabla_x \overline{\theta}(\tau)\|^2_{L^2(\Omega)} \\
\le \widetilde{C}t(M^2 + 2\|\widetilde{u}\|^2_{L^\infty(0,T;H^2(\Omega))}) \sup_{0\le\tau\le t} \|\overline{\theta}^\star(\tau)\|^2_{H^1_0(\Omega)}.
\end{aligned}$$

Proof. Using $\partial_t \overline{u}$ and $\overline{\theta}$ as test functions in the first and the second equations of (3.6) respectively we obtain

$$\begin{aligned}
&\partial_t \mathcal{E}(\overline{u}(t)) + \frac{1}{2} \partial_t \|\overline{\theta}(t)\|^2_{L^2(\Omega)} + \|\nabla_x \overline{\theta}(t)\|^2_{L^2(\Omega)} \\
&\quad = \int_\Omega (|\mathcal{D}(\varepsilon(v_1(t))) + \mathcal{D}(\varepsilon(\widetilde{u}(t)))|^2 - |\mathcal{D}(\varepsilon(v_2(t))) + \mathcal{D}(\varepsilon(\widetilde{u}(t)))|^2) \overline{\theta}(t)\, dx.
\end{aligned}$$

Using the Hölder inequality gives us that

$$\begin{aligned}
&\partial_t \mathcal{E}(\overline{u}(t)) + \frac{1}{2} \partial_t \|\overline{\theta}(t)\|^2_{L^2(\Omega)} + \|\nabla_x \overline{\theta}(t)\|^2_{L^2(\Omega)} \\
&\le \|\mathcal{D}(\varepsilon(v_1(t) - v_2(t)))\|_{L^4(\Omega)} \|\mathcal{D}(\varepsilon(v_1(t) + v_2(t) + 2\widetilde{u}(t)))\|_{L^4(\Omega)} \|\overline{\theta}(t)\|_{L^2(\Omega)}.
\end{aligned}$$

From properties of the operator $\mathcal{D}$, Korn's inequality and the Sobolev embedding theorem we get

$$\partial_t \mathcal{E}(\overline{u}(t)) + \frac{1}{2}\partial_t \|\overline{\theta}(t)\|^2_{L^2(\Omega)} + \|\nabla_x \overline{\theta}(t)\|^2_{L^2(\Omega)} \\ \leq \|v_1(t) - v_2(t))\|_{H^2(\Omega)} \|v_1(t) + v_2(t) + 2\widetilde{u}(t))\|_{H^2(\Omega)} \|\overline{\theta}(t)\|_{L^2(\Omega)}.$$

Applying Poincaré and Young inequalities, the property (1.2) and integrating over time we obtain

$$\|\nabla_x \overline{u}(t)\|^2_{L^2(\Omega)} + \|\overline{\theta}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x \overline{\theta}(\tau)\|^2_{L^2(\Omega)}\, d\tau \\ \leq C\int_0^t \|\overline{v}(\tau)\|^2_{H^2(\Omega)} (\|v_1(\tau)\|^2_{H^2(\Omega)} + \|v_2(\tau)\|^2_{H^2(\Omega)} + 4\|\widetilde{u}(\tau)\|^2_{H^2(\Omega)})\, d\tau.$$

From linearity of the problem (3.1) using Lemma 3.1 we can estimate

$$\|\nabla_x \overline{u}(t)\|^2_{L^2(\Omega)} + \|\overline{\theta}(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x \overline{\theta}(\tau)\|^2_{L^2(\Omega)}\, d\tau \\ \leq C\int_0^t \|\overline{\theta}^\star(\tau)\|^2_{H^1(\Omega)} (\|\theta_1^\star(\tau)\|^2_{H^1(\Omega)} + \|\theta_2^\star(\tau)\|^2_{H^1(\Omega)} + 4\|\widetilde{u}(\tau)\|^2_{H^2(\Omega)})\, d\tau.$$

Thus from the fact that $\theta_1^\star, \theta_2^\star \in B(M)$ it follows

$$\sup_{0\leq\tau\leq t} \|\nabla_x \overline{u}(\tau)\|^2_{L^2(\Omega)} + \sup_{0\leq\tau\leq t} \|\overline{\theta}(\tau)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x \overline{\theta}(\tau)\|^2_{L^2(\Omega)}\, d\tau \\ \leq Ct \sup_{0\leq\tau\leq t} \|\overline{\theta}^\star(\tau)\|^2_{H^1(\Omega)} \Big(2M^2 + 4 \sup_{0\leq\tau\leq T} \|\widetilde{u}(\tau)\|^2_{H^2(\Omega)}\Big)$$

and this inequality ends the proof of the first part of the assertion.

To prove the second inequality we differentiate the first equation of (3.6) with respect to time and use as a test function $\partial_t \overline{u}$ while the second equation of (3.6) is tested by $\partial_t \overline{\theta}$. Finally we obtain

$$2\mathcal{E}(\partial_t \overline{u}(t)) + \|\partial_t \overline{\theta}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\partial_t \|\nabla_x \overline{\theta}(t)\|^2_{L^2(\Omega)} \\ = \int_\Omega (|\mathcal{D}(\varepsilon(v_1(t))) + \mathcal{D}(\varepsilon(\widetilde{u}(t)))|^2 - |\mathcal{D}(\varepsilon(v_2(t))) + \mathcal{D}(\varepsilon(\widetilde{u}(t)))|^2) \partial_t \overline{\theta}(t)\, dx.$$

Similar argumentation as in the first part of the proof leads us to the desired inequality. □

Corollary 3.9. *For all sufficiently small $T^\star \leq T$ the operator $\mathcal{P}$ is a contraction. Indeed, from Proposition* 3.8 *we obtain that there exists a positive constant $C(T)$ such that for all $0 \leq t \leq T$*

$$\sup_{0\leq\tau\leq t} \|\overline{\theta}(\tau)\|^2_{H^1_0(\Omega)} \leq tC(T) \sup_{0\leq\tau\leq t} \|\overline{\theta}^\star(\tau)\|^2_{H^1_0(\Omega)}.$$

Thus we can choose such $T^\star \le T$ that it holds

$$\|\overline{\theta}\|_{L^\infty(0,T^\star;H_0^1(\Omega))} \le \alpha\|\overline{\theta}^\star(\tau)\|_{L^\infty(0,T^\star;H_0^1(\Omega))},$$

where $0 < \alpha < 1$.

Theorem 3.10. *Under the assumptions of Lemma* 2.1 *there exists a local in time unique solution*

$$\widehat{u} \in H^1(0,T^\star;H_0^1(\Omega)), \quad \widehat{\theta} \in H^1(0,T^\star;L^2(\Omega)) \cap L^\infty(0,T^\star;H_0^1(\Omega))$$

to the problem (2.4).

Proof. By the Banach fixed point theorem we obtain that the operator $\mathcal{P}\colon B(M) \to B(M)$ has a unique fixed point $\widehat{\theta} \in B(M)$. From lemma 3.1 we obtain that there exists a function $\widehat{u} \in H^1(0,T^\star;H_0^1(\Omega))$ which is the unique solution to (3.1). Moreover, it follows from Lemma 3.3 that the pair $(\widehat{u},\widehat{\theta})$ is the unique solution to (3.3), thus $(\widehat{u},\widehat{\theta})$ is the unique solution to (2.4). Additionally from Lemma 3.3 it follows that $\partial_t\widehat{\theta} \in L^2(0,T^\star;L^2(\Omega))$. □

Remark 3.11 (Strengthened regularity for (2.4)). The solution of the problem (2.4) belongs to the following spaces:

$$\widehat{u} \in L^\infty(0,T^\star;H^2(\Omega)), \quad \widehat{\theta} \in L^2(0,T^\star;H^2(\Omega)).$$

Indeed $\widehat{u}(t)$ for almost all $t \in (0,T^\star)$ satisfies an elliptic equation with right hand side $-\nabla_x\widehat{\theta}(t) \in L^2(\Omega)$, while $\widehat{\theta}$ satisfies a parabolic equation with right hand side $\big(-\mathrm{div}_x\partial_t\widehat{u} + |\mathcal{D}(\varepsilon(\widehat{u}))|^2 + \mathrm{div}_x\partial_t\widetilde{u}\big) \in L^2(0,T^\star;L^2(\Omega))$.

At the end of this section we formulate the local in time existence theorem for problem (1.1), which is a direct consequence of Lemma 2.1 and Theorem 3.10.

Theorem 3.12. *Under assumptions of Lemma* 2.1 *there exists a local in time unique solution* (u,θ) *for the problem* (1.1) *such that* $u \in L^\infty(0,T^\star;H^2(\Omega))$ *while* $\partial_t u \in L^2(0,T^\star;H^1(\Omega))$ *and* $\theta \in L^2(0,T^\star;H^2(\Omega)) \cap L^\infty(0,T^\star;H^1(\Omega))$ *while* $\partial_t\theta \in L^2(0,T^\star;L^2(\Omega))$.

4. Existence of a global in time solution for sufficiently small data

Our aim in this section is to choose the given data (boundary data, the initial condition and the vector of volume forces) in such a way to obtain a global in time solution to the problem (1.1). First let us make an observation.

Lemma 4.1. *The trace in normal direction of* $(\mathcal{D}(\varepsilon(\partial_t u(t,\,\cdot))) - \mathbb{I}\cdot\partial_t\theta(t,\,\cdot))$ *is a functional from* $H^{-1/2}(\partial\Omega)$ *for almost all* $t \in (0,T^\star)$ *and the following inequality holds*:

$$\begin{aligned}\|(\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\partial_t\theta(t))\overrightarrow{n}(\,\cdot\,)\|_{H^{-1/2}(\partial\Omega)} \\ \le C(\|\partial_t f(t)\|_{L^2(\Omega)} + \|\mathcal{D}(\varepsilon(\partial_t u(t))\|_{L^2(\Omega)} + \|\partial_t\theta(t)\|_{L^2(\Omega)})\end{aligned}$$

where $\mathbb{I}$ *denotes the* 3×3 *identity matrix and* $\overrightarrow{n}(x)$ *the unit outward normal to* $\partial\Omega$.

Proof. We use the fact (cf. [5]) that a function $\phi \cdot \overrightarrow{n}$ belongs to the space $H^{-1/2}(\partial\Omega)$ provided $\phi \in L^2(\Omega)$ and the weak divergence $\mathrm{div}_x\phi \in L^2(\Omega)$. Then it holds that

$$\|\phi \cdot \overrightarrow{n}\|_{H^{-1/2}(\partial\Omega)} \leq C(\|\phi\|_{L^2(\Omega)} + \|\mathrm{div}_x\phi\|_{L^2(\Omega)}).$$

Thus using a weak formulation of the first equation of (1.1) differentiated with respect to time and estimates from the previous section we easily obtain

$$\begin{aligned}
&\|(\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\,\partial_t\theta(t))\overrightarrow{n}\|_{H^{-1/2}(\partial\Omega)} \\
&\quad \leq C(\|\mathrm{div}_x(\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\partial_t\theta(t))\|_{L^2(\Omega)} + \|\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\partial_t\theta(t)\|_{L^2(\Omega)}) \\
&\quad \leq C(\|\partial_t f(t)\|_{L^2(\Omega)} + \|\mathcal{D}(\varepsilon(\partial_t u(t))\|_{L^2(\Omega)} + \|\partial_t\theta(t)\|_{L^2(\Omega)}). \qquad \square
\end{aligned}$$

To prove global in time existence of the solution to the problem (1.1) we need additionally some assumptions on the regularity and size of the data which we will specify at the end of the proof of Theorem 4.2 (inequalities (4.8)–(4.12)).

(A1) Let the norm $\|\theta_0\|^2_{H^1(\Omega)}$ be small enough.

(A2) Let the function $f \in L^1(\mathbb{R}_+; L^2(\Omega)) \cap H^1(\mathbb{R}_+; L^2(\Omega)) \cap L^6(\mathbb{R}_+; L^2(\Omega))$ and let the norms:

$$\|f|_{t=0}\|_{L^2(\Omega)}, \quad \|f\|_{L^1(\mathbb{R}_+;L^2(\Omega))}, \quad \|f\|_{H^1(\mathbb{R}_+;L^2(\Omega))}, \quad \|f\|_{L^6(\mathbb{R}_+;L^2(\Omega))}$$

be small enough.

(A3) Let the boundary condition

$$\begin{aligned}
u_D \in L^1(\mathbb{R}_+; H^{1/2}(\partial\Omega)) &\cap H^1(\mathbb{R}_+; H^{1/2}(\partial\Omega)) \\
&\cap L^4(\mathbb{R}_+; H^{3/2}(\partial\Omega)) \cap L^6(\mathbb{R}_+; H^{3/2}(\partial\Omega))
\end{aligned}$$

and let the norms:

$$\begin{aligned}
&\|u_D|_{t=0}\|_{H^{1/2}(\partial\Omega)}, \quad \|u_D\|_{L^1(\mathbb{R}_+;H^{1/2}(\partial\Omega))}, \quad \|u_D\|_{H^1(\mathbb{R}_+;H^{1/2}(\partial\Omega))}, \\
&\|u_D\|_{L^4(\mathbb{R}_+;H^{3/2}(\partial\Omega))}, \quad \|u_D\|_{L^6(\mathbb{R}_+;H^{3/2}(\partial\Omega))}
\end{aligned}$$

be small enough.

(A4) Let the boundary condition

$$\theta_D \in H^1(\mathbb{R}_+; H^{1/2}(\partial\Omega)) \cap W^{1,\infty}(\mathbb{R}_+; H^{1/2}(\partial\Omega))$$

and let the norms

$$\|\theta_D\|_{H^1(\mathbb{R}_+;H^{1/2}(\partial\Omega))}, \quad \|\theta_D\|_{W^{1,\infty}(\mathbb{R}_+;H^{1/2}(\partial\Omega))}$$

be small enough.

THEOREM 4.2. *If the given data satisfy the assumptions* (A1)–(A4)*, then there exists a global in time solution* (u,θ) *to* (1.1).

PROOF. In the proof we will use a convention that different constants will be denoted by the same letter C. We will at the same time deal with energetic estimates of two types. The first one comes out of testing $(1.1)_1$ (the first equation of (1.1)) by $\partial_t u$ while $(1.1)_2$ (the second equation of (1.1)) by θ. Integrating by parts we get the following:

$$\begin{aligned}(4.1)\quad \partial_t\mathcal{E}(u(t)) &+ \frac{1}{2}\partial_t\|\theta(t)\|^2_{L^2(\Omega)} + \|\nabla_x\theta(t)\|^2_{L^2(\Omega)}\\ &= \int_{\partial\Omega}(\mathcal{D}(\varepsilon(u(t)) - \mathbb{I}\,\theta(t))\overrightarrow{n}\partial_t u_D(t)\,dS + \int_\Omega f(t)\partial_t u(t)\,dx\\ &\quad+\int_{\partial\Omega}\theta_D(t)\frac{\partial\theta}{\partial\overrightarrow{n}}(t)\,dS + \int_\Omega|\mathcal{D}(\varepsilon(u(t))|^2\theta(t)\,dx.\end{aligned}$$

To obtain the second estimate we test $(1.1)_1$ differentiated with respect to time by $\partial_t u$ while $(1.1)_2$ by $\partial_t\theta$ and, as in the first estimate, we integrate by parts.

$$\begin{aligned}(4.2)\quad 2\mathcal{E}(\partial_t u(t)) &+ \|\partial_t\theta(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\partial_t\|\nabla_x\theta(t)\|^2_{L^2(\Omega)}\\ &= \int_{\partial\Omega}(\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\partial_t\theta(t))\,\overrightarrow{n}\partial_t u_D(t)dS + \int_\Omega\partial_t f(t)\partial_t u(t)\,dx\\ &\quad+\int_{\partial\Omega}\partial_t\theta_D(t)\frac{\partial\theta}{\partial\overrightarrow{n}}(t)\,dS + \int_\Omega|\mathcal{D}(\varepsilon(u(t))|^2\partial_t\theta(t)\,dx.\end{aligned}$$

We now add the estimates above and in order to close the resulting inequality we need to deal with the following terms: the first one is the integral with the volume force and its time derivative. We use Hölder inequality, then Poincaré inequality, property (1.2) and finally Young's inequality to obtain

$$\begin{aligned}&\int_\Omega(f(t)+\partial_t f(t))\partial_t u(t)\,dx\\ &\qquad\le\frac{1}{2}\mathcal{E}(\partial_t u(t)) + C(\|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|f(t)\|^2_{L^2(\Omega)} + \|\partial_t f(t)\|^2_{L^2(\Omega)}).\end{aligned}$$

Next we estimate the following boundary integral:

$$\begin{aligned}&\int_{\partial\Omega}(\mathcal{D}(\varepsilon(u(t)) - \mathbb{I}\,\theta(t))\overrightarrow{n}\partial_t u_D(t)\,dS\\ &\quad\le\|(\mathcal{D}(\varepsilon(u(t)) - \mathbb{I}\,\theta(t))\overrightarrow{n}\|_{H^{-1/2}(\partial\Omega)}\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)}\\ &\quad\le C(\|\mathrm{div}_x(\mathcal{D}(\varepsilon(u(t))) - \mathbb{I}\,\theta(t))\|_{L^2(\Omega)}\\ &\qquad+\|\mathcal{D}(\varepsilon(u(t)) - \mathbb{I}\,\theta(t)\|_{L^2(\Omega)})\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)}\\ &\quad\le C(\|f(t)\|_{L^2(\Omega)} + \|\mathcal{D}(\varepsilon(u(t))\|_{L^2(\Omega)} + \|\theta(t)\|_{L^2(\Omega)})\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)}\\ &\quad\le C(\|f(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^{1/2}(\partial\Omega)}\\ &\qquad+\|\theta_D(t)\|_{H^{1/2}(\partial\Omega)} + \|\nabla_x\theta(t)\|_{L^2(\Omega)})\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4}\|\nabla_x \theta(t)\|^2_{L^2(\Omega)} + C\|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)} \\
&\quad + C(\|f(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^{1/2}(\partial\Omega)} + \|\theta_D(t)\|_{H^{1/2}(\partial\Omega)})\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\leq \frac{1}{4}\|\nabla_x \theta(t)\|^2_{L^2(\Omega)} + C(\|f(t)\|^2_{L^2(\Omega)} \\
&\quad + \|u_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)}).
\end{aligned}
$$

We have used the fact which was stated in the proof of Lemma 4.1 and the equation $(1.1)_1$. Then we apply the elliptic estimates on $u(t)$ and Poincaré inequality. Further we estimate the next boundary integral using Lemma 4.1, definition of the energy $\mathcal{E}$, properties of the operator $\mathcal{D}$ and once again Young's inequality.

$$
\begin{aligned}
&\int_{\partial\Omega} (\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\,\partial_t\theta(t))\overrightarrow{n}\,\partial_t u_D(t)\, dS \\
&\quad \leq \|(\mathcal{D}(\varepsilon(\partial_t u(t)) - \mathbb{I}\,\partial_t\theta(t))\overrightarrow{n}\|_{H^{-1/2}(\partial\Omega)}\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\quad \leq C(\|\partial_t f(t)\|_{L^2(\Omega)} + \|\mathcal{D}(\varepsilon(\partial_t u(t))\|_{L^2(\Omega)} + \|\partial_t\theta(t)\|_{L^2(\Omega)})\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\quad \leq \frac{1}{2}\mathcal{E}(\partial_t u(t)) + \frac{1}{5}\|\partial_t\theta(t)\|^2_{L^2(\Omega)} \\
&\quad\quad + C\,\|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)} + C\|\partial_t f(t)\|_{L^2(\Omega)}\|\partial_t u_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\quad \leq \frac{1}{2}\mathcal{E}(\partial_t u(t)) + \frac{1}{5}\|\partial_t\theta(t)\|^2_{L^2(\Omega)} + C(\|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|\partial_t f(t)\|^2_{L^2(\Omega)}).
\end{aligned}
$$

The estimate below is obtained with the help of the Sobolev embedding theorem and ellipticity of the equation $(1.1)_1$.

$$
\begin{aligned}
\|\mathcal{D}(\varepsilon(u(t))\|_{L^4(\Omega)} &\leq C\|u(t)\|_{H^2(\Omega)} \\
&\leq C(\|\nabla_x\theta(t)\|_{L^2(\Omega)} + \|f(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^{3/2}(\partial\Omega)}).
\end{aligned} \tag{4.3}
$$

Now we are going to estimate the boundary integral with θ. We use once more the estimate for traces in the normal direction in the space $H^{-1/2}(\partial\Omega)$. Moreover we use the parabolicity of the equation $(1.1)_2$ and the inequality (4.3):

$$
\begin{aligned}
&\int_{\partial\Omega} (\partial_t\theta_D(t) + \theta_D(t))\frac{\partial\theta}{\partial\overrightarrow{n}}(t)\, dS \\
&\qquad \leq \left\|\frac{\partial\theta}{\partial\overrightarrow{n}}(t)\right\|_{H^{-1/2}(\partial\Omega)}\|\partial_t\theta_D(t) + \theta_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\qquad \leq C(\|\nabla_x\partial_t u\|_{L^2(\Omega)} + \|\mathcal{D}(\varepsilon(u(t))\|^2_{L^4(\Omega)} \\
&\qquad\quad + \|\partial_t\theta(t)\|_{L^2(\Omega)} + \|\theta_D(t)\|_{H^{1/2}(\partial\Omega)})\|\partial_t\theta_D(t) + \theta_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\qquad \leq C(\|\nabla_x\partial_t u\|_{L^2(\Omega)} + \|\nabla_x\theta(t)\|_{L^2(\Omega)} + \|\partial_t\theta(t)\|^2_{L^2(\Omega)} + \|f(t)\|^2_{L^2(\Omega)} \\
&\qquad\quad + \|u_D(t)\|^2_{H^{3/2}(\partial\Omega)} + \|\theta_D(t)\|_{H^{1/2}(\partial\Omega)})\|\partial_t\theta_D(t) + \theta_D(t)\|_{H^{1/2}(\partial\Omega)} \\
&\qquad \leq \frac{1}{2}\mathcal{E}(\partial_t u(t)) + \frac{1}{5}\|\partial_t\theta(t)\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+ C \sup_{0\le\tau\le t} \|\partial_t\theta_D(\tau) + \theta_D(\tau)\|_{H^{1/2}(\partial\Omega)} \|\nabla_x\theta(t)\|^2_{L^2(\Omega)} \\
&+ C(\|f(t)\|^4_{L^2(\Omega)} + \|u_D(t)\|^4_{H^{3/2}(\partial\Omega)} \\
&+ \|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|\partial_t\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)}).
\end{aligned}
$$

The next two estimates are the key to obtain global in time existence. To obtain the first one we apply Hölder inequality, estimates (4.3), Poincaré inequality and Young's inequality.

$$
\begin{aligned}
\int_\Omega |\mathcal{D}(\varepsilon(u(t)))|^2\theta(t)\,dx &\le \|\mathcal{D}(\varepsilon(u(t)))\|^2_{L^4(\Omega)} \|\theta(t)\|_{L^2(\Omega)} \\
&\le C(\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} + \|f(t)\|^2_{L^2(\Omega)} + \|u_D(t)\|^2_{H^{3/2}(\partial\Omega)}) \|\theta(t)\|_{L^2(\Omega)} \\
&\le C\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} \sup_{0\le\tau\le t} \|\theta(\tau)\|_{L^2(\Omega)} \\
&\quad + \frac{1}{4}\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} + C(\|f(t)\|^4_{L^2(\Omega)} + \|u_D(t)\|^4_{H^{3/2}(\partial\Omega)}) \\
&\quad + C(\|f(t)\|^2_{L^2(\Omega)} + \|u_D(t)\|^2_{H^{3/2}(\partial\Omega)}) \|\theta_D(t)\|_{H^{\frac{1}{2}}(\partial\Omega)} \\
&\le C\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} \sup_{0\le\tau\le t} \|\theta(\tau)\|_{L^2(\Omega)} + \frac{1}{4}\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} \\
&\quad + C(\|f(t)\|^4_{L^2(\Omega)} + \|u_D(t)\|^4_{H^{3/2}(\partial\Omega)} + \|\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)}).
\end{aligned}
$$

To obtain the following inequality we start similarly as in the previous estimates and with the help of the Gagliardo–Nirenberg–Sobolev inequality we can estimate as follows:

$$
\begin{aligned}
\int_\Omega |\mathcal{D}(\varepsilon(u(t))|^2\partial_t\theta(t)\,dx &\le \|\mathcal{D}(\varepsilon(u(t))\|^2_{L^4(\Omega)} \|\partial_t\theta(t)\|_{L^2(\Omega)} \\
&\le C\|\nabla_x u(t)\|^2_{L^4(\Omega)} \|\partial_t\theta(t)\|_{L^2(\Omega)} \\
&\le C\|\nabla_x u(t)\|^{3/2}_{H^1(\Omega)} \|\nabla_x u(t)\|^{1/2}_{L^2(\Omega)} \|\partial_t\theta(t)\|_{L^2(\Omega)}.
\end{aligned}
$$

Now, using ellipticity of $(1.1)_1$, we obtain that

$$
\begin{aligned}
\|\nabla_x u(t)\|^{3/2}_{H^1(\Omega)} &\le C(\|\nabla_x\theta(t)\|^{3/2}_{L^2(\Omega)} + \|f(t)\|^{3/2}_{L^2(\Omega)} + \|u_D(t)\|^{3/2}_{H^{3/2}(\partial\Omega)}), \\
\|\nabla_x u(t)\|^{\frac{1}{2}}_{L^2(\Omega)} &\le C(\|\theta(t)\|^{1/2}_{L^2(\Omega)} + \|f(t)\|^{1/2}_{L^2(\Omega)} + \|u_D(t)\|^{1/2}_{H^{1/2}(\partial\Omega)}).
\end{aligned}
$$

Using the last two inequalities we have

$$
\begin{aligned}
\int_\Omega |\mathcal{D}(\varepsilon(u(t))|^2\partial_t\theta(t)\,dx &\le C\{\|\partial_t\theta(t)\|_{L^2(\Omega)} \|\nabla_x\theta(t)\|^{3/2}_{L^2(\Omega)} \|\theta(t)\|^{1/2}_{L^2(\Omega)} \\
&+ \|\partial_t\theta(t)\|_{L^2(\Omega)} \|\nabla_x\theta(t)\|^{3/2}_{L^2(\Omega)} (\|f(t)\|^{1/2}_{L^2(\Omega)} + \|u_D(t)\|^{1/2}_{H^{1/2}(\partial\Omega)}) \\
&+ \|\partial_t\theta(t)\|_{L^2(\Omega)} \|\theta(t)\|^{1/2}_{L^2(\Omega)} (\|f(t)\|^{3/2}_{L^2(\Omega)} \\
&+ \|u_D(t)\|^{3/2}_{H^{3/2}(\partial\Omega)}) + \|\partial_t\theta(t)\|_{L^2(\Omega)} (\|f(t)\|^{1/2}_{L^2(\Omega)} + \|u_D(t)\|^{1/2}_{H^{1/2}(\partial\Omega)}) \\
&\cdot (\|f(t)\|^{3/2}_{L^2(\Omega)} + \|u_D(t)\|^{3/2}_{H^{3/2}(\partial\Omega)})\}.
\end{aligned}
$$

Application of Young's and Poncaré inequalities leads to the final estimate for crucial nonlinear term

$$\begin{aligned}
&\int_\Omega |\mathcal{D}(\varepsilon(u(t))|^2 \partial_t\theta(t)\,dx \le \frac{C}{2}\|\partial_t\theta(t)\|^2_{L^2(\Omega)} \sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|_{L^2(\Omega)}\\
&\quad + \frac{C}{2}\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} \sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^2(\Omega)} + \frac{C}{2}\|\partial_t\theta(t)\|^2_{L^2(\Omega)} \sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|^3_{L^2(\Omega)}\\
&\quad + C(\|f(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^{1/2}(\partial\Omega)}) + \frac{1}{5}\|\partial_t\theta(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla_x\theta(t)\|^2_{L^2(\Omega)}\\
&\quad + C(\|f(t)\|^6_{L^2(\Omega)} + \|u_D(t)\|^6_{H^{3/2}(\partial\Omega)} + \|\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)}) + \frac{1}{5}\|\partial_t\theta(t)\|^2_{L^2(\Omega)}\\
&\quad + C(\|f(t)\|_{L^2(\Omega)} + \|u_D(t)\|_{H^{1/2}(\partial\Omega)})(\|f(t)\|^3_{L^2(\Omega)} + \|u_D(t)\|^3_{H^{3/2}(\partial\Omega)})\\
&\le \|\partial_t\theta(t)\|^2_{L^2(\Omega)}\left(\frac{C}{2}\sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|_{L^2(\Omega)} + \frac{C}{2}\sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|^3_{L^2(\Omega)} + \frac{2}{5}\right)\\
&\quad + \|\nabla_x\theta(t)\|^2_{L^2(\Omega)}\left(\frac{C}{2}\sup_{0\le\tau\le t}\|\theta(\tau)\|_{L^2(\Omega)} + \frac{1}{4}\right) + C(\|f(t)\|^6_{L^2(\Omega)}\\
&\quad + \|u_D(t)\|^6_{H^{3/2}(\partial\Omega)} + \|\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|f(t)\|^2_{L^2(\Omega)} + \|u_D(t)\|^2_{H^{1/2}(\partial\Omega)}).
\end{aligned}$$

We can observe that constants C in the inequalities above depend only on the geometry of Ω and the coefficients of the operator $\mathcal{D}$.

Finally, we collect above inequalities to estimate the sum of (4.1) and (4.2).

$$\begin{aligned}
&\partial_t\mathcal{E}(u(t)) + \frac{1}{2}\partial_t\|\theta(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\partial_t\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\mathcal{E}(\partial_t u(t))\\
&\quad + \|\nabla_x\theta(t)\|^2_{L^2(\Omega)}\bigg(\frac{1}{4} - C_1\sup_{0\le\tau\le t}\|\partial_t\theta_D(\tau) + \theta_D(\tau)\|_{H^{1/2}(\partial\Omega)}\\
&\quad - C_2\sup_{0\le\tau\le t}\|\theta(\tau)\|^2_{L^2(\Omega)}\bigg)\\
&\quad + \|\partial_t\theta(t)\|^2_{L^2(\Omega)}\bigg(\frac{1}{5} - C_3\bigg(\sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|_{L^2(\Omega)}\\
&\quad + \sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|^3_{L^2(\Omega)}\bigg)\bigg)\\
&\le \mathbb{C}(\|f(t)\|_{L^2(\Omega)} + \|f(t)\|^2_{L^2(\Omega)} + \|f(t)\|^4_{L^2(\Omega)} + \|f(t)\|^6_{L^2(\Omega)}\\
&\quad + \|\partial_t f(t)\|^2_{L^2(\Omega)} + \|u_D(t)\|_{H^{1/2}(\partial\Omega)} + \|u_D(t)\|^2_{H^{1/2}(\partial\Omega)}\\
&\quad + \|u_D(t)\|^4_{H^{3/2}(\partial\Omega)} + \|u_D(t)\|^6_{H^{3/2}(\partial\Omega)} + \|\partial_t u_D(t)\|^2_{H^{1/2}(\partial\Omega)}\\
&\quad + \|\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)} + \|\partial_t\theta_D(t)\|^2_{H^{1/2}(\partial\Omega)}).
\end{aligned} \tag{4.4}$$

Additionally, $(1.1)_1$ and (1.2) provide us with the estimate

$$\mathcal{E}(u(0)) \le C(\|\nabla_x\theta_0\|^2_{L^2(\Omega)} + \|f|_{t=0}\|^2_{L^2(\Omega)} + \|u_D|_{t=0}\|^2_{H^{1/2}(\partial\Omega)}). \tag{4.5}$$

Moreover, from interpolation inequality and Young's inequality it follows that

$$\|f\|^4_{L^4(\mathbb{R}_+;L^2(\Omega))} \le C\|f\|_{L^2(\mathbb{R}_+;L^2(\Omega))}\|f\|^3_{L^6(\mathbb{R}_+;L^2(\Omega))} \le C(\|f\|^2_{L^2(\mathbb{R}_+;L^2(\Omega))} + \|f\|^6_{L^6(\mathbb{R}_+;L^2(\Omega))}). \tag{4.6}$$

Integration of (4.4) over time and application of (4.5) and (4.6) gives us

$$\begin{aligned} \mathcal{E}(u(t)) &+ \frac{1}{2}\|\theta(t)\|^2_{H^1(\Omega)} \\ &+ \frac{1}{2}\int_0^t \mathcal{E}(\partial_t u(\tau))\,d\tau + \int_0^t \|\theta(\tau)\|^2_{L^2(\Omega)}\,d\tau \\ &\cdot \left(\frac{1}{4} - C_1\|\theta_D\|_{W^{1,\infty}(\mathbb{R}_+;H^{1/2}(\partial\Omega))} - C_2 \sup_{0\le\tau\le t}\|\theta(\tau)\|^2_{L^2(\Omega)}\right) \\ &+ \int_0^t \|\partial_t\theta(\tau)\|^2_{L^2(\Omega)}\,d\tau \\ &\cdot \left(\frac{1}{4} - C_3\left(\sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|_{L^2(\Omega)} + \sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|^3_{L^2(\Omega)}\right)\right) \\ \le \mathbb{C}(&\|\theta_0\|^2_{H^1(\Omega)} + \|f|_{t=0}\|^2_{L^2(\Omega)} + \|u_D|_{t=0}\|^2_{H^{1/2}(\partial\Omega)} \\ &+ \|f\|_{L^1(\mathbb{R}_+;L^2(\Omega))} + \|f\|^2_{H^1(\mathbb{R}_+;L^2(\Omega))} \\ &+ \|f\|^6_{L^6(\mathbb{R}_+;L^2(\Omega))} + \|u_D\|_{L^1(\mathbb{R}_+;H^{1/2}(\partial\Omega))} \\ &+ \|u_D\|^2_{H^1(\mathbb{R}_+;H^{1/2}(\partial\Omega))} + \|u_D\|^4_{L^4(\mathbb{R}_+;H^{3/2}(\partial\Omega))} \\ &+ \|u_D\|^6_{L^6(\mathbb{R}_+;H^{3/2}(\partial\Omega))} + \|\theta_D\|^2_{H^1(\mathbb{R}_+;H^{1/2}(\partial\Omega))}) \end{aligned} \tag{4.7}$$

for all $t \in [0, T^\star)$, where $C_1, C_2, C_3, \mathbb{C} > 0$ are constants dependent only on the geometry of Ω and the coefficients of the operator $\mathcal{D}$. First we can see that the function θ_D should satisfy

$$\|\theta_D\|_{W^{1,\infty}(\mathbb{R}_+;H^{1/2}(\partial\Omega))} \le \delta < \frac{1}{4C_1} \tag{4.8}$$

for some $\delta \ge 0$. If the initial data θ_0 fulfils

$$\|\theta_0\|^2_{L^2(\Omega)} < \frac{1 - 4C_1\delta}{C_2}, \tag{4.9}$$

$$\|\nabla_x\theta_0\|_{L^2(\Omega)} + \|\nabla_x\theta_0\|^3_{L^2(\Omega)} < \frac{1}{4C_3}, \tag{4.10}$$

then there exists such a small t that

$$\frac{1}{4} - C_1\|\theta_D\|_{W^{1,\infty}(\mathbb{R}_+;H^{1/2}(\partial\Omega))} - C_2 \sup_{0\le\tau\le t}\|\theta(\tau)\|^2_{L^2(\Omega)} > 0,$$
$$\frac{1}{5} - C_3\left(\sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|_{L^2(\Omega)} + \sup_{0\le\tau\le t}\|\nabla_x\theta(\tau)\|^3_{L^2(\Omega)}\right) > 0$$

and left hand side of (4.7) is greater than zero. Next we note that the right hand side of (4.7) does not depend on the time variable t thus it controls supremum over t of the left hand side. Additionally, we denote by RHS right hand side of (4.7) and if it holds that

$$\text{RHS} < \left(\frac{1-4C_1\delta}{C_2}\right), \tag{4.11}$$

$$\text{RHS} < \min\left\{\left(\frac{1}{8C_3}\right)^2, \left(\frac{1}{8C_3}\right)^{2/3}\right\}, \tag{4.12}$$

then there does not occur finite a time blow up of the $H^1(\Omega)$ norm of the solution to (1.1). $\square$

5. Appendix. Auxiliary problem

As it was previously mentioned in this section we solve the auxiliary linear problem from thermoelasticity.

$$\begin{aligned} -\mathrm{div}_x\mathcal{D}(\varepsilon(u(t,x))) &= -\nabla_x\theta(t,x), \\ \partial_t\theta(t,x) - \Delta\theta(t,x) &= -\mathrm{div}_x\partial_t u(t,x) + h(t,x) \\ u(t,x)|_{\partial\Omega} &= 0, \\ \theta(t,x)_{\partial\Omega} &= 0, \\ \theta(0,x) &= \theta_0^\star(x). \end{aligned} \tag{5.1}$$

Theorem 5.1. *Let $h \in L^2((0,T)\times\Omega)$ and $\theta^\star \in H_0^1(\Omega)$. Then there exists a unique solution (u,θ) of the problem* (5.1) *belonging to $H^1(0,T;H_0^1(\Omega))$ and $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, respectively. Moreover, the following estimates hold*:

$$\begin{aligned} \sup_{0\le t\le T}\|\nabla_x u(t)\|^2_{L^2(\Omega)} + \sup_{0\le t\le T}\|\theta(t)\|^2_{L^2(\Omega)} + \|\nabla_x\theta\|^2_{L^2(0,T;L^2(\Omega))} \\ \le C(\|h\|^2_{L^2(0,T;L^2(\Omega))} + \|\theta_0^\star\|^2_{L^2(\Omega)}), \end{aligned}$$

$$\begin{aligned} \|\nabla_x\partial_t u\|^2_{L^2(0,T;L^2(\Omega))} + \|\partial_t\theta\|^2_{L^2(0,T;L^2(\Omega))} + \sup_{0\le t\le T}\|\nabla_x\theta(t)\|^2_{L^2(\Omega)} \\ \le C(\|h\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)}). \end{aligned}$$

To prove the theorem above we use the Galerkin approximation. As usual let $\{v_k\}_{k=1}^\infty$ be an orthogonal basis in $H_0^1(\Omega,\mathbb{R}^3)$ and an orthonormal basis in $L^2(\Omega,\mathbb{R}^3)$. Let $\{w_k\}_{k=1}^\infty$ be an orthogonal basis in $H_0^1(\Omega,\mathbb{R})$ and an orthonormal basis in $L^2(\Omega,\mathbb{R})$. For all natural numbers m we look for $u_m\colon[0,T]\to H_0^1(\Omega,\mathbb{R}^3)$ and $\theta_m\colon[0,T]\to H_0^1(\Omega,\mathbb{R})$ in the form

$$u_m(t) = \sum_{k=1}^m c_m^k(t)v_k, \qquad \theta_m(t) = \sum_{k=1}^m d_m^k(t)w_k$$

such that, for all $k = 1, \dots, m$,

$$\int_\Omega \mathcal{D}(\varepsilon(u_m(t)))\varepsilon(v_k)\,dx = -\int_\Omega \nabla_x\theta_m(t)v_k\,dx, \tag{5.2}$$

$$\int_\Omega \partial_t\theta_m(t)w_k dx + \int_\Omega \nabla_x\theta_m(t)\nabla_x w_k\,dx = \int_\Omega \partial_t u_m(t)\nabla_x w_k\,dx + \int_\Omega h(t)w_k\,dx, \tag{5.3}$$

$$d_k^m(0) = \int_\Omega \theta_0^\star w_k\,dx. \tag{5.4}$$

Fakt 5.2. *For each natural m there exist absolutely continuous functions c_m^k and d_m^k such that the functions u_m and θ_m defined above satisfy* (5.2)–(5.4).

Remark 5.3. One observes that from identities above we can compute $c_m^k(0)$ also. Moreover for every m it holds that

$$\begin{aligned}\|\nabla_x u_m(0)\|^2_{L^2(\Omega)} &= \Big\|\sum_{k=1}^m c_m^k(0)\nabla_x v_k\Big\|^2_{L^2(\Omega)}\\ &\le \|\theta_m(0)\|^2_{L^2(\Omega)} = \Big\|\sum_{k=1}^m d_m^k(0)w_k\Big\|^2_{L^2(\Omega)} \le \|\theta_0^\star\|^2_{L^2(\Omega)}\end{aligned}$$

and, analogously,

$$\|\nabla_x\theta_m(0)\|^2_{L^2(\Omega)} \le \|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)}.$$

Lemma 5.4 (Energetic estimates). *The sequences $\{u_m\}_{m=1}^\infty$ and $\{\theta_m\}_{m=1}^\infty$ are uniformly bounded in spaces $L^\infty(0,T;H_0^1(\Omega,\mathbb{R}^3)) \cap H^1(0,T;H_0^1(\Omega,\mathbb{R}^3))$ and $L^\infty(0,T;H_0^1(\Omega,\mathbb{R})) \cap H^1(0,T;L^2(\Omega,\mathbb{R}))$, respectively. Additionally the following estimates hold*:

$$\begin{aligned}\|u_m\|^2_{L^\infty(0,T;H_0^1(\Omega))} &+ \|\theta_m\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_x\theta_m\|^2_{L^2(0,T;L^2(\Omega))}\\ &\le C(\|h\|^2_{L^2(0,T;L^2(\Omega))} + \|\theta_0^\star\|^2_{L^2(\Omega)} + \|u_m(0)\|^2_{H_0^1(\Omega)}),\end{aligned} \tag{5.5}$$

$$\begin{aligned}\|\partial_t u_m\|^2_{L^2(0,T;H_0^1(\Omega))} &+ \|\partial_t\theta_m\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla_x\theta_m\|^2_{L^\infty(0,T;L^2(\Omega))}\\ &\le C(\|h\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)}).\end{aligned} \tag{5.6}$$

Proof. Multiplying (5.2) by $\partial_t c_m^k$ and (5.3) by d_m^k and summing over $k = 1, \dots, m$ gives

$$\int_\Omega \mathcal{D}(\varepsilon(u_m(t)))\varepsilon(\partial_t u_m(t))\,dx = -\int_\Omega \nabla_x\theta_m(t)\partial_t u_m(t)\,dx,$$

$$\int_\Omega \partial_t\theta_m(t)\theta_m(t)\,dx + \int_\Omega \nabla_x\theta_m(t)\nabla_x\theta_m(t)\,dx$$
$$= \int_\Omega \partial_t u_m(t)\nabla_x\theta_m(t)\,dx + \int_\Omega h\theta_m(t)\,dx.$$

We add the identities above and use Remark 1.1 to obtain

$$2\partial_t\mathcal{E}(u_m(t)) + \partial_t\|\theta_m(t)\|^2_{L^2} + 2\|\nabla_x\theta_m(t)\|^2_{L^2} = 2\int_\Omega h\theta_m(t)\,dx. \tag{5.7}$$

Thus integration over time and Young's inequality applied to the right hand side of (5.7) give us

$$\sup_{0\le\tau\le t}\mathcal{E}(u_m(\tau)) + \sup_{0\le\tau\le t}\|\theta_m(\tau)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x\theta_m(\tau)\|^2_{L^2(\Omega)}\,d\tau$$
$$\le C\bigg(\int_0^t \|h(t)\|^2_{L^2(\Omega)}d\tau + \|\theta_0^\star\|^2_{L^2(\Omega)} + \mathcal{E}(u_m(0))\bigg).$$

Using inequality (1.2) from Remark 1.1 and Poincaré inequality gives

$$\sup_{0\le\tau\le t}\|u_m(\tau)\|^2_{H^1_0(\Omega)} + \sup_{0\le\tau\le t}\|\theta_m(\tau)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla_x\theta_m(\tau)\|^2_{L^2(\Omega)}\,d\tau$$
$$\le C\bigg(\int_0^t \|h(t)\|^2_{L^2(\Omega)}\,d\tau + \|\theta_0^\star\|^2_{L^2(\Omega)} + \|u_m(0)\|^2_{H^1_0(\Omega)}\bigg),$$

which proves (5.5). To obtain (5.6) we first differentiate (5.2) and multiply by $\partial_t c^k_m$ and we multiply (5.3) by $\partial_t d^k_m$. As done previously, we sum both identities over $k = 1,\dots,m$ and similarly as in the first part of this proof we obtain

$$\mathcal{E}(\partial_t u_m(t)) + 2\|\partial_t\theta_m(t)\|^2_{L^2(\Omega)} + \partial_t\|\nabla_x\theta_m(t)\|^2_{L^2(\Omega)} = 2\int_\Omega h\partial_t\theta_m(t)\,dx.$$

This immediately results in

$$\int_0^t \|\partial_t u_m(\tau)\|^2_{H^1_0(\Omega)}\,d\tau + \int_0^t \|\partial_t\theta_m(\tau)\|^2_{L^2(\Omega)}d\tau + \sup_{0\le\tau\le t}\|\nabla_x\theta_m(\tau)\|^2_{L^2(\Omega)},$$
$$\le C\bigg(\int_0^t \|h(\tau)\|^2_{L^2(\Omega)}d\tau + \|\nabla_x\theta_0^\star\|^2_{L^2(\Omega)}\bigg).$$

The inequality above ends the proof. □

With the assertion from lemma above we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. We repeat the argumentation of the proof of theorem 2.9 in [3] and theorem 2 in [2]. First we approximate the problem (5.1) by (5.2)–(5.4). Lemma 5.4 gives us the required estimates to obtain weak convergence (weak-$*$ convergence in the case of L^∞ space) of our approximation, which means that there exist subsequences $\{u_{m_j}\}_{j=1}^\infty \subset \{u_m\}_{m=1}^\infty$ and

$\{\theta_{m_j}\}_{j=1}^\infty \subset \{\theta_m\}_{m=1}^\infty$ such that

$$\begin{aligned} u_{m_j} &\rightharpoonup u \quad \text{in } H^1(0,T;H_0^1(\Omega,\mathbb{R}^3)),\\ \theta_{m_j} &\rightharpoonup \theta \quad \text{in } L^\infty(0,T;H_0^1(\Omega,\mathbb{R}))\cap H^1(0,T;L^2(\Omega,\mathbb{R})). \end{aligned}$$

By the Rellich–Kondrachov theorem, the imbedding $H^1((0,T)\times\Omega)\subset L^2((0,T)\times\Omega)$ is compact and so the subsequences $\{u_{m_j}\}_{j=1}^\infty$ and $\{\theta_{m_j}\}_{j=1}^\infty$ are precompact in $L^2((0,T)\times\Omega)$ and it is possible to choose their subsequences which are strongly convergent in $L^2((0,T)\times\Omega)$ to the same limit (u,θ). Obviously functions (u,θ) solve (5.1) in the weak sense and satisfy required estimates.

It still remains to show uniqueness of the solution to (5.1). In order to do this we investigate the difference $(\overline{u},\overline{\theta}) := (u^1,\theta^1)-(u^2,\theta^2)$ where (u^1,θ^1) and (u^2,θ^2) are two different solutions to (5.1). Then similar calculations as in Lemma 5.4 lead us to

$$\begin{gathered} \|\overline{u}\|^2_{L^\infty(0,T;H_0^1(\Omega))} + \|\overline{\theta}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_x\overline{\theta}\|^2_{L^2(0,T;L^2(\Omega))} \le 0,\\ \|\partial_t\overline{u}\|^2_{L^2(0,T;H_0^1(\Omega))} + \|\partial_t\overline{\theta}\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla_x\overline{\theta}\|^2_{L^\infty(0,T;L^2(\Omega))} \le 0, \end{gathered}$$

which completes the proof of the theorem. □

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Leszek Bartczak
Warsaw University of Technology
Faculty of Mathematics and Information Science
pl. Politechniki 1
00-661 Warsaw, POLAND

E-mail address: l.bartczak@mini.pw.edu.pl