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AROUND ULAM'S QUESTION ON RETRACTIONS

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Abstract. It is known that the unit ball in infinitely dimensional Hilbert space can be retracted onto its boundary via a lipschitzian mapping. The magnitude of Lipschitz constant is only roughly estimated. The note contains a number of observations connected to this result and opens some new problems.

1. Introduction

One of the forms of Brouwer's Fixed Point Theorem states that for any $n = 1, 2, \ldots$, the $n-1$ dimensional sphere $S^{n-1} = \partial B^n$ is not the retract of the n-dimensional ball B^n.

There is a question raised around 1935 by S. Ulam which reads: "*Can one transform continuously the solid sphere of a Hilbert space into its boundary such that the transformation should be the identity on the boundary of the ball*". The problem has been included (Problem 36) in the famous collection known as *The Scottish Book*. The history and, probably, the most up to date information about the collection, and the problems contained, can be found in the book by

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D. Mauldin [10]. There is also a remark saying that the affirmative answer has been provided by Tychonoff.

It is not clear how the construction of Tychonoff looked like. Nowadays, the solution to the problem is mostly attributed to Kakutani who in 1943 showed a construction of a fixed point free mappings on the unit ball in l^2 and the way to construct the mapping with desired property (see [8]).

In the presently used terminology the Ulam's question should be reformulated. Let H be an infinitely dimensional Hilbert space with the unit ball B and the unit sphere S. *Is S the retract of B*? Due to Kakutani's result, the answer is "YES". However during years some new quantitative questions appeared closely related to this subject.

In 1983, Benyamini and Sternfeld [3] have proved that for any Banach space X of infinite dimension, there exists a mapping (a retraction) $R: B \to S$, such that $Rx = x$ for all $x \in S$ and such that $R \in \mathcal{L}(k)$. The last means that R satisfies for all $x, y \in B$ the Lipschitz condition

$$\|Rx - Ry\| \leq k\|x - y\|$$

with sufficiently large constant k. It opened (see [5]) a new direction of investigations called *the optimal retraction problem.*

Definition 1.1. For any infinitely dimensional Banach space X

$$k_0(X) = \inf [\, k : \text{there exists a retraction } R: B \to S,\ R \in \mathcal{L}(k)\,].$$

In spite of efforts of a number of researchers, the exact value of $k_0(X)$ is unknown for any space. More about this can be found in books [7], [4], [9] and the recent survey article [6]. The best known estimates are for the space l^1 and the space $C_0[0,1]$ of continuous functions vanishing at 0. (see [11], [1]),

$$4 \leq k_0(l^1) \leq 8, \qquad 3 \leq k_0(C_0[0,1]) \leq 2(2+\sqrt{2}) = 6.83\ldots$$

The Hilbert space case is the most resistant for finding good estimates. The published estimates for $k_0 = k_0(H)$ (see books and [2]) are

$$4.5\ldots \leq k_0(H) \leq 28.99\ldots,$$

so, the gap is large.

The aim of this note is to refresh the interest in the subject by proposing a new approach to study the mentioned case, presenting some basic observations and formulating some problems.

2. Basic tools

Let H be a Hilbert space with the scalar product $\langle \cdot\,, \cdot \rangle$, the unit ball B and the unit sphere S. To neglect the trivial case we assume that $\dim H \geq 2$. Fix any point $e \in S$ and let the subspace $E \subset H$ be the orthogonal complement of the one dimensional subspace spanned by e. Any point $x \in H$ can be uniquely represented as $x = (u, s)$ where $u \in E$ and $s = \langle x, e \rangle$ and obviously $\|x\|^2 = \|u\|^2 + s^2$. For any $t \in [-1, 1]$ let us accept the following terminology and notations:

- *the parallel hyperplanes* are $E_t = E + te$,
- *the parallel ball sections* are $B_t = E_t \cap B$,
- *the lenses* cut from B by E_t are

$$D_t = \{x \in B : x = (u, s),\ s \geq t\} = \{x \in B : \langle x, e \rangle \geq t\},$$

- *the spherical cups* cut by E_t are $S_t = D_t \cap S$,
- *We shall call the set $D \subset B$ a lipschtzian retract of B if there exists a mapping* (*a retraction*) *$R \colon B \to D$ such that $R \in L(k)$, for certain k and $Rx = x$ for all $x \in D$.*

The ball sections and lenses are closed convex sets, the spherical cups are closed but not convex if $t \neq 1$. If $\dim H < \infty$, then each ball section B_t is isometric to the $n-1$-dimensional ball of radius $\sqrt{1 - t^2}$ and in case of $\dim H = \infty$, B_t is isometric to the ball of the same radius in H.

Let us recall that for any closed and convex set $C \subset H$, there exists a mapping, *the nearest point projection*, $P_C \colon H \to C$, satisfying for all $x \in H$,

$$\|x - P_C x\| = \inf [\, \|x - z\| : z \in C\,].$$

The mapping P_C is *nonexpansive* meaning that for all $x, y \in H$

$$\|P_C x - P_C y\| \leq \|x - y\|.$$

For the unit ball B

$$P_B x = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \dfrac{x}{\|x\|} & \text{if } \|x\| \geq 1. \end{cases}$$

So, the nearest point projection on B coincides outside of B with the radial mapping $U \colon H \setminus \{0\} \to S$ defined as

$$Ux = \frac{x}{\|x\|}.$$

For U and for all $x, y \in H$, if $r \leq \|x\| \leq 1$, $r \leq \|y\| \leq 1$ then

$$\|Ux - Uy\| = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| P_B \frac{x}{r} - P_B \frac{y}{r} \right\| \leq \frac{1}{r} \|x - y\|.$$

Finally, in our constructions we shall use often the following fact (see [12]). Suppose, γ is a rectifiable curve laying on S, $\gamma \subset S$ with end points x, y, Then the *length* $l(\gamma)$ exceeds the *angle between* x and y,

$$l(\gamma) \geq \alpha(x, y) = \arccos \langle x, y \rangle.$$

3. Observations

The first observation is for both, finite and infinite case is,

Observation 3.1. *For any $t \in (-1, 1]$, S_t is the lipschitzian retract of B.*

In the proof we leave details of calculations to the reader.

Proof. Fix t as above. Observe first that D_t is the nonexpansive retract of B and the nearest point retraction has the form,

$$P_{D_t}x = P_{D_t}(u, s) = \begin{cases} (u, t) & \text{if } s \leq t \text{ and } \|u\| \leq \sqrt{1-t^2}, \\ \left(\sqrt{1-t^2}\dfrac{u}{\|u\|}, t\right) & \text{if } t \geq 0,\ -t \leq s \leq t \\ & \text{and } \|u\| \geq \sqrt{1-t^2}, \\ (u, s) & \text{if } s \geq t. \end{cases}$$

If $t < 0$, the second row in the above is not needed, the complete formula is given by the first and the third.

Consider the open cone

$$C_t = \left\{ x = (u, s) : \frac{\|u\|}{1-s} < \frac{\sqrt{1-t^2}}{1-t} = \sqrt{\frac{1+t}{1-t}} \right\}$$

and the retraction R_t of D_t onto $D_t \setminus C_t$ given by

$$R_t x = R_t(u, s) = \begin{cases} \left(u, 1 - \|u\|\sqrt{\dfrac{1-t}{1+t}}\right) & \text{if } (u, s) \in C_t, \\ (u, s) & \text{if } (u, s) \in D_t \setminus C_t. \end{cases}$$

One can easily check that for all $x, y \in D_t$ we have

$$\|R_t x - R_t y\| \leq \sqrt{\frac{2}{1+t}}\, \|x - y\|.$$

In other words $R_t \in \mathcal{L}(\sqrt{2/(1+t)})$.

The next step is to observe that, for any $x = (u, s) \in D_t \setminus C_t$, $\|x\| = \sqrt{\|u\|^2 + s^2} \geq \sqrt{(1+t)/2}$. Thus the radial mapping $Ux = x/\|x\|$ retracts $D_t \setminus C_t$ onto S_t and is of class $\mathcal{L}(\sqrt{2/(1+t)})$. Finally composing three mappings we get the retraction $R = U \circ R_t \circ P_{D_t} : B \to S_t$ which is lipschitzian of class $\mathcal{L}(2/(1+t))$. $\square$

Remark 3.2. If $\dim H = \infty$, then, due to mentioned result of Benyamini and Sternfeld, also $S_{-1} = S$ is the lipschitzian retract of B and for any $\varepsilon > 0$ there exists a retraction $R\colon B \to S$ of class $\mathcal{L}(k_0 + \varepsilon)$.

Let us introduce a function measuring minimal Lipschitz constants for retractions on S_t.

Definition 3.3. Let for $t \in [-1, 1]$

$$\kappa(t) = \inf\{\, k : \text{there exists a retraction } R_t\colon B \to S_t \text{ of class } \mathcal{L}(k)\,\}.$$

The basic properties of κ, common for the finite and infinite dimensional case, are the following:

- $\kappa(1) = 0$, but $\lim_{t\to 1^-} \kappa(t) = 1$,
- $1 < \kappa(t) \le 2/(1+t)$ for $t \in (-1, 1)$.

Both are easily justified by the facts presented above and the proof of the Observation 3.1. The differences appear if we consider the behavior of κ in the vicinity of $t = -1$.

Observation 3.4. *If* $\dim H < \infty$, *then* $\lim_{t\to -1} \kappa(t) = +\infty$.

Proof. Suppose the contrary. Then there exists a sequence of retractions $R_{t_n}\colon B \to S_{t_n}$ with $t_n \to -1$ such that $R_{t_n} \in \mathcal{L}(k)$ with a common value of k. Due to Arzelà Theorem, since all the mappings are equicontinuous, the sequence must contain an uniformly convergent subsequence. The limit of it would be a retraction $R\colon B \to S$ and we have a contradiction with finite dimensionality of H. $\square$

The more precise estimate is

Observation 3.5. *If* $\dim H < \infty$, *then*

$$\kappa(t) \ge \begin{cases} \dfrac{\arccos t}{\sqrt{1-t^2}} & \text{if } 1 > t \ge 0, \\[2ex] \dfrac{\pi - \arccos|t|}{\sqrt{1-t^2}} & \text{if } t \le 0. \end{cases}$$

Proof. Consider the case $t < 0$. For any $x = (u, t) \in B_t$ there is a point $y = (v, t) \in B_t \cap S$ such that $\|x - y\| \le \sqrt{1-t^2}$. Let $R_t\colon B \to S_t$ be a retraction of class $\mathcal{L}(k)$. The segment with end points x and y, $I = [x, y]$ is mapped by R_t onto a lipschtzian, so rectifiable, curve $\gamma = R_t(I)$. The length of γ satisfies $l(\gamma) \le k\sqrt{1-t^2}$. If $k\sqrt{1-t^2} < \pi - \arccos|t|$, then the image $R_t(B_t)$ does not cover a vicinity of e, there exists $\varepsilon > 0$, such that $\operatorname{dist}(e, R_t(B_t)) > \varepsilon$. Consequently, if $P_t = P_{E_t}$ is the nearest point (orthogonal) projection, then

$P \circ R_t(B_t)$ does not contain the center of B_t, the point $(0,t)$. Finally, if the above holds the mapping $Q\colon B_t \to B_t \cap S$ defined by

$$Qx = Q(u,t) = \left(\sqrt{1-t^2}\,\frac{P \circ R_t(u,t) - (0,t)}{\|P \circ R_t(u,t) - (0,t)\|}, t\right)$$

would be a retraction of the $(n-1)$-dimensional ball B_t, onto its boundary $(n-2)-$dimensional sphere $B_t \cap S$ which is a contradiction. The case $t \geq 0$ is proved the same way. □

For example for the halfsphere S_0 we get $\pi/2 \leq \kappa(0) \leq 2$.

Observation 3.6. *If we replace the parameter t by the angle $\alpha \in [0,\pi]$ with the natural interpretation $\cos\alpha = t$, our estimates can be reformulated as*

$$\frac{\alpha}{\sin\alpha} \leq \kappa(t) = \kappa(\cos\alpha) \leq \frac{2}{1+\cos\alpha}.$$

Observation 3.7. *The product $(1+t)\kappa(t)$ is nondecreasing for $t \in (-1,1)$.*

Proof. Fix $t \in (-1,1)$. Take $0 < \varepsilon < 1$ assuming only that if $t < 0$ that also $t + \varepsilon < 0$. Consider the ball $B(-\varepsilon e, r)$ where the radius r is chosen so that the unit ball B and $B(-\varepsilon e, r)$ have the same cross-section by E_t, $B_t = E_t \cap B = E_t \cap B(-\varepsilon e, r)$. Calculations show that $r = \sqrt{1 + 2t\varepsilon + \varepsilon^2}$. Similarly as for the unit ball we observe that there exists a lipschitzian retraction $\widetilde{R}$ of $B(-\varepsilon e, r)$ onto its spherical cup $S(-\varepsilon e, r) \cap B$. Moreover, such retraction can be constructed to have Lipschitz constant close to $\kappa(s)$ where

$$s = \frac{t+\varepsilon}{r} = \frac{t+\varepsilon}{\sqrt{1+2t\varepsilon+\varepsilon^2}}.$$

Since all the points $x \in B \setminus B(-\varepsilon e, r)$ satisfy $\|x\| \geq r - \varepsilon$, composing $\widetilde{R}$ with the radial mapping U we get a retraction $R = U \circ \widetilde{R}\colon B \to S_t$. Consequently, the Lipschitz constant of R estimates by the product of the Lipschitz constants of $\widetilde{R}$ and U and we get

$$\kappa(t) \leq \kappa(s)\frac{1}{r-\varepsilon} = \kappa\left(\frac{t+\varepsilon}{r}\right)\frac{1}{r-\varepsilon}.$$

Subtracting from both sides $\kappa(s)$ and dividing by $t - s < 0$ we get

$$\frac{\kappa(t)-\kappa(s)}{t-s} \geq \kappa(s)\,\frac{1-r+\varepsilon}{(r-\varepsilon)(t-s)}.$$

We leave to the reader passing to the limit with $\varepsilon \to 0$, $s \to t$ and the conclusion

$$\kappa'(t) \geq \kappa(t)\lim_{\varepsilon\to 0}\frac{1-r+\varepsilon}{(r-\varepsilon)(t-s)} = -\frac{\kappa(t)}{1+t}.$$

The above leads to $\kappa'(t)(1+t) + \kappa(t) = ((1+t)\kappa(t))' \geq 0$, which ends the proof. □

Remark 3.8. In the proof, the function κ has been treated as being continuous. Also the derivative κ' can be considered only as "right upper". Nevertheless the tricks used in the proof can be used to prove that the assumption of continuity is justified. We leave the technical details to the reader.

The above slightly improves the estimate from Observation 3.1. For $t \leq 0$ we have

$$\kappa(t) \leq \frac{\kappa(0)}{1+t} \leq \frac{2}{1+t}.$$

Observation 3.9. *If* $\dim H = \infty$, *then* $\kappa(t)$ *is bounded on* $[-1,1]$.

Proof. First observe that $\kappa(-1) = k_0$. It is enough to prove that $\kappa(t)$ is bounded in a vicinity of $t = -1$. Let us consider only $t \in (-1, -1/2)$. Let $R\colon B \to S$ be the retraction of class $\mathcal{L}(k)$, $k > k_0(H)$ and P_{D_t} be the nearest point projection of B onto D_t. So, the composition $P_{D_t} \circ R$ maps B onto $B_t \cup S_t$ keeping all the points in S_t fixed. Since B_t is isometric to the ball of radius $\sqrt{1-t^2}$, there exists a lipschitzian retraction R^* of B_t into its sphere $B_t \cap S$. Again, we can select R^* to be of class $\mathcal{L}(k)$. Let $Q\colon B_t \cup S_t \to S_t$ be defined as

$$Qx = Q(u,s) = \begin{cases} (R^*u, t) & \text{if } s = t, \\ (u,s) & \text{if } t < s. \end{cases}$$

Now, the composition $R_t = Q \circ P_{D_t} \circ R$ retracts B onto S_t. The composition $P_{D_t} \circ R$ is of class $\mathcal{L}(k)$ on B. Since Q acts on the nonconvex set $B_t \cup S_t$ the Lipschitz constant of Q must be evaluated. Let $x, y \in B_t \cup S_t$, $x = (u, s_1)$, $y = (v, s_2)$. Four cases should be taken into account. If $s_1 = s_2 = t$ then we have

$$\|Qx - Qy\| = \|R^*u - R^*v\| \leq k\|u - v\| = k\|x - y\|.$$

Obviously if both s_1, s_2 exceed t, $\|Qx - Qy\| = \|x - y\|$. Suppose $s_1 = t$. Two cases remain. If $t < s_2 < |t|$ then the nearest point to y in B_t is

$$z = (w,t) = P_{B_t} y = P_{B_t}(y, s_2) = \left(\sqrt{1-t^2}\,\frac{v}{\|v\|}, t\right) = R^*(w,t) = R^*z.$$

Now we have

$$\begin{aligned} \|Qx - Qy\| &\leq \|Qx - z\| + \|z - y\| = \|R^*x - R^*z\| + \|z - y\| \\ &\leq k\|x - z\| + \|z - y\| \leq (k+1)\|x - y\|. \end{aligned}$$

Finally, if $s_1 = t$, $s_2 \geq |t|$ since $\|x - y\| \geq 2|t|$ we get

$$\|Qx - Qy\| \leq 2 \leq 2\,\frac{\|x - y\|}{2|t|} \leq 2\,\|x - y\|.$$

Hence, in general we have $\|Qx - Qy\| \le (k+1)\|x-y\|$ and consequently

$$\begin{aligned}\|R_t x - R_t y\| &= \|Q \circ P_{D_t} \circ Rx - Q \circ P_{D_t} \circ Ry\| \\ &\le (k+1)\|P_{D_t} \circ Rx - P_{D_t} \circ Ry\| \le (k+1)k\|x-y\|.\end{aligned}$$

Because k can be taken close to $k_0 = k_0(H)$ and for all $t \ge -1/2$ we have the estimate $\kappa(t) \le 2/(1+t) \le 4 < k(k+1)$ we get the conclusion

$$\kappa(t) \le \min\left[\frac{2}{1+t}, k_0(k_0+1)\right]. \qquad \square$$

The above estimate is probably very imprecise. The exact formula for $\kappa(t)$ is a challenge. Especially, because there is a surprising evaluation from below. Let us begin with

Definition 3.10. $k_0^+ = \sup[\,\kappa(t) : -1 < t \le 1\,]$.

It occurs that $k_0^+ \ge k_0$ and moreover,

Observation 3.11. *If* $\dim H = \infty$, *then there exists* $-1 < a < 0$ *such that* $\kappa(t) \ge k_0$ *for all* $-1 < t \le a$.

Proof. Let $R_t\colon B \to S_t$, $t < 0$ be a retraction of class $\mathcal{L}(k)$. Consider R_t only as a mapping acting on B_t into S_t.As noticed in Observation 3.5, for any $x \in B_t$, $R_t x$ is the end point of a curve γ with the initial point $y \in B_t \cap S$ and of length $l(\gamma) \le k\sqrt{1-t^2}$. If t is sufficiently close to -1 the curve γ is contained in the part of S_t contained between two hyperplanes E_t and $E_{|t|}$. Indeed, it is enough, as a first estimate, to require that $k\sqrt{1-t^2} \le 2|t|$. This part of the sphere S is mapped by the nearest point projection P_{B_t} onto its, relative to E_t, boundary $B_t \cap S$. Thus the composition $Q = P_{B_t} \circ R_t\colon B_t \to B_t \cap S$ is a retraction of class $\mathcal{L}(k)$. Hence, $k_0^+ \ge k \ge k_0$. $\square$

In view of Observation 3.1, S_t are lipschitzian retractions of B. This has been proved in the easy and elementary way. The proof of Benyamini–Sterfeld result is much more technically complicated and advanced. The above Observation 3.9, indicates that in spite of this, the attempts of finding optimal retraction (having smallest possible Lipschitz constant) on S_t, for t close to -1 meet at least the same difficulties as in case of the whole S. It is also worth to notice that the slight modification of the function κ brings a different effect. Let

$$\kappa^*(t) = \inf[\,k : \text{there exists a mapping } T\colon B \to S \text{ of class } \mathcal{L}(k) \text{ with } \operatorname{Fix} T \supset S_t\,].$$

Then of course $\kappa^*(t) \le k_0$ and $\kappa^*(-1) = k_0$.

The value of a mentioned in Observation 3.9 can be (roughly) estimated.

OBSERVATION 3.12. *Let* $-1 < t < 0$. *Let* $\alpha(t) \in [0, \pi/2]$ *be the angle such that* $\sin\alpha = \sqrt{1-t^2}$. *The length of a shortest rectifiable curve* γ *contained in* S *and joining points belonging to* $B_t \cap S$ *and* $B_{|t|} \cap S$ *satisfies* $l(\gamma) \leq 2(\pi/2 - \alpha) = \pi - 2\alpha$. *Following the reasoning from Observation 3.11 we see that* $\kappa(t) \geq k_0$ *for all* t, *satisfying*

$$k_0 \leq k_0^+ \leq \frac{\pi - 2\alpha(t)}{\sin\alpha(t)} = \frac{\pi - 2\arcsin\sqrt{1-t^2}}{\sqrt{1-t^2}},$$

So, a can be estimated as follows:

OBSERVATION 3.13. *Let* $\alpha \in [0, \pi/2]$ *be the maximal angle for which the above inequality holds. Take* a *to satisfy* $\sin\alpha = \sqrt{1-a^2}$, $a = -\cos\alpha$ *and*

$$k_0 \sin\alpha = \pi - 2\alpha,$$

Replacing $\sin\alpha$ *by* α *we get*

$$\alpha \geq \frac{\pi}{k_0+2} \quad \textit{and} \quad a \geq -\cos\frac{\pi}{k_0+2}.$$

4. Conclusion

As declared, we presented here some problems. In spite of relatively elementary formulation, they require more precise investigations. In our opinion the following open questions can be raised:

- What is the precise formula for $\kappa(t)$ in both cases $\dim H < \infty$ and $\dim H = \infty$?
- Is $\kappa(t) = \text{const} = k_0 = k_0^+$ in the vicinity of -1?
- How far from -1 is the value $a = \sup[t : \kappa(t) \geq k_0]$?
- Can the properties of $\kappa(t)$ help with finding, a simpler that original, proof of Benyamini–Sternfeld Theorem for Hilbert space?

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