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ON THE STABILITY OF NEW IMPULSIVE ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study new impulsive ordinary differential equations and apply fixed point approach to establish existence and uniqueness theorem and derive an interesting stability result in the sense of generalized β-Ulam–Hyers–Rassias. At last, two examples are given to demonstrate the applicability of our result.

1. Introduction

Throughout this paper, $J = [0,T]$ and $C(J,\mathbb{R})$ denotes the space of all continuous functions from J into $\mathbb{R}$. Denote $PC(J,\mathbb{R}) := \{x\colon J \to \mathbb{R} : x \in C((t_k,t_{k+1}],\mathbb{R}),\ k = 0,1,\ldots,m$ and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 1,\ldots,m$, with $x(t_k^-) = x(t_k)\}$.

In addition to impulsive differential equations which the impulses are instantaneous, a new class of impulsive differential equations which the impulses are not instantaneous was reported in [9], [17]. It follows [9], [17], a function

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$x \in V := PC(J,\mathbb{R}) \bigcap_{i=0}^{m} C^1((s_i,t_{i+1}],\mathbb{R})$ is called a classical solution of the following modified impulsive differential equations:

$$
\begin{cases} x'(t) = f(t,x(t)), & t \in (s_i,t_{i+1}],\ i = 0,1,\ldots,m, \\ x(t) = g_i(t,x(t_i^+)), & t \in (t_i,s_i],\ i = 1,\ldots,m, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}
$$

if x satisfies $x(0) = x_0$, $x(t) = g_i(t,x(t_i^+))$, $t \in (t_i,s_i]$, $i = 1,\ldots,m$, and

$$
\begin{aligned} x(t) &= x_0 + \int_0^t f(s,x(s))\,ds, && t \in [0,t_1], \\ x(t) &= g_i(s_i,x(t_i^+)) + \int_{s_i}^t f(s,x(s))\,ds, && t \in (s_i,t_{i+1}],\ i = 1,\ldots,m. \end{aligned}
$$

In the present paper, we mainly apply fixed point approach to study a new stability of the following modified impulsive differential equations:

$$
\begin{cases} x'(t) = f(t,x(t)), & t \in (s_i,t_{i+1}],\ i = 0,1,\ldots,m, \\ x(t) = g_i(t,x(t_i^+)), & t \in (t_i,s_i],\ i = 1,\ldots,m, \end{cases} \tag{1.2}
$$

where $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \ldots < t_m \le s_m < t_{m+1} = T$ are pre-fixed numbers, $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_i\colon [t_i,s_i] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $i = 1,\ldots,m$, which is not instantaneous impulses.

Firstly, we introduce a generalized β-Ulam–Hyers–Rassias stability concept for the equation (1.2) which is partly motivated by the concepts of stability in [21], [23].

Let $0 < \beta \le 1$, $\psi \ge 0$ and $\varphi \in PC(J,\mathbb{R}_+)$ is nondecreasing. Consider

$$
\begin{cases} |y'(t) - f(t,y(t))| \le \varphi(t), & t \in (s_i,t_{i+1}],\ i = 0,1,\ldots,m, \\ |y(t) - g_i(t,y(t_i^+))| \le \psi, & t \in (t_i,s_i],\ i = 1,\ldots,m. \end{cases} \tag{1.3}
$$

Definition 1.1. The equation (1.2) is generalized β-Ulam–Hyers–Rassias stable with respect to (φ,ψ) if there exists $c_{f,\beta,g_i,\varphi} > 0$ such that for each solution $y \in V$ of the inequality (1.3) there exists a solution $x \in V$ of the equation (1.2) with

$$
|y(t) - x(t)|^\beta \le c_{f,\beta,g_i,\varphi}(\psi^\beta + \varphi^\beta(t)), \quad t \in J.
$$

Remark 1.2. A function $y \in V$ is a solution of the inequality (1.3) if and only if there is $G \in \bigcap_{i=0}^{m} C^1((s_i,t_{i+1}],\mathbb{R})$ and $g \in \bigcap_{i=1}^{m} C([t_i,s_i],\mathbb{R})$ (which depend on y) such that:

(a) $|G(t)| \le \varphi(t)$, $t \in \bigcup_{i=0}^{m} (s_i,t_{i+1}]$ and $|g(t)| \le \psi$, $t \in \bigcup_{i=1}^{m} (t_i,s_i]$;
(b) $y'(t) = f(t,y(t)) + G(t)$, $t \in (s_i,t_{i+1}]$, $i = 0,1,\ldots,m$;
(c) $y(t) = g_i(t,y(t_i^+)) + g(t)$, $t \in (t_i,s_i]$, $i = 1,\ldots,m$.

Remark 1.3. Obviously, if $y \in V$ is a solution of the inequality (1.3) then y is a solution of the following integral inequality:

$$\begin{cases} |y(t) - g_i(t, y(t_i^+))| \le \psi, & t \in (t_i, s_i],\ i = 1, \dots, m; \\ \left| y(t) - y(0) - \int_0^t f(s, y(s))\, ds \right| \le \int_0^t \varphi(s)\, ds, & t \in [0, t_1]; \\ \left| y(t) - g_i(s_i, y(t_i^+)) - \int_{s_i}^t f(s, y(s))\, ds \right| \le \psi + \int_{s_i}^t \varphi(s)\, ds, & \\ & t \in (s_i, t_{i+1}],\ i = 1, \dots, m. \end{cases} \tag{1.4}$$

Secondly, we use the same idea and extend to study generalized β-Ulam–Hyers–Rassias stability of the equation

$$\begin{cases} x'(t) = \lambda x(t) + f(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, \dots, m,\ \lambda > 0, \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \dots, m. \end{cases} \tag{1.5}$$

Definition 1.4. The equation (1.5) is generalized β-Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,\beta,g_i,\varphi} > 0$ such that for each solution $y \in V$ of the inequality

$$\begin{cases} |y'(t) - \lambda y(t) - f(t, y(t))| \le \varphi(t), & t \in (s_i, t_{i+1}],\ i = 0, 1, \dots, m, \\ |y(t) - g_i(t, y(t))| \le \psi, & t \in (t_i, s_i],\ i = 1, \dots, m, \end{cases} \tag{1.6}$$

there exists a solution $x \in V$ of the equation (1.5) with

$$|y(t) - x(t)|^\beta \le c_{f,\beta,g_i,\varphi}(\psi^\beta + \varphi^\beta(t)), \quad t \in J.$$

Just like Remark 1.3, if $y \in V$ is a solution of the inequality (1.6) then y is a solution of the following integral inequality

$$\begin{cases} |y(t) - g_i(t, y(t))| \le \psi, & t \in (t_i, s_i],\ i = 1, \dots, m; \\ \left| y(t) - e^{\lambda t} y(0) - \int_0^t e^{\lambda(t-s)} f(s, y(s)) ds \right| \le \int_0^t e^{\lambda(t-s)} \varphi(s)\, ds, & \\ & t \in [0, t_1]; \\ \left| y(t) - e^{\lambda(t-s_i)} g_i(s_i, y(s_i)) - \int_{s_i}^t e^{\lambda(t-s)} f(s, y(s))\, ds \right| & \\ \quad \le e^{\lambda(t-s_i)} \psi + \int_{s_i}^t e^{\lambda(t-s)} \varphi(s)\, ds, \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m. & \end{cases} \tag{1.7}$$

For more recent results on Ulam's type stability, the readers can refer to the monographs of [5], [10]–[13], [19], and other works [1], [3], [4], [6], [8], [15], [16], [18], [20], [22], [24] by using fixed point approach and classical analysis methods.

The rest of this paper is organized as follows. In Section 2, Banach fixed-point theorem for generalized complete metric spaces is used to derive existence and uniqueness of stable solution for the equation (1.2). In Section 3, extension type theorem for the equation (1.5) is given. At last, two examples are given to demonstrate the applicability of our result.

2. Main results

For a nonempty set X, a function $d\colon X \times X \to [0,\infty]$ is called a generalized metric on X if and only if d satisfies:

(a) $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(c) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Definition 2.1 (see Jung et al. [14] or Balachandran [2]). Suppose E is a vector space over $\mathbb{K}$. A function $\|\cdot\|_\beta\colon E \to [0,\infty)$ $(0 < \beta \le 1)$ is called a β-norm if and only if it satisfies:

(a) $\|x\|_\beta = 0$ if and only if $x = 0$;
(b) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$;
(c) $\|x + y\|_\beta \le \|x\|_\beta + \|y\|_\beta$.

Theorem 2.2 (see [7]). *Let (X,d) be a generalized complete metric space. Assume that $\Lambda\colon X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < +\infty$ for some $x \in X$, then the followings are true*:

(a) *The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*
(b) *x^* is the unique fixed point of Λ in $X^* = \{y \in X \mid d(\Lambda^k x, y) < \infty\}$;*
(c) *If $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1-L}\, d(\Lambda y, y).$$

Next, we introduce the following space of piecewise continuous functions

$$X = \{g\colon J \to \mathbb{R} \mid g \in PC(J,\mathbb{R})\}, \tag{2.1}$$

endowed with the generalized metric on X defined by

$$\begin{aligned} d(g,h) = \inf\{C_1 + C_2 \in [0,+\infty] \mid \\ |g(t) - h(t)|^\beta \le (C_1 + C_2)(\varphi^\beta(t) + \psi^\beta) \text{ for all } t \in J\}, \end{aligned} \tag{2.2}$$

where $C_1 \in \{C \in [0,+\infty] \mid |g(t) - h(t)|^\beta \le C\varphi^\beta(t)$ for all $t \in (s_i, t_{i+1}]$, $i = 0, 1, \ldots, m\}$, and $C_2 \in \{C \in [0,+\infty] \mid |g(t) - h(t)|^\beta \le C\psi^\beta$ for all $t \in (t_i, s_i]$, $i = 1, \ldots, m\}$. It is easy to verify that (X,d) is a complete generalized metric space.

Now, we are ready to study the stability result of the equation (1.2).

Theorem 2.3. *Assume that the following conditions*:

(H_1) $f \in C(J \times \mathbb{R}, \mathbb{R})$;

(H_2) *There exists a positive constant* L_f *such that*

$$|f(t,u_1)-f(t,u_2)|\le L_f|u_1-u_2|,$$

for each $t\in J$ *and all* $u_1,u_2\in\mathbb{R}$;

(H_3) $g_i\in C([t_i,s_i]\times\mathbb{R},\mathbb{R})$ *and there are positive constants* L_{g_i}, $i=1,\dots,m$ *such that*

$$|g_i(t,u_1)-g_i(t,u_2)|\le L_{g_i}|u_1-u_2|,$$

for each $t\in[t_i,s_i]$ *and all* $u_1,u_2\in\mathbb{R}$;

(H_4) *Let* $\varphi\in C(J,\mathbb{R}_+)$ *be a nondecreasing function. There exists* $c_\varphi>0$ *such that*

$$\int_0^t\varphi(s)ds\le c_\varphi\varphi(t),\quad \textit{for each } t\in J,$$

are satisfied. If there exists a function $y\in V$ *satisfing* (1.3)*, then there exists a unique solution* $y_0\colon J\to\mathbb{R}$ *such that*

$$y_0(t)=\begin{cases}x(0)+\displaystyle\int_0^t f(s,y_0(s))\,ds, & t\in[0,t_1],\\ g_i(t,y_0(t_i^+)), & t\in(t_i,s_i],\ i=1,\dots,m,\\ g_i(s_i,y_0(t_i^+))+\displaystyle\int_{s_i}^t f(s,y_0(s))\,ds, & t\in(s_i,t_{i+1}],\ i=1,\dots,m,\end{cases}\tag{2.3}$$

and

$$|y(t)-y_0(t)|^\beta\le\frac{(1+c_\varphi^\beta)(\varphi^\beta(t)+\psi^\beta)}{1-\rho},\quad t\in J,\tag{2.4}$$

provided that

$$\rho:=\max\Big\{L_{g_i}^\beta+L_f^\beta c_\varphi^\beta\ \Big|\ i=1,\dots,m\Big\}<1.\tag{2.5}$$

Proof. Define an operator $\Lambda\colon X\to X$ by

$$(\Lambda x)(t)=\begin{cases}x(0)+\displaystyle\int_0^t f(s,x(s))\,ds, & t\in[0,t_1],\\ g_i(t,x(t_i^+)), & t\in(t_i,s_i],\ i=1,\dots,m,\\ g_i(s_i,x(t_i^+))+\displaystyle\int_{s_i}^t f(s,x(s))\,ds, & t\in(s_i,t_{i+1}],\ i=1,\dots,m.\end{cases}\tag{2.6}$$

for all $x\in X$ and $t\in[0,T]$. Clearly, Λ is a well defined operator according to (H_1).

We show that Λ is strictly contractive on X. Note that the definition of (X,d), for any $g,h\in X$, it is possible to find $C_1,C_2\in[0,\infty]$ such that

$$|g(t)-h(t)|^\beta\le\begin{cases}C_1\varphi^\beta(t), & t\in(s_i,t_{i+1}],\ i=0,1,\dots,m,\\ C_2\psi^\beta, & t\in(t_i,s_i],\ i=1,\dots,m.\end{cases}\tag{2.7}$$

It is easy to see that (2.7) is equivalent to

$$|g(t)-h(t)| \leq \begin{cases} C_1^{1/\beta}\varphi(t), & t\in(s_i,t_{i+1}],\ i=0,1,\ldots,m, \\ C_2^{1/\beta}\psi, & t\in(t_i,s_i],\ i=1,\ldots,m. \end{cases} \tag{2.8}$$

By the definition of Λ in (2.6), (H_2), (H_3), and (2.8), we obtain the following three possible cases:

Case 1. $t\in[0,t_1]$, one has

$$\begin{aligned}|(\Lambda g)(t)-(\Lambda h)(t)|^\beta &= \bigg|\int_0^t f(s,g(s))\,ds-\int_0^t f(s,h(s))\,ds\bigg|^\beta \\ &\leq L_f^\beta\bigg(\int_0^t |g(s)-h(s)|\,ds\bigg)^\beta \leq L_f^\beta\bigg[C_1^{1/\beta}\int_0^t \varphi(s)\,ds\bigg]^\beta \\ &\leq L_f^\beta\Big[C_1^{1/\beta}c_\varphi\varphi(t)\Big]^\beta = L_f^\beta c_\varphi^\beta C_1\varphi^\beta(t).\end{aligned}$$

Case 2. For $t\in(t_i,s_i]$,

$$|(\Lambda g)(t)-(\Lambda h)(t)|^\beta = |g_i(t,g(t_i^+))-g_i(t,h(t_i^+))|^\beta \leq L_{g_i}^\beta C_2\psi^\beta.$$

Case 3. For $t\in(s_i,t_{i+1}]$, one can compute

$$\begin{aligned}&|(\Lambda g)(t)-(\Lambda h)(t)|^\beta \\ &\quad= \bigg|g_i(s_i,g(t_i^+))+\int_{s_i}^t f(s,g(s))\,ds-g_i(s_i,h(t_i^+))-\int_{s_i}^t f(s,h(s))\,ds\bigg|^\beta \\ &\quad\leq |g_i(s_i,g(t_i^+))-g_i(s_i,h(t_i^+))|^\beta+\bigg|\int_{s_i}^t f(s,g(s))ds-\int_{s_i}^t f(s,h(s))\,ds\bigg|^\beta \\ &\quad\leq L_{g_i}^\beta|g(t_i^+)-h(t_i^+)|^\beta+L_f^\beta\bigg[\int_{s_i}^t |g(s)-h(s)|\,ds\bigg]^\beta \\ &\quad\leq L_{g_i}^\beta C_2\psi^\beta+L_f^\beta\bigg[C_1^{1/\beta}\int_0^t\varphi(s)\,ds\bigg]^\beta \leq L_{g_i}^\beta C_2\psi^\beta+L_f^\beta C_1c_\varphi^\beta\varphi^\beta(t) \\ &\quad\leq (L_{g_i}^\beta+L_f^\beta c_\varphi^\beta)(C_1+C_2)(\varphi^\beta(t)+\psi^\beta).\end{aligned}$$

Thus, we have

$$|(\Lambda g)(t)-(\Lambda h)(t)|^\beta \leq \max\Big\{L_{g_i}^\beta+L_f^\beta c_\varphi^\beta\ \Big|\ i=1,\ldots,m\Big\}(C_1+C_2)(\varphi^\beta(t)+\psi^\beta),$$

for $t\in J$. In other words,

$$d(\Lambda g,\Lambda h)\leq \rho(C_1+C_2)(\varphi^\beta(t)+\psi^\beta).$$

Hence, we derive

$$d(\Lambda g,\Lambda h)\leq \rho d(g,h)$$

for any $g,h\in X$, and since the condition (2.5), the strictly continuous property is shown.

Let us take $g_0 \in X$. From the piecewise continuous property of g_0 and Λg_0, it follows that there exists a constant $0 < G_1 < \infty$ such that

$$\begin{aligned}|(\Lambda g_0)(t) - g_0(t)|^\beta &= \left| x(0) + \int_0^t f(s, g_0(s))\, ds - g_0(t) \right|^\beta \\ &\le G_1 \varphi^\beta(t) \le G_1(\varphi^\beta(t) + \psi^\beta), \quad t \in [0, t_1].\end{aligned}$$

There exists a constant $0 < G_2 < \infty$ such that

$$\begin{aligned}|(\Lambda g_0)(t) - g_0(t)|^\beta &= \left| g_i(t, g_0(t_i^+)) - g_0(t) \right|^\beta \\ &\le G_2 \psi^\beta \le G_2(\varphi^\beta(t) + \psi^\beta), \quad t \in (t_i, s_i],\ i = 1, \dots, m.\end{aligned}$$

There exists a constant $0 < G_3 < \infty$ such that

$$\begin{aligned}|(\Lambda g_0)(t) - g_0(t)|^\beta &= \left| g_i(s_i, g_0(t_i^+)) + \int_{s_i}^t f(s, g_0(s))\, ds - g_0(t) \right|^\beta \\ &\le G_3(\varphi^\beta(t) + \psi^\beta), \quad t \in (s_i, t_{i+1}],\ i = 1, \dots, m,\end{aligned}$$

since f, g_i and g_0 are bounded on J and $\varphi^\beta(\,\cdot\,) + \psi^\beta > 0$. Thus, (2.2) implies that

$$d(\Lambda g_0, g_0) < \infty.$$

By using Banach fixed point theorem, there exists a continuous function $y_0\colon J \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$ and $\Lambda y_0 = y_0$, that is, y_0 satisfies equation (2.3) for every $t \in J$.

Next, we check that $\{g \in X \mid d(g_0, g) < \infty\} = X$. For any $g \in X$, since g and g_0 are bounded on J and $\min_{t \in J}(\varphi^\beta(t) + \psi^\beta) > 0$, there exists a constant $0 < C_g < \infty$ such that $|g_0(t) - g(t)|^\beta \le C_g(\varphi^\beta(t) + \psi^\beta)$, for any $t \in J$. Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is, $\{g \in X \mid d(g_0, g) < \infty\} = X$. Hence, we conclude that y_0 is the unique continuous function with the property (2.3). On the other hand, from (1.4) and (H_4) it follows that

$$d(y, \Lambda y) \le 1 + c_\varphi^\beta. \tag{2.9}$$

Thus, we derive

$$d(y, y_0) \le \frac{d(\Lambda y, y)}{1 - \rho} \le \frac{1 + c_\varphi^\beta}{1 - \rho},$$

which means that (2.4) is true for $t \in J$. The proof is done. □

3. Extension

Now we adopt the same idea in the above section and extend to study generalized β-Ulam–Hyers–Rassias stability of the equation (1.5).

Consider the following impulsive ordinary differential equations with constant coefficients

$$\begin{cases} x'(t) = \lambda x(t) + f(t, x(t)), & t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, m,\ \lambda > 0, \\ x(t) = g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \ldots, m, \\ x(0) = x_0 \in \mathbb{R}. \end{cases} \tag{3.1}$$

A function $x \in V$ is called a classical solution of (3.1) if x satisfies $x(0) = x_0$, $x(t) = g_i(t, x(t))$, $t \in (t_i, s_i]$, $i = 1, \ldots, m$, and

$$x(t) = e^{\lambda t} x_0 + \int_0^t e^{\lambda(t-s)} f(s, x(s))\, ds, \qquad t \in [0, t_1],$$

$$x(t) = e^{\lambda(t-s_i)} g_i(s_i, x(s_i)) + \int_{s_i}^t e^{\lambda(t-s)} f(s, x(s))\, ds, \quad t \in (s_i, t_{i+1}],\ i = 1, \ldots, m.$$

Theorem 3.1. *Assume that* (H_1)–(H_4) *are satisfied and a function* $y \in V$ *satisfies* (1.6)*. Then there exists a unique solution* $y_0 \colon J \to \mathbb{R}$ *such that*

$$y_0(t) = \begin{cases} e^{\lambda t} x(0) + \int_0^t e^{\lambda(t-s)} f(s, y_0(s))\, ds, & t \in [0, t_1], \\ g_i(t, y_0(t)), & t \in (t_i, s_i],\ i = 1, \ldots, m, \\ e^{\lambda(t-s_i)} g_i(s_i, y_0(s_i)) + \int_{s_i}^t e^{\lambda(t-s)} f(s, y_0(s))\, ds, & \\ & t \in (s_i, t_{i+1}],\ i = 1, \ldots, m, \end{cases} \tag{3.2}$$

and

$$|y(t) - y_0(t)|^\beta \le \frac{e^{\beta\lambda T}(1 + c_\varphi^\beta)(\varphi^\beta(t) + \psi^\beta)}{1 - \rho}, \quad t \in J, \tag{3.3}$$

provided that

$$\rho_\lambda := e^{\beta\lambda T} \max\Big\{ L_{g_i}^\beta + L_f^\beta c_\varphi^\beta \ \Big|\ i = 1, \ldots, m \Big\} < 1. \tag{3.4}$$

Proof. Just like the proof in Theorem 2.3, we define an operator $\Lambda_\lambda \colon X \to X$ by

$$(\Lambda_\lambda x)(t) = \begin{cases} e^{\lambda t} x(0) + \int_0^t e^{\lambda(t-s)} f(s, x(s))\, ds, & t \in [0, t_1], \\ g_i(t, x(t)), & t \in (t_i, s_i],\ i = 1, \ldots, m, \\ e^{\lambda(t-s_i)} g_i(s_i, x(s_i)) + \int_{s_i}^t e^{\lambda(t-s)} f(s, x(s))\, ds, & \\ & t \in (s_i, t_{i+1}],\ i = 1, \ldots, m. \end{cases} \tag{3.5}$$

for all $x \in X$ and $t \in [0, T]$.

Clearly, Λ_λ is a well defined operator according to (H_1). We only need to verify that Λ_λ is strictly contractive on X. In fact, it follows from the definition of Λ_λ in (3.5), (H_2), (H_3), and (2.8), we obtain that

Case 1. For $t \in [0, t_1]$,

$$|(\Lambda_\lambda g)(t) - (\Lambda_\lambda h)(t)|^\beta \le e^{\beta\lambda T} L_f^\beta c_\varphi^\beta C_1 \varphi^\beta(t).$$

Case 2. For $t \in (t_i, s_i]$,

$$|(\Lambda_\lambda g)(t) - (\Lambda_\lambda h)(t)|^\beta = |g_i(t, g(t)) - g_i(t, h(t))|^\beta \le L_{g_i}^\beta C_2 \psi^\beta.$$

Case 3. For $t \in (s_i, t_{i+1}]$,

$$\begin{aligned}|(\Lambda_\lambda g)(t) &- (\Lambda_\lambda h)(t)|^\beta \\ &\le e^{\beta\lambda T} L_{g_i}^\beta |g(s_i) - h(s_i)|^\beta + e^{\beta\lambda T} L_f^\beta \left[\int_{s_i}^t |g(s) - h(s)|\, ds\right]^\beta \\ &\le e^{\beta\lambda T} L_{g_i}^\beta C_2 \psi^\beta + e^{\beta\lambda T} L_f^\beta \left[C_1^{1/\beta} \int_0^t \varphi(s)\, ds\right]^\beta \\ &\le e^{\beta\lambda T} L_{g_i}^\beta C_2 \psi^\beta + e^{\beta\lambda T} L_f^\beta C_1 c_\varphi^\beta \varphi^\beta(t) \\ &\le e^{\beta\lambda T} (L_{g_i}^\beta + L_f^\beta c_\varphi^\beta)(C_1 + C_2)(\varphi^\beta(t) + \psi^\beta).\end{aligned}$$

From above, we have $d(\Lambda_\lambda g, \Lambda_\lambda h) \le \rho_\lambda d(g, h)$, for any $g, h \in X$, and since the condition (3.4), the strictly continuous property is shown.

Let us take $g_0 \in X$. Proceeding the same procedure in Theorem 2.3, $d(\Lambda_\lambda g_0, g_0) < \infty$. By using Banach fixed point theorem, there exists a continuous function $y_0 \colon J \to \mathbb{R}$ such that $\Lambda_\lambda^n g_0 \to y_0$ in (X, d) as $n \to \infty$ and $\Lambda_\lambda y_0 = y_0$, that is, y_0 satisfies equation (3.2) for every $t \in J$, which is the unique continuous function.

On the other hand, from (1.7) and (H_4) it follows that

$$d(y, \Lambda_\lambda y) \le e^{\beta\lambda T}(1 + c_\varphi^\beta).$$

Finally,

$$d(y, y_0) \le \frac{e^{\beta\lambda T}(1 + c_\varphi^\beta)}{1 - \rho_\lambda},$$

which means that (3.3) is true for $t \in J$. The proof is done. □

4. Example

In this section, we present two examples, which indicate how our theorems can be applied to concrete problems.

Example 4.1. Consider

$$\begin{cases} x'(t) = \dfrac{|x(t)|}{8 + e^t}, & t \in (0, 1], \\ x(t) = \dfrac{|x(1^+)|}{(3 + e^{t-1})(1 + |x(1^+)|)}, & t \in (1, 2], \end{cases} \tag{4.1}$$

and

$$\begin{cases} \left| y'(t) - \dfrac{|y(t)|}{8+e^t} \right| \le e^t, & t \in [0,1], \\ \left| y(t) - \dfrac{|y(1^+)|}{(3+e^{t-1})(1+|y(1^+)|)} \right| \le 1, & t \in (1,2]. \end{cases}$$

We put $\beta = 1/2$, $J = [0,2]$ and $0 = t_0 = s_0 < t_1 = 1 < s_1 = 2$. Denote

$$\begin{aligned} f(t,x(t)), &= \frac{|x(t)|}{8+e^t} && \text{with } \ L_f = \frac{1}{9} \ \text{ for } t \in [0,1], \\ g_1(t,x(t)) &= \frac{|x(1^+)|}{(3+e^{t-1})(1+|x(1^+)|)} && \text{with } L_{g_1} = \frac{1}{4} \ \text{ for } t \in (1,2]. \end{aligned}$$

We put $\varphi(t) = e^t$ and $\psi = 1$. Then we choose $c_\varphi = 1$ satisfying the condition $\int_0^t e^s \, ds \le e^t$. Moreover, $L_{g_1}^\beta + L_f^\beta c_\varphi = 5/6 < 1$.

Now all the assumptions of Theorem 2.3 are satisfied. Thus, (4.1) has a unique solution $y_0\colon [0,2] \to \mathbb{R}$ such that $|y(t) - y_0(t)|^{1/2} \le 12(e^{t/2} + 1)$, for all $t \in [0,2]$.

Example 4.2. Consider

$$\begin{cases} x'(t) = x(t) + \dfrac{|x(t)|}{35+e^t}, & t \in (0,1], \\ x(t) = \dfrac{|x(t)|}{(24+e^{t-1})(1+|x(t)|)}, & t \in (1,2], \end{cases}$$

and

$$\begin{cases} \left| y'(t) - y(t) - \dfrac{|y(t)|}{35+e^t} \right| \le e^t, & t \in [0,1], \\ \left| y(t) - \dfrac{|y(t)|}{(24+e^{t-1})(1+|y(t)|)} \right| \le 1, & t \in (1,2]. \end{cases}$$

Let $\lambda = 1$, $\beta = 1/2$, $T = 2$, $J = [0,2]$ and $0 = t_0 = s_0 < t_1 = 1 < s_1 = 2$. Denote

$$\begin{aligned} f(t,x(t)) &= \frac{|x(t)|}{35+e^t} && \text{with } \ L_f = \frac{1}{36} \ \text{ for } t \in [0,1], \\ g_1(t,x(t)) &= \frac{|x(t)|}{(24+e^{t-1})(1+|x(t)|)} && \text{with } L_{g_1} = \frac{1}{25} \ \text{ for } t \in (1,2]. \end{aligned}$$

We put $\varphi(t) = e^t$ and $\psi = 1$. Set $c_\varphi = 1$, we have $\int_0^t e^s \, ds \le e^t$. Obviously,

$$e^{\beta\lambda T}(L_{g_1}^\beta + L_f^\beta c_\varphi) = \frac{11}{30} \times e \approx 0.9968 < 1.$$

By Theorem 3.1, there exists a unique solution $y_0\colon [0,2] \to \mathbb{R}$ such that

$$y_0(t) = \begin{cases} e^t x(0) + \displaystyle\int_0^t e^{t-s} \frac{|y_0(s)|}{35+e^s} \, ds, & t \in [0,1], \\ \dfrac{|y_0(t)|}{(24+e^{t-1})(1+|y_0(t)|)}, & t \in (1,2], \end{cases}$$

and

$$|y(t) - y_0(t)|^{1/2} \leq \frac{60e}{30 - 11e}(e^{t/2} + 1) \quad \text{for all } t \in [0, 2].$$

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