Topological Methods in Nonlinear Analysis
Volume 46, No. 1, 2015, 135–163

© 2015 Juliusz Schauder Centre for Nonlinear Studies
Nicolaus Copernicus University

EIGENVALUE, BIFURCATION, CONVEX SOLUTIONS FOR MONGE–AMPÈRE EQUATIONS

Guowei Dai — Ruyun Ma

This paper is devoted to the memory of my father,
who died from myocardial infarction on the 3th of October 2011.
His smile is in my soul, for ever and ever. Guowei Dai.

Abstract. In this paper we study the following eigenvalue boundary value problem for Monge-Ampère equations

$$\begin{cases} \det(D^2u) = \lambda^N f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We establish global bifurcation results for the problem with $f(u) = u^N + g(u)$ and Ω being the unit ball of $\mathbb{R}^N$. More precisely, under some natural hypotheses on the perturbation function $g\colon [0,+\infty) \to [0,+\infty)$, we show that $(\lambda_1, 0)$ is a bifurcation point of the problem and there exists an unbounded continuum of convex solutions, where λ_1 is the first eigenvalue of the problem with $f(u) = u^N$. As the applications of the above results, we consider with determining interval of λ, in which there exist convex solutions for this problem in unit ball. Moreover, we also get some results about the existence and nonexistence of convex solutions for this problem on general domain by domain comparison method.

2010 *Mathematics Subject Classification.* 34C23, 34D23, 35J60.

Key words and phrases. Eigenvalue, bifurcation, convex solution, Monge–Ampère equation.

Research supported by NNSF of China (No. 11261052, 11401477), the Fundamental Research Funds for the Central Universities (No. DUT15RC(3)018) and Scientific Research Project of the Higher Education Institutions of Gansu Province (No. 2014A-009).

1. Introduction

The Monge-Ampère equations are a type of important fully nonlinear elliptic equations [10], [25]. The study of Monge–Ampère equations has been received considerable attentions in recent years. Historically, the study of Monge–Ampère equations is motivated by Minkowski problem and Weyl problem. Existence and regularity results may be found in [3]–[5], [10], [14], [16], [20]–[22], [27] and the reference therein.

We consider the following real Monge–Ampère equations

$$\begin{cases} \det(D^2u) = \lambda^N f(-u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $D^2u = (\partial^2 u/(\partial x_i \partial x_j))$ is the Hessian matrix of u, B is the unit ball of $\mathbb{R}^N$, λ is a positive parameter and $f\colon [0,+\infty) \to [0,+\infty)$ is a continuous function. The study of problem (1.1) in general domains of $\mathbb{R}^N$ may be found in [3] and [10]. Kutev [15] investigated the existence of strictly convex radial solutions of problem (1.1) when $f(s) = s^p$. Delano [8] treated the existence of convex radial solutions of problem (1.1) for a class of more general functions, namely $\lambda \exp f(|x|, u, |\nabla u|)$.

In [11], [15], the authors have showed that problem (1.1) can reduce to the following boundary value problem

$$\begin{cases} ((u')^N)' = \lambda^N N r^{N-1} f(-u), & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{1.2}$$

By a solution of problem (1.2) we understand that it is a function which belongs to $C^2[0,1]$ and satisfies (1.2). It has been known that any negative solution of problem (1.2) is strictly convex in $(0,1)$ so long as f does not vanish on any entire interval (see [11]). Wang [26], Hu-Wang [11] also established several criteria for the existence, multiplicity and nonexistence of strictly convex solutions for problem (1.2) by using fixed index theorem. However, there is no any information on the bifurcation points and the optimal intervals for the parameter λ so as to ensure existence of single or multiple convex solutions. Fortunately, Lions [17] have proved the existence of the first eigenvalue λ_1 of problem (1.1) with $f(u) = u^N$ via constructive proof.

Motivated by above, we shall establish a global bifurcation theorem for problem (1.2) with $f(u) = u^N + g(u)$, i.e.

$$\begin{cases} ((u')^N)' = \lambda^N N r^{N-1}((-u)^N + g(-u)), & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $g\colon [0,+\infty) \to [0,+\infty)$ satisfies $\lim_{s\to 0^+} g(s)/s^N = 0$. Concretely, we shall show that $(\lambda_1, 0)$ is a bifurcation point of problem (1.3) and there exists an unbounded continuum of convex solutions.

In global bifurcation theory of differential equations, it is well known that a change of the index of the trivial solution implies the existence of a branch of nontrivial solutions, bifurcating from the set of trivial solutions and which is either unbounded or returns to the set of trivial solution. Hence, the index formula of an isolated zero is very important in the study of the bifurcation phenomena for semi-linear differential equations. However, problem (1.3) is a type of nonlinear equation. Hence, the common index formula involving linear map cannot be used here. In order to overcome this difficulty, we shall study an auxiliary eigenvalue problem, which has an independent interest, and establish an index formula for it. Then by use of the index formula of the auxiliary problem, we prove an index formula involving problem (1.3) which guarantees $(\lambda_1, 0)$ is a bifurcation point of nontrivial solutions of problem (1.3).

Based on the above global bifurcation results, we investigate the existence of strictly convex solutions of problem (1.2). We shall give the optimal intervals for the parameter λ so as to ensure existence of single or multiple strictly convex solutions. In order to study the exact multiplicity of convex solutions for problem (1.2), we introduce the concept of stable solution. Then by Implicit Function Theorem and stability properties, under some more strict assumptions of f, we can show that the convex solution branch of problem (1.2) can be a smooth curve. Our results extend the corresponding results of [11], [17], [26].

On the basis of results on unit ball, we also study problem (1.1) on a general domain Ω, i.e.

$$\begin{cases} \det(D^2u) = \lambda^N f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded convex domain of $\mathbb{R}^N$ with smooth boundary and $0 \in \operatorname{Int}\Omega$.

It is well known [10] that problem (1.4) is elliptic only when the Hessian matrix D^2u is positive definite and it is therefore natural to confine our attention to convex solutions and nonnegative functions f with $f(s) > 0$ for $s > 0$. Obviously, any convex solution of problem (1.4) is negative and strictly convex.

In [27], the authors have proved a lemma concerning the comparison between domains for problem (1.4) with $f(s) = e^s$ by sub-supersolution method. We shall show that this lemma is also valid for problem (1.4). Using this domain comparison lemma and the results on unit ball, we can prove some existence and nonexistence of convex solutions for problem (1.4).

The rest of this paper is arranged as follows. In Section 2, we study an auxiliary problem and prove a key index formula. In Section 3, we establish a global bifurcation theorem for problem (1.3). In Section 4, we give the intervals for the parameter λ which ensure existence of single or multiple strictly convex solutions for problem (1.2) under some suitable assumptions of nonlinearity f. In Section 5, under some more strict assumptions of f, we prove the exact multiplicity of convex solutions for problem (1.2). In Section 6, we prove some existence and nonexistence of convex solutions for problem (1.4).

2. A key preliminary result

In this section, we shall study an auxiliary eigenvalue problem and prove a key index formula that will be used in the next section.

Let $p \in [2, +\infty)$. Consider the following auxiliary problem

$$\begin{cases} -(|v'(r)|^{p-2}v'(r))' = \mu^{p-1}(p-1)r^{p-2}|v(r)|^{p-2}v(r), & r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{2.1}$$

Let X be the Banach space $C[0,1]$ with the norm

$$\|v\| = \sup_{r\in[0,1]} |v(r)|.$$

Define the map $T^p_\mu \colon X \to X$ by

$$T^p_\mu v = \int_1^r \varphi_{p'}\bigg(\int_s^0 \mu^{p-1}(p-1)\tau^{p-2}\varphi_p(v)\,d\tau\bigg)\,ds, \quad 0 \le r \le 1,$$

where $\varphi_p(s) = |s|^{p-2}s$, $p' = p/(p-1)$. It is not difficult to verify that T^p_μ is continuous and compact. Clearly, problem (2.1) can be equivalently written as $v = T^p_\mu v$.

Firstly, we show that the existence and uniqueness theorem is valid for problem (2.1).

Lemma 2.1. *If (μ, v) is a solution of* (2.1) *and v has a double zero, then $v \equiv 0$.*

Proof. Let v be a solution of problem (2.1) and $r_* \in [0,1]$ be a double zero. We note that v satisfies

$$v(r) = \int_{r_*}^r \varphi_{p'}\bigg(\int_s^{r_*} (p-1)\mu^{p-1}\tau^{p-2}\varphi_p(v)\,d\tau\bigg)ds.$$

Firstly, we consider $r \in [0, r_*]$. Then we have

$$|v(r)| \le \varphi_{p'}\bigg(\int_r^{r_*} (p-1)\mu^{p-1}\tau^{p-2}\varphi_p(|v|)\,d\tau\bigg).$$

Furthermore, it follows from above that

$$\varphi_p(|v|) \quad \le \quad \mu^{p-1}\int_r^{r_*}(p-1)\tau^{p-2}\varphi_p(|v|)\,d\tau.$$

By the modification of Gronwall–Bellman inequality [13, Lemma 2.2], we get $v \equiv 0$ on $[0, r^*]$. Similarly, we can get $v \equiv 0$ on $[r^*, 1]$ and the proof is completed. □

Set $W_c^{1,p}(0,1) := \{v \in W^{1,p}(0,1) \mid v'(0) = v(1) = 0\}$ with the norm

$$\|v\|_w = \left(\int_0^1 |v'|^p \, dr\right)^{1/p} + \left(\int_0^1 (p-1) r^{p-2} |v|^p \, dr\right)^{1/p}.$$

Then it is easy to verify that $W_c^{1,p}(0,1)$ is a real Banach space.

DEFINITION 2.2. We call that $v \in W_c^{1,p}(0,1)$ is the weak solution of problem (2.1), if

$$\int_0^1 |v'|^{p-2} v' \phi' \, dr = (p-1)\mu^{p-1} \int_0^1 r^{p-2} |v|^{p-2} v \phi \, dr$$

for any $\phi \in W_c^{1,p}(0,1)$.

For the regularity of weak solution, we have the following result.

LEMMA 2.3. *Let v be a weak solution of problem* (2.1)*, then v satisfies problem* (2.1)*.*

In order to prove Lemma 2.3, we need the following technical result.

PROPOSITION 2.4. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. For a given $x_0 \in \mathbb{R}$, if f is continuous in some neighbourhood U of x_0, differential in $U \setminus \{x_0\}$ and $\lim_{x \to x_0} f'(x)$ exists, then f is differential at x_0 and $f'(x_0) = \lim_{x \to x_0} f'(x)$.*

PROOF. The conclusion is a direct corollary of Lagrange Mean Theorem, we omit its proof here. □

PROOF OF LEMMA 2.3. According to Definition 2.2, we have

$$-(|v'(r)|^{p-2} v'(r))' = \mu^{p-1}(p-1) r^{p-2} |v(r)|^{p-2} v(r) \quad \text{in } (0,1)$$

in the sense of distribution, i.e.

$$-(|v'(r)|^{p-2} v'(r))' = \mu^{p-1}(p-1) r^{p-2} |v(r)|^{p-2} v(r) \quad \text{in } (0,1) \setminus I,$$

for some $I \subset (0,1)$, which satisfies $\operatorname{meas}\{I\} = 0$. Furthermore, by virtue of the compact embedding of $W_c^{1,p}(0,1) \hookrightarrow C^\alpha[0,1]$ with some $\alpha \in (0,1)$ (see [9]), we obtain that $v \in C^\alpha[0,1]$. Thus, we have that $\lim_{r \to r_0} \mu^{p-1}(p-1) r^{p-2} |v(r)|^{p-2} v(r)$ exists for any $r_0 \in I$. Letting $u := -\varphi_p(v')$, we have

$$\lim_{r \to r_0} u'(r) = \lim_{r \to r_0} \mu^{p-1}(p-1) r^{p-2} |v(r)|^{p-2} v(r).$$

The above relation follows that $\lim_{r \to r_0} u'(r)$ exists for any $r_0 \in I$. Thus, Proposition 2.4 follows that $u \in C^1(0,1)$, which implies that v satisfies problem (2.1).□

Define the functional J on $W_c^{1,p}(0,1)$ by

$$J(v) = \int_0^1 \frac{1}{p}|v'(r)|^p\,dr - \mu^{p-1}\frac{p-1}{p}\int_0^1 r^{p-2}|v|^p\,dr.$$

It is not difficult to verify that the critical points of J are the weak solutions of problem (2.1). Taking

$$f_1(v) := \int_0^1 \frac{1}{p}|v'(r)|^p\,dr \quad \text{and} \quad f_2(v) := \frac{p-1}{p}\int_0^1 r^{p-2}|v|^p\,dr,$$

consider the following eigenvalue problem

$$A(v) = \eta B(v), \tag{2.2}$$

where $A = \partial f_1$ and $B = \partial f_2$ denote the sub-differential of f_1 and f_2, respectively (refer to [6] for the details of sub-differential).

By some simple computations, we can show that

$$\frac{f_1(v)}{f_2(v)} \geq \frac{1}{(p-1)} \tag{2.3}$$

for any $v \in W_c^{1,p}(0,1)$ and $v \not\equiv 0$. Moreover, we have the following result.

Lemma 2.5. *Put* $\eta_1(p) = \inf_{v\in W_c^{1,p}(0,1),\, v\not\equiv 0} f_1(v)/f_2(v)$. *Then we have that*:

(a) (2.2) *has no nontrivial solution for* $\eta \in (0, \eta_1(p))$;
(b) $\eta_1(p)$ *is simple, i.e.* (2.2) *has a positive solution and the set of all solutions of* (2.2) *is an one dimensional linear subspace of* $W_c^{1,p}(0,1)$;
(c) (2.2) *has a positive solution if and only if* $\eta = \eta_1(p)$.

Proof. Let $W_c^{1,p}(0,1) =: V$. We denote by $\Phi(V)$ the family of all proper lower semi-continuous convex functions φ from V into $(-\infty, +\infty]$, where "proper" means that the effective domain $D(\varphi) = \{x \in V \mid \varphi(x) < +\infty\}$ of φ is not empty.

Next, we verify the conditions (A0)–(A4) of [12]. Clearly, we have that $f_1,\ f_2 \in \Phi(V)$, $D(f_1) = D(f_2) = V$ and $V \subset L^1_{\rm loc}(0,1)$, i.e. condition (Al) is satisfied (by taking $\Omega = (0,1)$). Let $R(v) := f_2(v)/f_1(v)$. Then we have $R(|v|) \geq R(v)$ for all $v \in V$. It is easy to see that $f_1(v) \geq 0$ for all $v \in V$ and $f_1(v) = 0$ if and only if $v = 0$. Note that (2.3) implies that there exists $u \in V$ such that $u \neq 0$ and $R(u) = \sup\{R(v) \mid v \in V,\ v \neq 0\}$. So condition (A2) is verified. Taking $\alpha = p$, we have $f_i(tv) = t\alpha f_i(v)$ for all $v \in V^+ = \{w \in V \mid w(r) \geq 0$ for almost every $r \in (0,1)\}$, for all $t > 0$, $i = 1,2$. Thus, condition (A3) is satisfied. For any $u,\ v \in V^+$, we define $(u \vee w)(r) = \max(u(r), w(r))$, $(u \wedge w)(r) = \min(u(r), w(r))$, $I_1 = \{r \in [0,1] \mid u(r) \geq w(r)\}$ and $I_2 = \{r \in$

$[0,1] \mid u(r) < w(r)\}$. Then we have

$$\begin{aligned} f_1(u \vee w) + f_1(u \wedge w) &= \int_0^1 \frac{1}{p} |(u \vee w)'(r)|^p \, dr + \int_0^1 \frac{1}{p} |(u \wedge w)'(r)|^p \, dr \\ &= \int_{I_1} \frac{1}{p} |u'|^p \, dr + \int_{I_2} \frac{1}{p} |w'|^p \, dr + \int_{I_1} \frac{1}{p} |w'|^p \, dr + \int_{I_2} \frac{1}{p} |u'|^p \, dr \\ &= \int_0^1 \frac{1}{p} |u'|^p \, dr + \int_0^1 \frac{1}{p} |w'|^p \, dr = f_1(u) + f_1(w). \end{aligned}$$

Similarly, we can also show that $f_2(u \vee w) + f_2(u \wedge w) = f_2(u) + f_2(w)$. Hence, condition (A4) is verified. Finally, Lemmas 2.1 and 2.3 imply that every nonnegative nontrivial solution u of (2.2) belongs to $C(0,1) \cap L^\infty(0,1)$ and satisfies $u(r) > 0$ for all $r \in (0,1)$. So condition (A0) is verified.

Now, by Theorem I of [12], we can obtain (a) and (b). Finally, we prove (c). Suppose now that (2.2) with $\eta > \eta_1$ has a positive solution v, and let u be a positive solution of (2.2) corresponding to $\eta_1(p)$. Lemmas 2.1 and 2.3 imply that every positive solution w of (2.2) satisfies $w \in C^1[0,1]$ and $w'(1) < 0$. By virtue of this fact and the fact that tv is also a solution of (2.2) for any real number t, we may assume without loss of generality that $u \le v$. It is not difficult to verify that A and B are monotone operators. The rest of proof is similar to that of [12, Theorem II]. □

Let $\eta = \mu^{p-1}$, Lemma 2.5 shows the following result.

LEMMA 2.6. *Put* $\mu_1(p) = (\eta_1(p))^{1/(p-1)}$. *Then we have that*:

(a) (2.1) *has no nontrivial solution for* $\mu \in (0, \mu_1(p))$;

(b) $\mu_1(p)$ *is simple*;

(c) (2.1) *has a positive solution if and only if* $\mu = \mu_1(p)$.

Moreover, we have the following result.

LEMMA 2.7. *If* (μ, u) *satisfies* (2.1) *with* $\mu \ne \mu_1(p)$ *and* $u \not\equiv 0$, *then* u *must change sign.*

PROOF. Suppose that u is not changing-sign. Without loss of generality, we can assume that $u \ge 0$ in $(0,1)$. Lemmas 2.1 and 2.3 imply that $u > 0$ in $(0,1)$. Lemma 2.6 implies $\mu = \mu_1(p)$ and $u = cv_1$ for some positive constant c, where v_1 is the positive eigenfunction corresponding to $\mu_1(p)$ with $\|v_1\| = 1$. This is a contradiction. □

In addition, we also have that $\mu_1(p)$ is also isolated.

LEMMA 2.8. $\mu_1(p)$ *is the unique eigenvalue in* $(0, \delta_p)$ *for some* $\delta_p > \mu_1(p)$.

PROOF. Lemma 2.6 has shown that $\mu_1(p)$ is left-isolated. Assume by contradiction that there exists a sequence of eigenvalues $\lambda_n \in (\mu_1(p), \delta_p)$ which converge to $\mu_1(p)$. Let v_n be the corresponding eigenfunctions. Define

$$\psi_n := \frac{v_n}{\left((p-1)\int_0^1 r^{p-2}|v_n|^p\,dr\right)^{1/p}}.$$

Clearly, ψ_n are bounded in $W_c^{1,p}(0,1)$ so there exists a subsequence, denoted again by ψ_n, and $\psi \in W_c^{1,p}(0,1)$ such that $\psi_n \rightharpoonup \psi$ in $W_c^{1,p}(0,1)$ and $\psi_n \to \psi$ in $C^\alpha[0,1]$. Since functional f_1 is sequentially weakly lower semi-continuous, we have that

$$\int_0^1 |\psi'|^p\,dr \le \liminf_{n\to+\infty}\int_0^1 |\psi_n'|^p\,dr = \mu_1^{p-1}(p).$$

On the other hand, $(p-1)\int_0^1 r^{p-2}|\psi_n|^p\,dr = 1$ and $\psi_n \to \psi$ in $C^\alpha[0,1]$ imply that $(p-1)\int_0^1 r^{p-2}|\psi|^p\,dr = 1$. Hence, $\int_0^1 |\psi'|^p\,dr = \eta_1(p)$ via Lemma 2.5. Then Lemmas 2.1 and 2.5 show that $\psi > 0$ in $(0,1)$. Thus $\psi_n \ge 0$ for n large enough which contradicts the conclusion of Lemma 2.7. □

Next, we show that the principle eigenvalue function $\mu_1\colon [2,+\infty) \to \mathbb{R}$ is continuous.

Lemma 2.9. *The eigenvalue function $\mu_1\colon [2,+\infty)\to\mathbb{R}$ is continuous.*

Proof. It is sufficient to show that $\eta_1(p)\colon [2,+\infty)\to\mathbb{R}$ is continuous because of $\mu_1(p) = (\eta_1(p))^{1/(p-1)}$.

From the variational characterization of $\eta_1(p)$ it follows that

$$\eta_1(p) = \sup\left\{\lambda>0 \;\middle|\; \lambda(p-1)\int_0^1 r^{p-2}|v|^p\,dr \le \int_0^1 |v'|^p\,dr \text{ for all } v\in C_c^\infty[0,1]\right\}, \tag{2.4}$$

where $C_c^\infty[0,1] = \{v \in C^\infty[0,1] \mid v'(0) = v(1) = 0\}$, as $C_c^\infty[0,1]$ is dense in $W_c^{1,p}(0,1)$ (see [1]).

Let $\{p_j\}_{j=1}^\infty$ be a sequence in $[2,+\infty)$ which converge to $p\ge 2$. We shall show that

$$\lim_{j\to+\infty}\eta_1(p_j) = \eta_1(p). \tag{2.5}$$

To do this, let $v \in C_c^\infty[0,1]$. Then, due to (2.4), we get that

$$\eta_1(p_j)(p_j-1)\int_0^1 r^{p_j-2}|v|^{p_j}\,dr \le \int_0^1 |v'|^{p_j}\,dr.$$

On applying the Dominated Convergence Theorem we find that

$$\limsup_{j\to+\infty}\eta_1(p_j)(p-1)\int_0^1 r^{p-2}|v|^p\,dr \le \int_0^1 |v'|^p\,dr. \tag{2.6}$$

Relation (2.6), the fact that v is arbitrary and (2.4) yield

$$\limsup_{j\to+\infty} \eta_1(p_j) \le \eta_1(p).$$

Thus, to prove (2.5) it suffices to show that

$$\liminf_{j\to+\infty} \eta_1(p_j) \ge \eta_1(p). \tag{2.7}$$

Let $\{p_k\}_{k=1}^\infty$ be a subsequence of $\{p_j\}_{j=1}^\infty$ such that $\lim\limits_{k\to+\infty} \eta_1(p_k) = \liminf\limits_{j\to+\infty} \eta_1(p_j)$.

Let us fix $\varepsilon_0 > 0$ so that $p - \varepsilon_0 > 1$ and for each $0 < \varepsilon < \varepsilon_0$ and $k \in \mathbb{N}$ large enough, $p - \varepsilon < p_k < p + \varepsilon$. For $k \in \mathbb{N}$, let us choose $v_k \in W_c^{1,p_k}(0,1)$ such that $v_k > 0$ in $(0,1)$,

$$\int_0^1 |v_k'|^{p_k}\, dr = 1 \tag{2.8}$$

and

$$\int_0^1 |v_k'|^{p_k}\, dr = \eta_1(p_k)(p_k - 1) \int_0^1 r^{p_k-2} |v_k|^{p_k}\, dr. \tag{2.9}$$

For $0 < \varepsilon < \varepsilon_0$ and $k \in \mathbb{N}$ large enough, (2.7), (2.8) and (2.9) imply that

$$\begin{aligned} \|v_k\|_{W_c^{1,p_k}(0,1)} \\ \le 1 + \max\left\{ \left(\frac{1}{\lim\limits_{k\to+\infty} \eta_1(p_k)} \right)^{1/(p+\varepsilon)}, \left(\frac{1}{\lim\limits_{k\to+\infty} \eta_1(p_k)} \right)^{1/(p-\varepsilon)} \right\}. \end{aligned} \tag{2.10}$$

This shows that $\{v_k\}_{k=1}^\infty$ is a bounded sequence in $W_c^{1,p_k}(0,1)$, hence, also in $W_c^{1,p-\varepsilon}(0,1)$. Passing to a subsequence if necessary, we can assume that $v_k \rightharpoonup v$ in $W_c^{1,p-\varepsilon}(0,1)$ and hence that $v_k \to v$ in $C^\alpha[0,1]$ with $\alpha = 1 - 1/(p-\varepsilon)$ because the embedding of $W^{1,p-\varepsilon}(0,1) \hookrightarrow C^\alpha[0,1]$ is compact. Thus,

$$|v_k|^{p_k} \to |v|^p. \tag{2.11}$$

We note that (2.9) implies that

$$\eta_1(p_k)(p_k - 1) \int_0^1 r^{p_k-2} |v_k|^{p_k}\, dr = 1 \tag{2.12}$$

for all $k \in \mathbb{N}$. Thus letting $k \to +\infty$ in (2.12) and using (2.11), we find that

$$\liminf_{j\to+\infty} \eta_1(p_j)(p-1) \int_0^1 r^{p-2} |v|^p\, dr = 1. \tag{2.13}$$

On the other hand, since $v_k \rightharpoonup v$ in $W_c^{1,p-\varepsilon}(0,1)$, from (2.8) and the Hölder's inequality we obtain that

$$\|v'\|_{p-\varepsilon}^{p-\varepsilon} \le \liminf_{k\to+\infty} \|v_k'\|_{p-\varepsilon}^{p-\varepsilon} \le 1,$$

where $\|\cdot\|_p$ denotes the normal of $L^p(0,1)$. Now, letting $\varepsilon \to 0^+$, we find

$$\|v'\|_p \le 1. \tag{2.14}$$

Clearly, (2.13), (2.14) and $v \in W_c^{1,p-\varepsilon}(0,1)$ follow that $v \in W_c^{1,p}(0,1)$.

Consequently, combining (2.13) with (2.14) we obtain that

$$\liminf_{j\to+\infty} \eta_1(p_j)(p-1)\int_0^1 r^{p-2}|v|^p\,dr \geq \int_0^1 |v'|^p\,dr.$$

This together with the variational characterization of $\eta_1(p)$ implies (2.7) and hence (2.5). This concludes the proof of the lemma. $\square$

We have known that $I-T_\mu^p$ is a completely continuous vector field in X. Thus, the Leray–Schauder degree $\deg(I-T_\mu^p, B_r(0), 0)$ is well defined for arbitrary r-ball $B_r(0)$ and $\mu \in (0,\delta_p)\setminus\{\mu_1(p)\}$, where δ_p comes from Lemma 2.8. By an argument similar to that of Lemma 4.3 of [7], we can get the following theorem.

THEOREM 2.10. *For fixed $p \geq 2$ and all $r > 0$, we have that*

$$\deg(I-T_\mu^p, B_r(0), 0) = \begin{cases} 1 & \text{if } \mu \in (0,\mu_1(p)), \\ -1 & \text{if } \mu \in (\mu_1(p),\delta_p). \end{cases}$$

3. Global bifurcation result

With a simple transformation $v = -u$, problem (1.3) can be written as

$$\begin{cases} ((-v')^N)' = \lambda^N N r^{N-1}(v^N + g(v)), & r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{3.1}$$

Let $X^+ := \{v \in X \mid v(r) \geq 0\}$ with the norm of X. Define the map $T_g\colon X^+ \to X^+$ by

$$T_g v(r) = \int_r^1 \bigg(\int_0^s N\tau^{N-1}((v(\tau))^N + g(v(\tau)))\,d\tau\bigg)^{1/N} ds, \quad 0 \leq r \leq 1.$$

It is not difficult to verify that T_g is continuous and compact. Clearly, problem (3.1) can be equivalently written as

$$v = \lambda T_g v.$$

Now, we show that the existence and uniqueness theorem is valid for problem (3.1).

LEMMA 3.1. *If (λ, v) is a solution of* (3.1) *in $\mathbb{R}\times X^+$ and v has a double zero, then $v \equiv 0$.*

PROOF. Let v be a solution of problem (3.1) and $r_* \in [0,1]$ be a double zero. We note that

$$v(r) = \lambda \int_r^{r_*} \bigg(\int_{r_*}^s N\tau^{N-1}((v(\tau))^N + g(v(\tau)))\,d\tau\bigg)^{1/N} ds.$$

Firstly, we consider $r \in [0, r_*]$. Then we have that

$$|v(r)| \leq \lambda \bigg(\int_r^{r_*} N\tau^{N-1} |((v(\tau))^N + g(v(\tau)))| \, d\tau \bigg)^{1/N},$$

furthermore,

$$\begin{aligned} |v(r)|^N &\leq \lambda^N \int_r^{r_*} N\tau^{N-1} |((v(\tau))^N + g(v(\tau)))| \, d\tau \\ &\leq \lambda^N \int_r^{r^*} N\tau^{N-1} \bigg| 1 + \frac{g(v(\tau))}{(v(\tau))^N} \bigg| \, |v(\tau)|^N \, d\tau. \end{aligned}$$

According to the assumptions on g, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$|g(s)| \leq \varepsilon s^N \quad \text{for any } s \in [0, \delta].$$

Hence, we have that

$$|v(r)|^N \leq \lambda^N \int_r^{r^*} N \bigg(1 + \varepsilon + \max_{s \in [\delta, \|v\|]} \bigg| \frac{g(s)}{s^N} \bigg| \bigg) |v(\tau)|^N \, d\tau.$$

By the modification of the Gronwall–Bellman inequality [13, Lemma 2.2], we get $v \equiv 0$ on $[0, r^*]$. Similarly, using the Gronwall–Bellman inequality [2], [9], we can get $v \equiv 0$ on $[r^*, 1]$ and the proof is completed. $\square$

Now, we consider the following eigenvalue problem

$$\begin{cases} ((-v')^N)' = \lambda^N N r^{N-1} v^N, & r \in (0, 1), \\ v'(0) = v(1) = 0. \end{cases} \tag{3.2}$$

As Lions [17] showed, the first eigenvalue λ_1 is positive, simple and the corresponding eigenfunctions are positive in $(0, 1)$ and concave on $[0, 1]$. Moreover, we also have the following result.

Lemma 3.2. *If $(\mu, \varphi) \in (0, +\infty) \times (C^2[0, 1] \setminus \{0\})$ satisfies and $\mu \neq \lambda_1$, then φ must change sign.*

Proof. By way of contradiction, we may suppose that φ is not changing-sign. Without loss of generality, we can assume that $\varphi \geq 0$ in $(0, 1)$. Lemma 3.1 follows that $\varphi > 0$ in $(0, 1)$. Theorem 1 of [17] implies $\mu = \lambda_1$ and $\varphi = \theta\psi_1$ for some positive constant θ, where ψ_1 is the positive eigenfunction corresponding to λ_1 with $\|\psi_1\| = 1$. We have a contradiction. $\square$

Next, we show that λ_1 is also isolated.

Lemma 3.3. *λ_1 is isolated; that is to say, λ_1 is the unique eigenvalue in $(0, \delta)$ for some $\delta > \lambda_1$.*

Proof. Theorem 1 of [17] has shown that λ_1 is left-isolated. Assume by contradiction that there exists a sequence of eigenvalues $\lambda_n \in (\lambda_1, \delta)$ which converge to λ_1. Let v_n be the corresponding eigenfunctions. Let $w_n := v_n/\|v_n\|_{C^1[0,1]}$, then w_n should be the solutions of the following problem

$$w = \lambda_n \int_r^1 \bigg(\int_0^s N\tau^{N-1} w^N \, d\tau\bigg)^{1/N} ds.$$

Clearly, w_n are bounded in $C^1[0,1]$ so there exists a subsequence, denoted again by w_n, and $\psi \in X$ such that $w_n \to \psi$ in X. It follows that

$$\psi = \lambda_1 \int_r^1 \bigg(\int_0^s N\tau^{N-1} \psi^N \, d\tau\bigg)^{1/N} ds.$$

Then Theorem 1 of [17] follows that $\psi = \theta\psi_1$ for some positive constant θ in $(0,1)$. Thus $w_n \geq 0$ for n large enough contradicts v_n changing-sign in $(0,1)$ which is implied by Lemma 3.2. □

Define $T_N : X^+ \to X^+$ by

$$T_N v := \int_r^1 \bigg(\int_0^s N\tau^{N-1} v^N \, d\tau\bigg)^{1/N} ds, \quad 0 \leq r \leq 1.$$

Clearly, $I - T_N$ is a completely continuous vector field in X^+. Thus, the Leray–Schauder degree $\deg(I - T_N, B_r(0), 0)$ is well defined for arbitrary r-ball $B_r(0)$ of X^+ and $\mu \in (0,\delta) \setminus \{\lambda_1\}$, where δ comes from Lemma 3.3.

Lemma 3.4. *Let λ be a constant with $\lambda \in (0,\delta)$. Then, for arbitrary $r > 0$,*

$$\deg(I - \lambda T_N, B_r(0), 0) = \begin{cases} 1 & \text{if } \lambda \in (0,\lambda_1), \\ -1 & \text{if } \lambda \in (\lambda_1, \delta). \end{cases}$$

Proof. Taking $p = N + 1$ and $\mu = \lambda$ in T_μ^p, we can see that $\lambda_1 = \mu_1(p)$. Furthermore, it is not difficult to verify that $\lambda T_N(v) = T_\mu^p(v)$ for any $v \in X^+$. By Theorem 2.10, we can deduce this lemma. □

Theorem 3.5. *$(\lambda_1, 0)$ is a bifurcation point of (3.1) and the associated bifurcation branch $\mathcal{C}$ in $\mathbb{R} \times X^+$ whose closure contains $(\lambda_1, 0)$ is either unbounded or contains a pair $(\overline{\lambda}, 0)$ where $\overline{\lambda}$ is an eigenvalue of (3.2) and $\overline{\lambda} \neq \lambda_1$.*

Proof. Suppose that $(\lambda_1, 0)$ is not a bifurcation point of problem (3.1). Then there exist $\varepsilon > 0$, $\rho_0 > 0$ such that for $|\lambda - \lambda_1| \leq \varepsilon$ and $0 < \rho < \rho_0$ there is no nontrivial solution of the equation

$$v - \lambda T_g v = 0$$

with $\|v\| = \rho$. From the invariance of the degree under a compact homotopy we obtain that

$$\deg(I - \lambda T_g, B_\rho(0), 0) \equiv \text{constant} \quad \text{for } \lambda \in [\lambda_1 - \varepsilon, \lambda_1 + \varepsilon]. \tag{3.3}$$

By taking ε smaller if necessary, in view of Lemma 3.3, we can assume that there is no eigenvalue of (3.2) in $(\lambda_1, \lambda_1 + \varepsilon]$. Fix $\lambda \in (\lambda_1, \lambda_1 + \varepsilon]$. We claim that the equation

$$v - \lambda \int_r^1 \left(\int_0^s N\tau^{N-1}(v^N + tg(v))\, d\tau \right)^{1/N} ds = 0 \tag{3.4}$$

has no solution v with $\|v\| = \rho$ for every $t \in [0,1]$ and ρ sufficiently small. Suppose on the contrary, let $\{v_n\}$ be the nontrivial solutions of equation (3.4) with $\|v_n\| \to 0$ as $n \to +\infty$.

Let $w_n := v_n/\|v_n\|$, then w_n should be the solutions of the following problem

$$w(t) = \lambda \int_r^1 \left(\int_0^s N\tau^{N-1} \left(w^N + t\frac{g(v)}{\|v_n\|^N} \right) d\tau \right)^{1/N} ds. \tag{3.5}$$

Let $\widetilde{g}(v) = \max\limits_{0 \le s \le v} |g(s)|$, then $\widetilde{g}$ is nondecreasing with respect to v and

$$\lim_{v \to 0^+} \frac{\widetilde{g}(v)}{v^N} = 0. \tag{3.6}$$

Further it follows from (3.6) that

$$\frac{|g(v)|}{\|v\|^N} \le \frac{\widetilde{g}(v)}{\|v\|^N} \le \frac{\widetilde{g}(\|v\|)}{\|v\|^N} \to 0 \quad \text{as } \|v\| \to 0. \tag{3.7}$$

By (3.5), (3.7) and the compactness of T_g, we obtain that for some convenient subsequence $w_n \to w_0$ as $n \to +\infty$. Now (λ, w_0) verifies problem (3.2) and $\|w_0\| = 1$. This implies that λ is an eigenvalue of (3.2). This is a contradiction.

From the invariance of the degree under homotopies and Lemma 3.4 we then obtain that

$$\deg(I - \lambda T_g(\,\cdot\,), B_r(0), 0) = \deg(I - \lambda T_N(\,\cdot\,), B_r(0), 0) = -1. \tag{3.8}$$

Similarly, for $\lambda \in [\lambda_1 - \varepsilon, \lambda_1)$ we find that

$$\deg(I - \lambda T_g(\,\cdot\,), B_r(0), 0) = 1. \tag{3.9}$$

Relations (3.8) and (3.9) contradict (3.3) and hence $(\lambda_1, 0)$ is a bifurcation point of problem (3.1).

By standard arguments in global bifurcation theory (see [23]), we can show the existence of a global branch of solutions of problem (3.1) emanating from $(\lambda_1, 0)$. Our conclusion is proved. □

Next, we shall prove that the first choice of the alternative of Theorem 3.5 in X^+ which are positive in (0,1). Set $K^+ = \mathbb{R} \times P^+$ under the product topology.

THEOREM 3.6. *There exists an unbounded continuum $\mathcal{C} \subseteq (K^+ \cup \{(\lambda_1, 0)\})$ of solutions to problem* (3.1) *emanating from* $(\lambda_1, 0)$.

Proof. For any $(\lambda, v) \in \mathcal{C}$, Lemma 3.1 implies that either $v \equiv 0$ or $v > 0$ in (0,1). Thus, we have $\mathcal{C} \subseteq (K^+ \cup \{(\lambda_1, 0)\})$.

Now, we prove that the first choice of the alternative of Theorem 3.5 when $n \to +\infty$ with $(\lambda_n, v_n) \in \mathcal{C}$, $v_n \not\equiv 0$ and $\overline{\lambda}$ is another eigenvalue of (3.2). Let $w_n := v_n/\|v_n\|$, then w_n should be the solutions of the following problem

$$w = \lambda_n \int_r^1 \bigg(\int_0^s N\tau^{N-1} \bigg(w^N + \frac{g(v)}{\|v_n\|^N} \bigg) \, d\tau \bigg)^{1/N} ds. \tag{3.10}$$

By an argument similar to that of Theorem 3.5, we obtain that for some convenient subsequence $w_n \to w_0$ as $n \to +\infty$. It is easy to see that $(\overline{\lambda}, w_0)$ verifies problem (3.2) and $\|w_0\| = 1$. Lemma 3.2 follows w_0 must change sign, and as a consequence for some n large enough, w_n must change sign, and this is a contradiction. $\square$

Remark 3.7. Clearly, the proof of Theorem 3.6 also shows that $(\lambda_1, 0)$ is the unique bifurcation point from $(\lambda, 0)$ of the positive solutions of problem (3.1).

Finally, we give a key lemma that will be used later.

Lemma 3.8. Let $b_2(r) \geq b_1(r) > 0$ for $r \in (0,1)$ and $b_i(r) \in C([0,1])$, $i = 1, 2$. Also let u_1, u_2 be solutions of the following differential problems

$$\begin{cases} ((-u')^N)' = b_i(r) u^N, \quad i = 1, 2, \\ u'(0) = u(1) = 0, \end{cases}$$

respectively. If $u_1(r) \neq 0$ in $(0,1)$, then either there exists $\tau \in (0,1)$ such that $u_2(\tau) = 0$ or $b_2 = b_1$ and $u_2(r) = \mu u_1(r)$ for some constant $\mu \neq 0$ and almost every $r \in (0,1)$.

Proof. If $u_2(r) \neq 0$ in $(0,1)$, then we can assume without loss of generality that $u_1(r) > 0$, $u_2(r) > 0$ in $(0,1)$. Then from problem (3.11), we can easily show that u_1 and u_2 are strictly decreasing concave functions in $(0,1)$. Moreover, it is easy to check that the conclusion of Lemma 3.1 is also valid for problem (3.11). So we have $u_1'(r) < 0$ and $u_2'(r) < 0$ for $r \in (0,1]$.

By some simple calculations, we have that

$$\begin{aligned} &\int_0^1 \bigg(\frac{u_1^{N+1}(-u_2')^N}{u_2^N} - u_1(-u_1')^N \bigg)' dr \\ &= \int_0^1 \bigg(w u_1^{N+1} + \bigg((-u_1')^{N+1} + N \bigg(\frac{-u_1 u_2'}{u_2} \bigg)^{N+1} - (N+1) u_1^N u_1' \bigg(\frac{-u_2'}{u_2} \bigg)^N \bigg) \bigg) dr, \end{aligned}$$

where $w = b_2 - b_1$. The left-hand side of (3.12) equals

$$\lim_{r \to 1^-} \frac{u_1^{N+1}(-u_2')^N}{u_2^N} := H.$$

We prove that $H = 0$. By L'Hospital rule, we have that

$$\begin{aligned} H &= \lim_{r\to 1^-} \frac{u_1^{N+1}(-u_2')^N}{u_2^N} = \lim_{r\to 1^-} \frac{(N+1)u_1^N u_1'(-u_2')^N + u_1^{N+1}((-u_2')^N)'}{N u_2^{N-1} u_2'} \\ &= \lim_{r\to 1^-} \frac{(N+1)u_1^N u_1'(-u_2')^N + u_1^{N+1} b_2 u_2^N}{N u_2^{N-1} u_2'} \\ &= \lim_{r\to 1^-} \frac{(N+1)u_1^N u_1'(-u_2')^N}{N u_2^{N-1} u_2'} + \lim_{r\to 1^-} \frac{u_1^{N+1} b_2 u_2^N}{N u_2^{N-1} u_2'} \\ &= \lim_{r\to 1^-} \frac{(N+1)u_1'(-u_2')^N}{N u_2'} \lim_{r\to 1^-} \frac{u_1^N}{u_2^{N-1}}. \end{aligned}$$

If $N = 1$, then $H = 0$. If $1 < N \le 2$, applying the L'Hospital rule again, we obtain that

$$\lim_{r\to 1^-} \frac{u_1^N}{u_2^{N-1}} = \lim_{r\to 1^-} \frac{N u_1'}{(N-1)u_2'} \lim_{r\to 1^-} \frac{u_1^{N-1}}{u_2^{N-2}}.$$

This implies that $H = 0$. If $k - 1 < N \le k$, then we continue this process k times to obtain $H = 0$.

Therefore, the left-hand side of (3.12) equals zero. Hence the right-hand side of (3.12) also equals zero. The Young's inequality implies that

$$(-u_1')^{N+1} + N\left(\frac{-u_1 u_2'}{u_2}\right)^{N+1} - (N+1)u_1^N u_1'\left(\frac{-u_2'}{u_2}\right)^N \ge 0,$$

and the equality holds if and only if

$$\left(\frac{-u_1'}{u_1}\right)^{N+1} = \left(\frac{-u_2'}{u_2}\right)^{N+1}.$$

It follows that there exists a constant $\mu \ne 0$ such that $u_2 = \mu u_1$ and $b_2 = b_1$. □

As an immediate consequence, we obtain the following Sturm type comparison lemma.

LEMMA 3.9. *Let $b_i(r) \in C([0,1])$, $i = 1, 2$ such that $b_2(r) \ge b_1(r) > 0$ for $r \in (0,1)$ and the inequality is strict on some subset of positive measure in $(0,1)$. Also let u_1, u_2 are solutions of (3.11) with $i = 1, 2$, respectively. If $u_1 \ne 0$ in $(0,1)$, then u_2 has at least one zero in $(0,1)$.*

4. Convex solutions

In this section, we shall investigate the existence and multiplicity of convex solutions of problem (1.2). With a simple transformation $v = -u$, problem (1.2) can be written as

$$\begin{cases} ((-v'(r))^N)' = \lambda^N N r^{N-1} f(v(r)), & r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{4.1}$$

Define the map $T_f\colon X^+ \to X^+$ by

$$T_f v(r) = \int_r^1 \left(\int_0^s N\tau^{N-1} f(v(r))\, d\tau \right)^{1/N} ds, \quad 0 \leq r \leq 1.$$

Similar to T_g, T_f is continuous and compact. Clearly, problem (4.1) can be equivalently written as

$$v = \lambda T_f v.$$

Let $f_0, f_\infty \in \mathbb{R} \setminus \mathbb{R}^-$ be such that

$$f_0^N = \lim_{s\to 0^+} \frac{f(s)}{s^N} \quad \text{and} \quad f_\infty^N = \lim_{s\to+\infty} \frac{f(s)}{s^N}.$$

Throughout this section, we always suppose that f satisfies the following signum condition

(f1) $f \in C(\mathbb{R} \setminus \mathbb{R}^-, \mathbb{R} \setminus \mathbb{R}^-)$ *with* $f(s)s^N > 0$ *for* $s > 0$.

Applying Theorem 3.6, we shall establish the existence of convex solutions of (1.2) as follows.

Theorem 4.1. *If* $f_0 \in (0, +\infty)$ *and* $f_\infty \in (0, +\infty)$, *then for any* $\lambda \in (\lambda_1/f_\infty, \lambda_1/f_0)$ *or* $\lambda \in (\lambda_1/f_0, \lambda_1/f_\infty)$, (1.2) *has at least one solution* u *such that it is negative, strictly convex in* $(0,1)$.

Proof. It suffices to prove that (4.1) has at least one solution v such that it is positive, strictly concave in $(0,1)$.

Clearly, $f_0 \in (0, +\infty)$ implies $f(0) = 0$. Hence, $v = 0$ is always the solution of problem (4.1). Let $\zeta \in C(\mathbb{R} \setminus \mathbb{R}^-, \mathbb{R} \setminus \mathbb{R}^-)$ be such that $f(s) = f_0^N s^N + \zeta(s)$ with $\lim_{s\to 0^+} \zeta(s)/s^N = 0$. Applying Theorem 3.6 to (4.1), we have that there exists an unbounded continuum $\mathcal{C}$ emanating from $(\lambda_1/f_0, 0)$, such that

$$\mathcal{C} \subseteq (\{(\lambda_1, 0)\} \cup (\mathbb{R} \times P^+)).$$

To complete this theorem, it will be enough to show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$. Let $(\mu_n, v_n) \in \mathcal{C}$ satisfy $\mu_n + \|v_n\| \to +\infty$. We note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since (0,0) is the only solution of (4.1) for $\lambda = 0$ and $\mathcal{C} \cap (\{0\} \times X^+) = \emptyset$.

We divide the rest of proofs into two steps.

Step 1. We show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim\limits_{n\to+\infty} \mu_n = +\infty$. On the other hand, we note that

$$((-v_n'(r))^N)' = \mu_n^N N r^{N-1} \widetilde{f}_n(r) v_n^N,$$

where

$$\widetilde{f}_n(r) = \begin{cases} \dfrac{f(v_n)}{v_n^N} & \text{if } v_n \neq 0, \\ f_0^N & \text{if } v_n = 0. \end{cases}$$

The signum condition (f1) implies that there exists a positive constant ϱ such that $\widetilde{f}_n(r) \geq \varrho$ for any $r \in [0,1]$. By Lemma 3.9, we get v_n must change sign in $(0,1)$ for n large enough, and this contradicts the fact that $v_n \in \mathcal{C}$.

Step 2. We show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$.

It follows from Step 1 that $\|v_n\| \to +\infty$. Let $\xi \in C(\mathbb{R} \setminus \mathbb{R}^-, \mathbb{R} \setminus \mathbb{R}^-)$ be such that $f(s) = f_\infty^N s^N + \xi(s)$. Then $\lim_{s\to+\infty} \xi(s)/s^N = 0$. Let $\widetilde{\xi}(v) = \max_{0\leq s\leq v} |\xi(s)|$. Then $\widetilde{\xi}$ is nondecreasing. Set $\overline{\xi}(v) = \max_{v/2\leq s\leq v} |\xi(s)|$. Then we have

$$\lim_{v\to+\infty} \frac{\overline{\xi}(v)}{v^N} = 0 \text{ and } \widetilde{\xi}(v) \leq \widetilde{\xi}\left(\frac{v}{2}\right) + \overline{\xi}(v).$$

It follows that

$$\lim_{v\to+\infty} \frac{\widetilde{\xi}(v)}{v^N} = 0. \tag{4.2}$$

We divide the equation

$$((-v_n')^N)' - \mu_n^N f_\infty^N r^{N-1} v_n^N = \mu_n^N r^{N-1} \xi(v_n)$$

by $\|v_n\|^N$ and set $\overline{v}_n = v_n/\|v_n\|$. Since $\overline{v}_n$ are bounded in X^+, after taking a subsequence if necessary, we have that $\overline{v}_n \rightharpoonup \overline{v}$ for some $\overline{v} \in X^+$. Moreover, from (4.2) and the fact that $\widetilde{\xi}$ is nondecreasing, we have that

$$\lim_{n\to+\infty} \frac{\xi(v_n(r))}{\|v_n\|^N} = 0 \tag{4.3}$$

since

$$\frac{|\xi(v_n(r))|}{\|v_n\|^N} \leq \frac{\widetilde{\xi}(|v_n(r)|)}{\|v_n\|^N} \leq \frac{\widetilde{\xi}(\|v_n(r)\|)}{\|v_n\|^N}.$$

By the continuity and compactness of T_f, it follows that

$$((-\overline{v}')^N)' - \overline{\lambda}^N f_\infty^N r^{N-1} \overline{v}^N = 0,$$

where $\overline{\lambda} = \lim_{n\to+\infty} \lambda_n$, again choosing a subsequence and relabeling it if necessary.

It is clear that $\|\overline{v}\| = 1$ and $\overline{v} \in \overline{\mathcal{C}} \subseteq \mathcal{C}$ since $\mathcal{C}$ is closed in $\mathbb{R} \times X^+$. Therefore, $\overline{\lambda} f_\infty = \lambda_1$, so $\overline{\lambda} = \lambda_1/f_\infty$. Therefore, $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$. □

REMARK 4.2. From the proof of Theorem 4.1, we can see that if $f_0, f_\infty \in (0,+\infty)$ then there exist $\lambda_2 > 0$ and $\lambda_3 > 0$ such that (1.2) has at least one strictly convex solution for all $\lambda \in (\lambda_2, \lambda_3)$ and has no convex solution for all $\lambda \in (0, \lambda_2) \cup (\lambda_3, +\infty)$.

PROOF. Clearly, $f_0, f_\infty \in (0,+\infty)$ implies that there exists a positive constant M such that

$$\left|\frac{f(s)}{s^N}\right| \leq M \quad \text{for any } s > 0.$$

It is sufficient to show that there exists $\lambda_2 > 0$ such that (1.2) has no convex solution for all $\lambda \in (0, \lambda_2)$. Suppose on the contrary that there exists one pair $(\mu, v) \in \mathcal{C}$ such that $\mu \in (0, 1/M^{1/N})$. Let $w = v/\|v\|$. Obviously, one has that

$$1 = \|w\| = \left\|\mu \int_r^1 \left(\int_0^s N\tau^{N-1}\left(\frac{f(v)}{\|v\|^N}\right) d\tau\right)^{1/N} ds\right\| \leq M^{1/N}\mu < 1.$$

This is a contradiction. □

From the proof of Theorem 4.1 and Remark 4.2, we can deduce the following two corollaries.

Corollary 4.3. *Assume that there exists a positive constant $\rho > 0$ such that*

$$\frac{f(s)}{s^N} \geq \rho \quad \textit{for any } s > 0.$$

Then there exists $\zeta_* > 0$ such that problem (1.2) *has no convex solution for any $\lambda \in (\zeta_*, +\infty)$.*

Corollary 4.4. *Assume that there exists a positive constant $\varrho > 0$ such that*

$$\left|\frac{f(s)}{s^N}\right| \leq \varrho \quad \textit{for any } s > 0.$$

Then there exists $\eta_* > 0$ such that problem (1.2) *has no convex solution for any $\lambda \in (0, \eta_*)$.*

Theorem 4.5. *If $f_0 \in (0, +\infty)$ and $f_\infty = 0$, then for any $\lambda \in (\lambda_1/f_0, +\infty)$,* (1.2) *has at least one solution u such that it is negative, strictly convex in* $(0, 1)$.

Proof. In view of Theorem 4.1, we only need to show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(+\infty, +\infty)$. Suppose on the contrary that there exists μ_M such that $(\mu_M, 0)$ is a blow up point (see Definition 1.1 of [24]) and $\mu_M < +\infty$. Then there exists a sequence $\{\mu_n, v_n\}$ such that $\lim_{n\to+\infty} \mu_n = \mu_M$ and $\lim_{n\to+\infty} \|v_n\| = +\infty$ as $n \to +\infty$. Let $w_n = v_n/\|v_n\|$ and w_n should be the solutions of the following problem

$$w = \mu_n \int_r^1 \left(\int_0^s N\tau^{N-1}\left(\frac{f(v_n)}{\|v_n\|^N}\right) d\tau\right)^{1/N} ds.$$

Similar to (4.3), we can show that

$$\lim_{n\to+\infty} \frac{f(v_n(r))}{\|v_n\|^N} = 0.$$

By the compactness of T_f, we obtain that for some convenient subsequence $w_n \to w_0$ as $n \to +\infty$. Letting $n \to +\infty$, we obtain that $w_0 \equiv 0$. This contradicts $\|w_0\| = 1$. □

REMARK 4.6. Under the assumptions of Theorem 4.5, in view of Corollary 4.4, we can see that there exists $\lambda_4 > 0$ such that problem (1.2) has at least one strictly convex solution for all $\lambda \in (\lambda_4, +\infty)$ and has no convex solution for all $\lambda \in (0, \lambda_4)$.

THEOREM 4.7. *If $f_0 \in (0, +\infty)$ and $f_\infty = \infty$, then for any $\lambda \in (0, \lambda_1/f_0)$, (1.2) has at least one solution u such that it is negative, strictly convex in $(0,1)$.*

PROOF. Considering of the proof of Theorem 4.1, we only need to show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(0, +\infty)$. Clearly, $f_\infty = +\infty$ implies that $f(s) \geq M^N s^N$ for some positive constant M and s large enough.

To complete the proof, it suffices to show that the unique blow up point of $\mathcal{C}$ is $\lambda = 0$. Suppose on the contrary that there exists $\widehat{\lambda} > 0$ such that $(\widehat{\lambda}, 0)$ is a blow up point of $\mathcal{C}$. Then there exists a sequence $\{\lambda_n, v_n\}$ such that $\lim_{n\to+\infty} \lambda_n = \widehat{\lambda}$ and $\lim_{n\to+\infty} \|v_n\| = +\infty$. Let $w_n = v_n/\|v_n\|$. Clearly, one has that

$$w_n = \lambda_n \int_r^1 \left(\int_0^s N\tau^{N-1} \left(\frac{f(v_n)}{v_n^N} \frac{v_n^N}{\|v_n\|^N} \right) d\tau \right)^{1/N} ds.$$

Take $M = 64/\widehat{\lambda} + 1$. For $r \in [1/4, 3/4]$, by virtue of Lemma 2.3 of [11], we have that

$$\begin{aligned}
|w_n| &\geq M\lambda_n \int_r^1 \left(\int_0^s N\tau^{N-1} |w_n|^N \, d\tau \right)^{1/N} ds \\
&\geq M\|w_n\|\lambda_n \int_r^1 \left(\int_0^s N\tau^{N-1} (1-\tau)^N \, d\tau \right)^{1/N} ds \\
&\geq M\|w_n\|\lambda_n (1-r) \left(\int_0^r N\tau^{N-1} (1-\tau)^N \, d\tau \right)^{1/N} \\
&\geq M\|w_n\|\lambda_n (1-r)^2 \left(\int_0^r N\tau^{N-1} \, d\tau \right)^{1/N} \\
&\geq M\|w_n\|\lambda_n r(1-r)^2 \geq \frac{M\|w_n\|\lambda_n}{64}.
\end{aligned} \tag{4.4}$$

It is obvious that (4.4) follows $M\lambda_n \leq 64$. Thus, we get that $M \leq 64/\widehat{\lambda}$. While, this is impossible because of $M = 64/\widehat{\lambda} + 1$. □

REMARK 4.8. Clearly, Theorem 4.7 and Corollary 4.3 imply that if $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$ then there exists $\lambda_5 > 0$ such that (1.2) has at least one strictly convex solution for all $\lambda \in (0, \lambda_5)$ and has no convex solution for all $\lambda \in (\lambda_5, +\infty)$.

THEOREM 4.9. *If $f_0 = 0$ and $f_\infty \in (0, +\infty)$, then for any $\lambda \in (\lambda_1/f_\infty, +\infty)$, (1.2) has at least one solution u such that it is negative, strictly convex in $(0,1)$.*

Proof. If (λ, v) is any solution of (4.1) with $\|v\| \not\equiv 0$, dividing (4.1) by $\|v\|^{2N}$ and setting $w = v/\|v\|^2$ yield

$$
\begin{cases} ((-w'(r))^N)' = \lambda^N N r^{N-1}\left(\dfrac{f(v)}{\|v\|^{2N}}\right), & r \in (0,1), \\ w'(0) = w(1) = 0. \end{cases} \tag{4.5}
$$

Define

$$
\widetilde{f}(w) = \begin{cases} \|w\|^{2N} f\left(\dfrac{w}{\|w\|^2}\right) & \text{if } w \neq 0, \\ 0 & \text{if } w = 0. \end{cases} \tag{4.6}
$$

Clearly, (4.5) is equivalent to

$$
\begin{cases} ((-w'(r))^N)' = \lambda^N r^{N-1} \widetilde{f}(w), & r \in (0,1), \\ w'(0) = w(1) = 0. \end{cases} \tag{4.7}
$$

It is obvious that $(\lambda, 0)$ is always the solution of (4.7). By the simple computation, we can show that $\widetilde{f}_0 = f_\infty$ and $\widetilde{f}_\infty = f_0$.

Now applying Theorem 4.5 and the inversion $w \to w/\|w\|^2 = v$, we can achieve our conclusion. $\square$

Remark 4.10. Under the assumptions of Theorem 4.9, we note there exists $\lambda_6 > 0$ such that (1.2) has at least one strictly convex solution for all $\lambda \in (\lambda_6, +\infty)$ and has no convex solution for all $\lambda \in (0, \lambda_6)$.

Next, we shall need the following topological lemma.

Lemma 4.11 (see [19]). *Let X be a Banach space and let C_n be a family of closed connected subsets of X. Assume that:*

(a) *there exist $z_n \in C_n$, $n = 1, 2, \ldots$, and $z^* \in X$, such that $z_n \to z^*$;*
(b) *$r_n = \sup\{\|x\| \mid x \in C_n\} = +\infty$;*
(c) *for every $R > 0$, $\left(\bigcup_{n=1}^{+\infty} C_n\right) \cap B_R$ is a relatively compact set of X, where*

$$
B_R = \{x \in X \mid \|x\| \leq R\}.
$$

Then there exists an unbounded component $\mathfrak{C}$ in $\mathfrak{D} = \limsup_{n\to+\infty} C_n$ and $z^* \in \mathfrak{C}$.

Theorem 4.12. *If $f_0 = 0$ and $f_\infty = 0$, then there exists $\lambda_* > 0$ such that for any $\lambda \in (\lambda_*, +\infty)$, (1.2) has at least two solutions u_1 and u_2 such that they are negative, strictly convex in $(0,1)$.*

PROOF. Define

$$f^n(s) = \begin{cases} \dfrac{1}{n^N} s^N, & s \in \left[0, \dfrac{1}{n}\right], \\ \left(f\left(\dfrac{2}{n}\right) - \dfrac{1}{n^{2N}}\right) ns + \dfrac{2}{n^{2N}} - f\left(\dfrac{2}{n}\right), & s \in \left(\dfrac{1}{n}, \dfrac{2}{n}\right), \\ f(s), & s \in \left[\dfrac{2}{n}, +\infty\right). \end{cases}$$

Now, consider the following problem

$$\begin{cases} ((u'(r))^N)' = \lambda^N N r^{N-1} f^n(-u(r)) & \text{in } 0 < r < 1, \\ u'(0) = u(1) = 0. \end{cases}$$

Clearly, we can see that $\lim_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = 1/n$ and $f_\infty^n = f_\infty = 0$. Theorem 4.5 implies that there exists a sequence unbounded continua $\mathcal{C}_n$ emanating from $(n\lambda_1, 0)$ and joining to $(+\infty, +\infty)$.

Taking $z_n^1 = (n\lambda_1, 0)$ and $z_n^2 = (+\infty, +\infty)$, we have $z_n^1, z_n^2 \in \mathcal{C}_n$ and $z_n^1 \to (+\infty, 0)$, $z_n^2 \to (+\infty, +\infty)$. The compactness of T_f implies that $\left(\bigcup_{n=1}^{+\infty} \mathcal{C}_n\right) \cap B_R$ is pre-compact. So Lemma 4.11 implies that there exists an unbounded component $\mathcal{C}$ of $\limsup_{n\to+\infty} \mathcal{C}_n$ such that $(+\infty, 0) \in \mathcal{C}$ and $(+\infty, +\infty) \in \mathcal{C}$. By an argument similar to that of Theorem 4.5, we can show that $\mathcal{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$. □

REMARK 4.13. From Theorem 4.12 and Corollary 4.4, we can also see that there exists $\lambda_7 > 0$ such that (1.2) has at least one strictly convex solution for all $\lambda \in [\lambda_7, \lambda_*]$ and has no convex solution for all $\lambda \in (0, \lambda_7)$.

THEOREM 4.14. *If $f_0 = 0$ and $f_\infty = \infty$, then for any $\lambda \in (0, +\infty)$,* (1.2) *has at least one solution u such that it is negative, strictly convex in* $(0, 1)$.

PROOF. Using an argument similar to that of Theorem 4.12, in view of the conclusion of Theorem 4.7, we can easily get the results of this theorem. □

THEOREM 4.15. *If $f_0 = \infty$ and $f_\infty = 0$, then for any $\lambda \in (0, +\infty)$,* (1.2) *has at least one solution u such that it is negative, strictly convex in* $(0, 1)$.

PROOF. By an argument similar to that of Theorem 4.9 and the conclusions of Theorem 4.13, we can prove it. □

THEOREM 4.16. *If $f_0 = \infty$ and $f_\infty \in (0, +\infty)$, then for any $\lambda \in (0, \lambda_1/f_\infty)$,* (1.2) *has at least one solution u such that it is negative, strictly convex in* $(0, 1)$.

PROOF. By an argument similar to that of Theorem 4.9 and the conclusion of Theorem 4.7, we can obtain it. □

Remark 4.17. Similarly to Remark 4.8, there exists $\lambda_8 > 0$ such that (1.2) has at least one strictly convex solution for all $\lambda \in (0, \lambda_8)$ and has no convex solution for all $\lambda \in (\lambda_8, +\infty)$.

Theorem 4.18. *If $f_0 = \infty$ and $f_\infty = \infty$, then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$,* (1.2) *has at least two solutions u_1 and u_2 such that they are negative, strictly convex in* $(0, 1)$.

Proof. Define

$$f^n(s) = \begin{cases} n^N s^N, & s \in \left[0, \dfrac{1}{n}\right], \\ \left(f\left(\dfrac{2}{n}\right) - 1\right) ns + 2 - f\left(\dfrac{2}{n}\right), & s \in \left(\dfrac{1}{n}, \dfrac{2}{n}\right), \\ f(s), & s \in \left[\dfrac{2}{n}, +\infty\right). \end{cases}$$

By the conclusions of Theorem 4.7 and an argument similar to that of Theorem 4.12, we can prove there exists an unbounded component $\mathcal{C}$ of solutions of problem (1.2) such that $(0, 0) \in \mathcal{C}$ and $(0, +\infty) \in \mathcal{C}$. By an argument similar to that of Theorem 4.7, we can show that $\mathcal{C} \cap ((0, +\infty) \times \{0\}) = \emptyset$. By arguments similar to those of Theorems 4.7 and 4.12, we can show that there exists $\mu_* > 0$ such that $\mathcal{C} \cap ((\mu_*, +\infty) \times X^+) = \emptyset$. □

Remark 4.19. By Theorem 4.17 and Corollary 4.3, we can see that there exists $\lambda_9 > 0$ such that (1.2) has at least one strictly convex solution for all $\lambda \in [\lambda^*, \lambda_9]$ and has no convex solution for all $\lambda \in (\lambda_9, +\infty)$.

Remark 4.20. Clearly, the conclusions of Theorem 1.1 of [26] and Theorem 5.1 of [11] are the corollaries of Theorems 4.1, 4.5, 4.7, 4.9, 4.12–4.15, 4.17.

Remark 4.21. Let $f(s) = e^s$. It can be easily verified that $f_0 = \infty$ and $f_\infty = \infty$. This fact with Remark 4.18 implies that there is no solution of problem (1.2) with λ large enough, and for sufficiently small λ there are two strictly convex solutions. Set $\mu := \lambda^{1/2}$. Through a scaling, we can show that problem (1.2) is equivalent to

$$\begin{cases} \det(D^2 u) = e^{-u} & \text{in } B_\mu(0), \\ u = 0 & \text{on } \partial B_\mu(0), \end{cases} \tag{4.8}$$

where $B_\mu(0)$ denotes the set of $\{x \in \mathbb{R}^N \mid |x| \leq \mu\}$. Hence there is no solution of problem (4.8) with μ large enough, and for sufficiently small μ there are two strictly convex solutions. Obviously, this result improves the corresponding one of [27, Theorem 3.1]. So Theorem 3.1 of [27] is our corollary of Theorem 4.17.

Remark 4.22. Obviously, the results of Theorems 4.1, 4.5, 4.7, 4.9, 4.12–4.15 and 4.17 are also valid on $B_R(0)$ for any $R > 0$.

5. Exact multiplicity of convex solutions

In this section, under some more strict assumptions of f, we shall show that the unbounded continuum which are obtained in Section 4 may be smooth curves. We just show the case of $f_0 \in (0, +\infty)$ and $f_\infty = 0$. Other cases are similar.

Firstly, we study the local structure of the bifurcation branch $\mathcal{C}$ near $(\lambda_1, 0)$, which is obtained in Theorem 3.5. Let $\mathbb{E} = \mathbb{R} \times X^+$, $\Phi(\lambda, v) := v - \lambda T_g(v)$ and

$$\mathcal{S} := \overline{\{(\lambda, v) \in \mathbb{E} \mid \Phi(\lambda, v) = 0, v \neq 0\}}^{\mathbb{E}}.$$

In order to formulate and prove main results of this section, it is convenient to introduce López–Gómez's notations [18]. Given any $\lambda \in \mathbb{R}$ and $0 < s < +\infty$, we consider an open neighbourhood of $(\lambda_1, 0)$ in $\mathbb{E}$ defined by

$$\mathbb{B}_s(\lambda_1, 0) := \{(\lambda, v) \in \mathbb{E} \mid \|v\| + |\lambda - \lambda_1| < s\}.$$

Let X_0 be a closed subspace of X such that $X = \operatorname{span}\{\psi_1\} \oplus X_0$. According to the Hahn–Banach theorem, there exists a linear functional $l \in (X^+)*$, here $(X^+)*$ denotes the dual space of X^+, such that

$$l(\psi_1) = 1 \quad \text{and} \quad X_0 = \{v \in X^+ \mid l(v) = 0\}.$$

Finally, for any $0 < \varepsilon < +\infty$ and $0 < \eta < 1$, we define

$$K^+_{\varepsilon,\eta} := \{(\lambda, v) \in \mathbb{E} \mid |\lambda - \lambda_1| < \varepsilon,\ l(v) > \eta\|v\|\}.$$

Applying an argument similar to that of [18, Lemma 6.4.1], we may obtain the following result, which localizes the possible solutions of (1.3) bifurcating from $(\lambda_1, 0)$.

LEMMA 5.1. *For every $\eta \in (0, 1)$ there exists a number $\delta_0 > 0$ such that, for each $0 < \delta < \delta_0$,*

$$((\mathcal{S} \setminus \{(\lambda_1, 0)\}) \cap \mathbb{B}_\delta(\lambda_1, 0)) \subset K^+_{\varepsilon,\eta}.$$

Moreover, for each $(\lambda, v) \in (\mathcal{S} \setminus \{(\lambda_1, 0)\}) \cap (\mathbb{B}_\delta(\lambda_1, 0))$, there are $s \in \mathbb{R}$ and unique $y \in X_0$ such that

$$v = s\psi_1 + y \quad \textit{and} \quad s > \eta\|v\|.$$

Furthermore, for these solutions (λ, v), $\lambda = \lambda_1 + o(1)$ and $y = o(s)$ as $s \to 0^+$.

REMARK 5.2. From Lemma 5.1, we can see that $\mathcal{C}$ near $(\lambda_1, 0)$ is given by a curve $(\lambda(s), v(s)) = (\lambda_1 + o(1), s\psi_1 + o(s))$ for s near 0^+.

The primary result in this section is the following theorem.

THEOREM 5.3. *Let $f \in C^1(\mathbb{R} \setminus \mathbb{R}^-, \mathbb{R} \setminus \mathbb{R}^-)$ satisfy the assumptions of Theorem 4.5. Suppose $f'(s) < Nf(s)/s$ for any $s > 0$. Then for any $\lambda \in (\lambda_1/f_0, +\infty)$,* (1.2) *has exactly one solution u such that it is negative, strictly convex in* $(0, 1)$.

Remark 5.4. Clearly, the assumption $f'(s) < Nf(s)/s$ for $s > 0$ is equivalent to $f(s)/s^N$ is decreasing for $s > 0$.

We use the stability properties to prove Theorem 5.3. Let

$$Y := \{v \in C^2(0,1) \mid v'(0) = v(1) = 0\}.$$

For any $\phi \in Y$ and convex solution u of (1.2), by some simple computations, we can show that the linearized equation of (1.2) about u at the direction ϕ is

$$\begin{cases} (-\phi'(-v')^{N-1})' - \lambda^N r^{N-1} f'(v)\phi = \mu\phi/N & \text{in } (0,1), \\ \phi'(0) = \phi(1) = 0, \end{cases} \tag{5.1}$$

where $v = -u$. Hence, the linear stability of a solution u of (1.2) can be determined by the linearized eigenvalue problem (5.1). A solution u of (1.2) is stable if all eigenvalues of (5.1) are positive, otherwise it is unstable. We define the *Morse index* $M(u)$ of a solution u of (1.2) to be the number of negative eigenvalues of (5.1). A solution u of (1.2) is degenerate if 0 is an eigenvalue of (5.1), otherwise it is non-degenerate.

The following lemma is our main stability result for the negative steady state solution.

Lemma 5.5. *Suppose that f satisfies the conditions of Theorem* 5.3. *Then any negative solution u of* (1.2) *is stable, hence, non-degenerate and Morse index* $M(u) = 0$.

Proof. Let u be a negative solution of (1.2), and let (μ_1, φ_1) be the corresponding principal eigen-pairs of (5.1) with $\varphi_1 > 0$ in $(0,1)$. We notice that $v := -u$ and ϕ_1 satisfy the equations

$$\begin{cases} ((-v'(r))^N)' - \lambda^N N r^{N-1} f(v(r)) = 0 & \text{in } (0,1), \\ v'(0) = v(1) = 0 \end{cases} \tag{5.2}$$

and

$$\begin{cases} (-\phi_1'(-v')^{N-1})' - \lambda^N r^{N-1} f'(v)\phi_1 = \mu_1\phi_1/N & \text{in } (0,1), \\ \phi_1'(0) = \phi_1(1) = 0. \end{cases} \tag{5.3}$$

Multiplying (5.3) by $-v$ and (5.2) by $-\varphi_1$, subtracting and integrating, we obtain

$$\mu_1 \int_0^1 \varphi_1 v\,dr = N \int_0^1 \lambda^N r^{N-1} \varphi_1 (Nf(v) - f'(v)v)\,dr.$$

Since $v > 0$ and $\varphi_1 > 0$ in $(0,1)$, then $\mu_1 > 0$ and the negative steady state solution u must be stable. $\square$

PROOF OF THEOREM 5.3. Define $F\colon \mathbb{R}\times X^+ \to X^+$ by

$$F(\lambda, v) = ((-v'(r))^N)' - \lambda^N N r^{N-1} f(v(r)),$$

where $v = -u$. From Lemma 5.5, we know that any convex solution v of (1.2) is stable. Therefore, at any solution (λ^*, v^*), we can apply Implicit Function Theorem to $F(\lambda, v) = 0$, and all the solutions of $F(\lambda, v) = 0$ near (λ^*, v^*) are on a curve $(\lambda, v(\lambda))$ with $|\lambda - \lambda^*| \le \varepsilon$ for some small $\varepsilon > 0$. Furthermore, by virtue of Remark 5.1, the unbounded continuum $\mathcal{C}$ is a curve, which has been obtained from Theorem 4.5. $\square$

From Theorem 5.3, we can see that for $\lambda > \lambda_1/f_0$ there exists a unique negative solution u_λ with $M(u_\lambda) = 0$. In addition, we also have the following result.

THEOREM 5.6. *Under the assumptions of Theorem* 5.3, *we also have that u_λ decreasing with respect to λ.*

PROOF. Since u_λ is differentiable with respect to λ (as a consequence of Implicit Function Theorem), letting $v_\lambda = -u_\lambda$, then $dv_\lambda/d\lambda$ satisfies

$$\bigg(\bigg(\bigg(-\frac{dv_\lambda}{d\lambda}\bigg)'(r)\bigg)(-(v_\lambda)'(r))^{N-1}\bigg)' = \lambda^N r^{N-1} f'(v_\lambda)\frac{dv_\lambda}{d\lambda} + N\lambda^{N-1} r^{N-1} f(v_\lambda).$$

By an argument similar to that of Lemma 5.5, we can show that

$$\int_0^1 (\lambda(f'(v_\lambda)v_\lambda - Nf(v_\lambda))\frac{dv_\lambda}{d\lambda} + Nf(v_\lambda)v_\lambda)\,dr = 0.$$

Assumptions of f imply $dv_\lambda/d\lambda \ge 0$. Therefore, we have $du_\lambda/d\lambda \le 0$. $\square$

REMARK 5.7. From Theorem 5.6, we can also get that (1.2) has no convex solution for all $\lambda \in (0, \lambda_1/f_0]$ under the assumptions of Theorem 5.3. In this sense, we get the optical interval for the parameter λ which ensures the existence of single strictly convex solution for (1.2) under the assumptions of Theorem 5.3.

REMARK 5.8. Note that the results of Theorems 5.3 and 5.6 have extended the corresponding results to [17, Proposition 3], in the case of $\Omega = B$.

REMARK 5.9. Clearly, the results of Theorem 5.6 are better than the corresponding results of [11, Theorem 3.1], if we assume $f \in C^1(\mathbb{R}\setminus\mathbb{R}^-, \mathbb{R}\setminus\mathbb{R}^-)$ in the Theorem 3.1 of [11]. Moreover, we do not need f is increasing.

PROOF. It is sufficient to show that the assumption (3.9) of [11] implies $f'(s) < Nf(s)/s$ for $s > 0$. Luckily, for any $s > 0$ and $t \in (0,1)$, by the assumption (3.9) of [11], we have

$$\begin{aligned} f'(s) &= \lim_{t\to 1}\frac{f(s)-f(ts)}{(1-t)s} \le \lim_{t\to 1}\frac{f(s)-[(1+\eta)t]^N f(s)}{(1-t)s} \\ &< \lim_{t\to 1}\frac{f(s)-t^N f(s)}{(1-t)s} = \lim_{t\to 1}\frac{(1+t+\ldots+t^{N-1})f(s)}{s} = \frac{Nf(s)}{s}, \end{aligned}$$

where $\eta > 0$ comes from the assumption (3.9) of [11]. □

6. Convex solutions on general domain

In this section, we extend the results in Section 4 to the general domain Ω by domain comparison method.

Through out this section, we assume that

(f2) $f\colon [0,+\infty) \to [0,+\infty)$ is C^2 and increasing;

(f3) $f(s) > 0$ for $s > 0$.

We use sub-supersolution method to construct a solution by iteration in an arbitrary domain. Note that 0 is always a sup-solution of problem (1.4). So we only need to find a sub-solution.

By an argument similar to that of [27, Lemma 3.2], we may obtain the following lemma.

Lemma 6.1. *If we have a strictly convex function $u_* \in C^3(\overline{\Omega})$, such that $\det(D^2u_*) \ge \lambda^N f(-u_*)$ in Ω and $u_* \le 0$ on $\partial\Omega$, then problem* (1.4) *has a convex solution u in Ω.*

As an immediate consequence, we obtain the following comparison.

Lemma 6.2. *Given two bounded convex domains Ω_1 and Ω_2 such that $\Omega_1 \subset \Omega_2$. If we have a convex solution u of problem* (1.4) *in Ω_2, then there exists a convex solution v of problem* (1.4) *in Ω_1, or equivalently if there is no convex solution of problem* (1.4) *in Ω_1, then there is no convex solution of problem* (1.4) *in Ω_2.*

Our main results are the following two theorems.

Theorem 6.3. *Assume that* (f2) *and* (f3) *hold.*

(a) *If $f_0 \in (0,+\infty)$ and $f_\infty \in (0,+\infty)$, then there exist $\lambda_2 > 0$ and $\lambda_3 > 0$ such that* (1.4) *has at least one convex solution for all $\lambda \in (\lambda_2,\lambda_3)$.*

(b) *If $f_0 \in (0,+\infty)$ and $f_\infty = 0$, then there exists $\lambda_4 > 0$ such that* (1.4) *has at least one convex solution for all $\lambda \in (\lambda_4,+\infty)$.*

(c) *If $f_0 \in (0,+\infty)$ and $f_\infty = +\infty$, then there exists $\lambda_5 > 0$ such that* (1.4) *has at least one convex solution for all $\lambda \in (0,\lambda_5)$.*

(d) *If $f_0 = 0$ and $f_\infty \in (0,+\infty)$, then there exists $\lambda_6 > 0$ such that* (1.4) *has at least one convex solution for all $\lambda \in (\lambda_6,+\infty)$.*

(e) *If $f_0 = 0$ and $f_\infty = 0$, then there exist $\lambda_7 > 0$ and $\lambda_* > 0$ such that* (1.4) *has at least two convex solutions for all $\lambda \in (\lambda_*, +\infty)$, one convex solution for all $\lambda \in [\lambda_7, \lambda_*]$.*

(f) *If $f_0 = 0$ (or $+\infty$) and $f_\infty = +\infty$ (or 0), then for any $\lambda \in (0, +\infty)$,* (1.4) *has one convex solution.*

(g) *If $f_0 = +\infty$ and $f_\infty \in (0, \infty)$, then there exists $\lambda_8 > 0$ such that* (1.4) *has at least one convex solution for all $\lambda \in (0, \lambda_8)$.*

(h) *If $f_0 = +\infty$ and $f_\infty = +\infty$, then there exist $\lambda_9 > 0$ and $\lambda^* > 0$ such that* (1.4) *has at least two convex solutions for all $\lambda \in (0, \lambda^*)$, has at least one convex solution for all $\lambda \in [\lambda^*, \lambda_9]$.*

PROOF. We only give the proof of (a) since the proofs of (b)–(h) can be given similarly. It is obvious that there exists a positive constant R_1 such that $\Omega \subseteq B_{R_1}(0)$. Theorem 4.1, Remarks 4.2 and 4.21 that there exist $\lambda_2 > 0$ and $\lambda_3 > 0$ such that problem (1.4) with $\Omega = B_{R_1}(0)$ has at least a strictly convex solution for all $\lambda \in (\lambda_2, \lambda_3)$. Using Lemma 6.2, we have that problem (1.4) has at least a convex solution for all $\lambda \in (\lambda_2, \lambda_3)$. □

THEOREM 6.4. *Assume that* (f2) *and* (f3) *hold.*

(a) *If $f_0 \in (0, +\infty)$ and $f_\infty \in (0, +\infty)$, then there exist $\mu_2 > 0$ and $\mu_3 > 0$ such that* (1.4) *has no convex solution for all $\lambda \in (0, \mu_2) \cup (\mu_3, +\infty)$.*

(b) *If $f_0 \in (0, +\infty)$ and $f_\infty = 0$, then there exists $\mu_4 > 0$ such that* (1.4) *has no convex solution for all $\lambda \in (0, \mu_4)$.*

(c) *If $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$, then there exists $\mu_5 > 0$ such that* (1.4) *has no convex solution for all $\lambda \in (\mu_5, +\infty)$.*

(d) *If $f_0 = 0$ and $f_\infty \in (0, +\infty)$, then there exists $\mu_6 > 0$ such that* (1.4) *has no convex solution for all $\lambda \in (0, \mu_6)$.*

(e) *If $f_0 = 0$ and $f_\infty = 0$, then there exists $\mu_7 > 0$ such that* (1.4) *has no convex solution for all $\lambda \in (0, \mu_7)$.*

(f) *If $f_0 = +\infty$ and $f_\infty \in (0, \infty)$, then there exists $\mu_8 > 0$ such that* (1.4) *has no convex solution for all $\lambda \in (\mu_8, +\infty)$.*

(g) If $f_0 = +\infty$ and $f_\infty = +\infty$, then there exists $\mu_9 > 0$ such that (1.4) has no convex solution for all $\lambda \in (\mu_9, +\infty)$.

PROOF. We also only give the proof of (a) since the proofs of (b)–(g) can be given similarly. It is obvious that there exists a positive constant R_2 such that $B_{R_2}(0) \subseteq \Omega$. Theorem 4.1, Remarks 4.2 and 4.21 imply that there exist $\mu_2 > 0$ and $\mu_3 > 0$ such that problem (1.4) with $\Omega = B_{R_2}(0)$ has no convex solution for all $\lambda \in (0, \mu_2) \cup (\mu_3, +\infty)$. Using Lemma 6.2 again, we have that problem (1.4) has no convex solution for all $\lambda \in (0, \mu_2) \cup (\mu_3, +\infty)$. □

REMARK 6.5. From Theorems 6.3 and 6.4, we can easily see that $\mu_9 \geq \lambda_9 \geq \lambda^*$. Set $\mu := \lambda^{1/2}$. Through a scaling, we can show that problem (1.4) is equivalent to

$$\begin{cases} \det(D^2u) = f(-u) & \text{in } \mu\Omega, \\ u = 0 & \text{on } \partial\mu\Omega. \end{cases} \tag{6.1}$$

In the case of $f(s) = e^s$ in (6.1), Zhang and Wang [27, Theorem 12], has shown that $\mu_9 = \lambda_9 = \lambda^*$. Unfortunately, we do not know whether this relation also holds for the general case of $f_0 = +\infty$ and $f_\infty = +\infty$.

Acknowledgments. The authors express their gratitude to Professor Hai-yan Wang for kindly pointing out this problem and hospitality. The authors are also very grateful to an anonymous referee for his or her very valuable comments and suggestions.

References

[1] R.A. Admas, *Sobolev Spaces*, New-York, Academic Press, 1975.

[2] H. Brezis, *Operateurs Maximaux Monotone et Semigroup de Contractions dans les Espase de Hilbert*, Math. Studies, vol. 5, North-Holland, Amsterdam, 1973.

[3] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations*, Part I. Monge-Ampère equation, Comm. Pure Appl. Math. **37** (1984). 369–402.

[4] S.Y. Cheng and S.T. Yao, *On the regularity of the Monge-Ampère equation* $\det(\partial^2 u/\partial x_i \partial x_j) = F(x,u)$, Comm. Pure Appl. Math. **30** (1977), 41–68.

[5] S.Y. Cheng and S.T. Yao, *The real Monge-Ampère equations and affine fiat structures*, (Proc. of the 1980 Beijing Symp. on Differential Geometry and Differential Equations), (S.S. Cheng and W.T. Wu, eds.), Science Press Beijing 1982, Gordon and Breach, New York, 1982.

[6] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[7] G. Dai, R. Ma and H. Wang, *Eigenvalues, bifurcation and one-sign solutions for the periodic p-Laplacian*, Commun. Pure Appl. Anal. **12** (2013), 2839–2872.

[8] Ph. Delano, *Radially symmetric boundary value problems for real and complex elliptic Monge-Ampère equations*, J. Differential Equations **58** (1985), 318–344.

[9] L.C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Rhode Island, 1998.

[10] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag, Berlin, Heidelberg, 2001.

[11] S. Hu and H. Wang, *Convex solutions of BVP arising from Monge-Ampère equations*, Discrete Contin. Dynam. Syst. **16** (2006), 705–720.

[12] T. Idogawa and M. Ôtani, *The first eigenvalues of some abstact elliptic operator*, Funkc. Ekvac. **38** (1995), 1–9.

[13] B. Im, E. Lee and Y.H. Lee, *A global bifurcation phenomena for second order singular boundary value problems*, J. Math. Anal. Appl. **308** (2005), 61–78.

[14] N.V. Krylov, *On degenerate nonlinear elliptic equations*, Mat. Sbornik **120** (1983), 311–330.

[15] N.D. Kutev, *Nontrivial solutions for the equations of Monge-Ampère type*, J. Math. Anal. Appl. **132** (1988), 424–433.

[16] P.L. Lions, *Sur les équations de Monge-Ampère* I, Manuscripta Math. **41** (1983), 1–43; II, Arch. Rat. Mech. Anal. Announced in C.R. Acad. Sci. Paris **293** (1981), 589–592.

[17] ______, *Two remarks on Monge-Ampère equations*, Ann. Mat. Pura Appl. (4) **142** (1985), 263–275.

[18] J. LÓPEZ-GÓMEZ, *Spectral theory and nonlinear functional analysis*, Chapman and Hall/CRC, Boca Raton, 2001.

[19] R. MA AND Y. AN, *Global structure of positive solutions for nonlocal boundary value problems involving integral conditions*, Nonlinear Anal. **71** (2009), 4364–4376.

[20] A.V. POGORELOV, *On the regularity of generalized solutions of the equation* $\det(\partial^2 u/\partial x_i \partial x_j) = \phi(x_1, \ldots, x_n) > 0$, Soviet Math. Dokl. **12** (1971), 1436–1440.

[21] ______, *The Diriehlet problem for the n-dimensional analogue of the Monge-Ampère equation*, Soviet. Math. Dokl. **12** (1971), 1727–1731.

[22] ______, THE MINKOWSKI MULTIDIMENSIONAL PROBLEM, J. Wiley, New-York, 1978.

[23] P.H. RABINOWITZ, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.

[24] J. SHI, Blow up points of solution curves for a semilinear problem, Topol. Methods Nonlinear Anal. **15** (2000), 251–266.

[25] K. TSO, *On a real Monge-Ampère functional*, Invent. Math. **101** (1990), 425–448.

[26] H. WANG, *Convex solutions of boundary value problems*, J. Math. Anal. Appl. **318** (2006), 246–252.

[27] Z. ZHANG AND K. WANG, *Existence and non-existence of solutions for a class of Monge-Ampère equations*, J. Differential Equations **246** (2009), 2849–2875.

Manuscript received March 12, 2013

GUOWEI DAI
School of Mathematical Sciences
Dalian University of Technology
Dalian, 116024, P.R. CHINA

E-mail address: daiguowei@dlut.edu.cn

RUYUN MA
Department of Mathematics
Northwest Normal University
Lanzhou, 730070, P.R. CHINA