Topological **M**ethods in **N**onlinear **A**nalysis
Volume 46, No. 1, 2015, 1–15

POSITIVE SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS PARAMETRIC EQUATIONS

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Abstract. We consider a nonlinear parametric Dirichlet problem driven by a nonhomogeneous differential operator which includes as special cases the p-Laplacian, the (p,q)-Laplacian and the generalized p-mean curvature operator. Using variational methods, we prove a bifurcation-type theorem describing the dependence of positive solutions on the parameter.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Dirichlet eigenvalue problem

$$(\mathrm{P})_\lambda \qquad -\operatorname{div} a(Du(z)) = \lambda f(z,u(z)) \quad \text{in } \Omega, \qquad u\,|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0.$$

In $(\mathrm{P})_\lambda$ the map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is strictly monotone and satisfies certain other regularity conditions. The precise conditions on $a(\,\cdot\,)$ are stated in hypotheses H(a) below. They provide a unifying framework to treat equations driven by the p-Laplacian, the (p,q)-Laplacian differential operator and the generalized p-mean curvature differential operator. Also, $\lambda > 0$ is a parameter and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function (i.e. for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \to f(z,x)$ is continuous), which is strictly $(p-1)$-sublinear in the x-variable near $+\infty$. We prove a bifurcation-type result describing precisely the dependence of positive solutions of $(\mathrm{P})_\lambda$ on the parameter $\lambda > 0$. Recently

2010 *Mathematics Subject Classification.* 35J20, 35J25, 35J60, 35J92.
Key words and phrases. Nonlinear regularity, mountain pass theorem, bifurcation-type theorem, strong comparison principle, nonhomogeneous differential operator, positive solutions.

positive solutions for nonlinear eigenvalue problems driven by the p-Laplacian, were obtained by Brock, Itturiaga and Ubilla [5], Hu and Papageorgiou [13] and Perera [18]. In contrast to our differential operator here, the p-Laplacian is $(p-1)$-homogeneous and this is a feature that helps the analysis of the equation. Finally we should mention the recent work of Cardinali, Papageorgiou, Rubbioni [6], where an analogous result was proved for a Neumann logistic equation driven by the p-Laplacian.

2. Hypotheses – auxiliary results

Let $\vartheta \in C^1(0,\infty)$ be such that

$$\tag{2.1} \begin{aligned} &0 < \widehat{c} \le \frac{t\vartheta'(t)}{\vartheta(t)} \le c_0 && \text{for all } t > 0, \\ &c_1 t^{p-1} \le \vartheta(t) \le c_2(t^{q-1} + t^{p-1}) && \text{for all } t > 0 \end{aligned}$$

and some $c_1, c_2 > 0$, $1 < q < p$. The hypotheses on the map $a(\,\cdot\,)$ are the following:

H(a) $a(y) = a_0(||y||)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0, a(0) = 0$ and

(i) $a_0 \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ and $\displaystyle\lim_{s\to 0^+} \frac{sa'(s)}{a(s)} > -1$;

(ii) $\displaystyle ||\nabla a(y)|| \le c_3 \frac{\vartheta(||y||)}{||y||}$ for all $y \in \mathbb{R}^N \setminus \{0\}$ and some $c_3 > 0$;

(iii) $\displaystyle (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \ge \frac{\vartheta(||y||)}{||y||} ||\xi||^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$.

Remark 2.1. Let

$$G_0(t) = \int_0^t a_0(s)s\,ds, \quad t \ge 0.$$

Evidently $G_0(\,\cdot\,)$ is strictly convex and strictly increasing. For all $y \in \mathbb{R}^N$ we set $G(y) = G_0(||y||)$. Then $G(\,\cdot\,)$ is convex, $G(0) = 0$ and for all $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$\nabla G(y) = G_0'(||y||)\frac{y}{||y||} = a_0(||y||)y = a(y).$$

Hence $G(\,\cdot\,)$ is the primitive of $a(\,\cdot\,)$. Since $G(\,\cdot\,)$ is convex and $G(0) = 0$, we have

$$\tag{2.2} G(y) \le (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N.$$

From hypotheses H(a) and (2.2), (2.3), we easily deduce the following properties of the map $a(\,\cdot\,)$.

Lemma 2.2. *If hypotheses* H(a) *hold, then*:

(a) $y \to a(y)$ *is maximal monotone and strictly monotone*;
(b) $||a(y)|| \le c_4(1 + ||y||^{p-1})$ *for all* $y \in \mathbb{R}^N$ *and some* $c_4 > 0$;
(c) $(a(y), y)_{\mathbb{R}^N} \ge c_1||y||^p/(p-1)$ *for all* $y \in \mathbb{R}^N$ (*see* (2.2)).

From this lemma and the integral form of the mean value theorem, we obtain the following growth conditions on $G(\cdot)$.

Corollary 2.3. *If hypotheses* H(a) *hold, then*

$$\frac{c_1}{p(p-1)}||y||^p \le G(y) \le c_5(1+||y||^p) \quad \textit{for all } y \in \mathbb{R}^N \textit{ and some } c_5 > 0.$$

Examples 2.4. The following maps satisfy hypotheses H(a):

(a) $a(y) = ||y||^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}(||Du||^{p-2}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(b) $a(y) = ||y||^{p-2}y + ||y||^{q-2}y$ with $1 < q < p < \infty$.

This map corresponds to the (p,q)-differential operator

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

This differential operator is important in quantum physics (see Benci, D'Avenia, Fortunato and Pisani [3]) and in reaction diffusion equations and plasma physics (see Cherfils and Ilyasov [7]). Recently such equations were studied by Cingolani and Degiovanni [8], Li and Guo [16], Sun [20].

(c) $a(y) = (1 + ||y||^2)^{(p-2)/2}y$ with $1 < p < \infty$.

This map corresponds to the generalized p-mean curvature differential operator

$$\operatorname{div}((1 + ||Du||^2)^{(p-2)/2}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Such equations can be found in Pucci and Serrin [19].

(d) $a(y) = ||y||^{p-2}y + ||y||^{p-2}y/(1 + ||y||^p)$ with $1 < p < \infty$.

(e) $a(y) = ||y||^{p-2}y + \ln(1 + ||y||^p)y \quad$ with $1 < p < \infty$.

Let $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that

$$|f_0(z,x)| \le \alpha(z) + c|x|^{r-1} \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R},$$

with $\alpha \in L^\infty(\Omega)_+$, $c > 0$ and

$$1 < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

We set

$$F_0(z,x) = \int_0^x f_0(z,s)\,ds$$

and consider the C^1-functional $\varphi_0 \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))\,dz - \int_\Omega F_0(z,u(z))\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next result can be found in Gasinski and Papageorgiou [13].

Proposition 2.5. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, i.e. there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \textit{for all } h \in C_0^1(\overline{\Omega}), \quad ||h||_{C_0^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, i.e. there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \quad \textit{for all } h \in W_0^{1,p}(\Omega), \quad ||h|| \le \rho_1.$$

Remark 2.6. The first such result was proved by Brezis and Nirenberg [4] for the case when $G(y) = ||y||^2/2$ for all $y \in \mathbb{R}^N$. It was extended to the case $G(y) = ||y||^p/p$ for all $y \in \mathbb{R}^N$ with $1 < p < \infty$ by Garcia Azorero, Manfredi and Peral Alonso [9]. The proof of [13] differs from the proofs in [4], [9].

In the analysis of problem $(\mathrm{P})_\lambda$, we will use the ordered Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

The order cone of this space is

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \ge 0, \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\},$$

where $n(\,\cdot\,)$ is the outward unit normal on $\partial\Omega$.

Also throughout this work by $||\cdot||$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincare inequality, we have $||u|| = ||Du||_p$ for all $u \in W_0^{1,p}(\Omega)$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. If $u \in W_0^{1,p}(\Omega)$ then

$$u^\pm(\,\cdot\,) = u(\,\cdot\,)^\pm \in W_0^{1,p}(\Omega) \quad \text{and} \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

If $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example a Carathéodory function), then

$$N_h(u)(\,\cdot\,) = h(\cdot\, u(\,\cdot\,)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $A \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(1/p + 1/p' = 1)$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega (a(Du), Dy)_{\mathbb{R}^N}\,dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega). \tag{2.3}$$

Here by $\langle \cdot\,, \cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(\Omega), W^{-1,p'}(\Omega))$.

From Gasinski and Papageorgiou [10], we have

Proposition 2.7. *If hypotheses* H(a) *hold and* $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ *is the nonlinear map defined by* (2.3)*, then* A *is monotone continuous bounded* (*i.e. maps bounded sets to bounded sets) hence maximal monotone too and of type* $(S)_+$*, i.e. if* $u_n \xrightarrow{w} u$ *in* $W_0^{1,p}(\Omega)$ *and* $\limsup\limits_{n\to+\infty} \langle A(u_n), u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* $W_0^{1,p}(\Omega)$*.*

Let $h_1, h_2 \in L^\infty(\Omega)$. We write $h_1 \prec h_2$, if for any $K \subseteq \Omega$ compact, we can find $\varepsilon > 0$ suth that

$$h_1(z) + \varepsilon \le h_2(z) \quad \text{for a.a. } z \in K.$$

Evidently, if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

The next strong comparison principle extends Proposition 2.6 of Arcoya and Ruiz [2] which was proved for the particular case of the p-Laplacian.

Proposition 2.8. *If* $\xi \ge 0$*,* $h_1, h_2 \in L^\infty(\Omega), h_1 \prec h_2$ *and* $u \in C_0^1(\overline{\Omega})$*,* $v \in \operatorname{int} C_+$ *are solutions of*

$$\begin{aligned} -\operatorname{div} a(Du(z)) + \xi|u(z)|^{p-2}u(z)* &= h_1(z) \quad \text{in } \Omega,\\ -\operatorname{div} a(Dv(z)) + \xi|v(z)|^{p-2}v(z) &= h_2(z) \quad \text{in } \Omega, \end{aligned}$$

then $v - u \in \operatorname{int} C_+$*.*

Proof. We have

$$A(u) + \xi|u|^{p-2}u \le A(v) + \xi|v|^{p-2}v \quad \text{in } W^{-1,p'}(\Omega).$$

Acting with $(u - v)^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} \langle A(u) - A(v), (u - v)^+\rangle &+ \int_\Omega \xi(|u|^{p-2}u - |v|^{p-2}v)(u - v)^+\,dz \le 0,\\ &\Rightarrow \int_{\{u>v\}} (a(Du) - a(Dv), Du - Dv)_{\mathbb{R}^N}\,dz\\ &\quad + \int_{\{u>v\}} \xi(|u|^{p-2}u - |v|^{p-2}v)(u - v)\,dz \le 0,\\ &\Rightarrow |\{u > v\}|_N = 0, \quad \text{hence } u \le v \quad \text{(see Lemma 2.2)}. \end{aligned}$$

Next we show that $u(z) < v(z)$ for all $z \in \Omega$. To this end, we introduce the sets

$$E_0 = \{z \in \Omega : u(z) = v(z)\} \quad \text{and} \quad E = \{z \in \Omega : Du(z) = Dv(z) = 0\}.$$

Claim. $E_0 \subseteq E$.

Let $z_0 \in E_0$. Then the function $w = v - u$ attains its minimum at z_0 and so $Du(z_0) = Dv(z_0)$. If $Du(z_0) \neq 0$, then we can find $\rho > 0$ small such that $B_\rho(z_0) \subseteq \Omega$ and

$$||Du(z)|| > 0, \quad ||Dv(z)|| > 0, \quad (Du(z), Dv(z))_{\mathbb{R}^N} > 0 \quad \text{for all } z \in B_\rho(z_0).$$

We have $w = v - u \in C_+ \setminus \{0\}$ and $w(\,\cdot\,)$ satisfies the following linear elliptic equation in $B_\rho(z_0)$ (see Arcoya and Ruiz [2]):

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\left(\vartheta_{ij}(z)\frac{\partial w}{\partial z_j}\right) = -\xi(|v|^{p-2}v - |u|^{p-2}u) + h_2 - h_1 \quad \text{in } B_\rho(z_0). \tag{2.4}$$

In (2.4) the coefficients $\vartheta_{ij}(\,\cdot\,)$ are given by

$$\vartheta_{ij}(z) = \int_0^1 \frac{\partial a_i}{\partial y_j}((1-t)Du(z) + tDv(z))\,dt.$$

We have $\vartheta_{ij} \in C(\overline{B}_\rho(z_0))$ and by choosing $\rho > 0$ even smaller if necessary in (2.4), we can have the differential operator uniformly elliptic and the forcing term (i.e. right hand side) positive. Then the maximum principle of Vazquez [21] implies that $u(z) < v(z)$ for all $z \in \overline{B}_\rho(z_0)$, which contradicts the fact that $z_0 \in E_0$. This proves the Claim.

Since $v \in \operatorname{int} C_+$, we have that E is compact and E_0 being a closed subset of E (see the Claim), itself is also compact. Therefore we can find $\Omega_1 \subseteq \Omega$ a smooth open set such that

$$E_0 \subseteq \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega.$$

We can find $\varepsilon \in (0,1)$ such that

$$u(z) + \varepsilon \le v(z) \quad \text{for all } z \in \partial\Omega_1 \quad \text{and} \quad h_1(z) + \varepsilon \le h_2(z) \tag{2.5}$$

for almost all $z \in \Omega_1$.

We choose $\delta \in (0, \varepsilon)$ small such that

$$\xi||s|^{p-2}s - |s'|^{p-2}s'| \le \varepsilon \tag{2.6}$$

for all $s, s' \in [-||v||_\infty, ||u||_\infty]$, with $|s - s'| \le \delta$. Then we have

$$\begin{aligned} -\operatorname{div} a(D(u+\delta)) + \xi|u+\delta|^{p-2}(u+\delta) &= -\operatorname{div} a(Du) + \xi|u+\delta|^{p-2}(u+\delta) \\ &= \xi[|u+\delta|^{p-2}(u+\delta) - |u|^{p-2}u] + h_1 \\ &\le h_1 + \varepsilon \le h_2 \qquad \text{(see (2.5) and (3.3))} \\ &= -\operatorname{div} a(Dv) + \xi|v|^{p-2}v \quad \text{in } \Omega_1, \\ &\Rightarrow u + \delta \le v \quad \text{in } \Omega_1 \qquad \text{(see Pucci, Serrin [19]).} \end{aligned}$$

Since $E_0 \subseteq \Omega_1$, it follows that $E_0 = \emptyset$ and so $u(z) < v(z)$ for all $z \in \Omega$.

Next, let $z_0 \in \partial\Omega$. Since $\partial\Omega$ is by hypothesis a C^2 -manifold, we can find $\rho > 0$ small such that

$$B_{2\rho}(\widehat{z}) \subseteq \Omega \quad \text{and} \quad z_0 \in \partial B_{2\rho}(\widehat{z}) \cap \partial\Omega \quad (\text{with } \widehat{z} \in \Omega).$$

Invoking Lemma 2 of Lewis [15], we can find $\widehat{w} \in C^1(B_{2\rho}(\widehat{z}))$ such that

$$-\operatorname{div}(\Theta(z)D\widehat{w}(z)) = 0 \qquad \text{in } B_{2\rho}(\widehat{z}) \setminus \overline{B}_\rho(\widehat{z})(\Theta(z) = (\vartheta_{ij}(z))_{i,j=1}^N), \tag{2.7}$$

$$\widehat{w}\,|_{\partial B_\rho(\widehat{z})} = 1, \qquad \widehat{w}|_{\partial B_{2\rho}(\widehat{z})} = 0,\ 0 < \widehat{w} < 1 \text{ in } B_{2\rho}(\widehat{z}) \setminus \overline{B}_\rho(\widehat{z}), \tag{2.8}$$

and $||D\widehat{w}(z)|| \geq \widehat{c} > 0$ for all $z \in B_{2\rho}(\widehat{z}) \setminus \overline{B}_\rho(\widehat{z})$.

From the previous part of the proof we have $w(z) > 0$ for all $z \in \Omega$. Hence

$$m_\rho = \min[w(z) : z \in \partial B_\rho(\widehat{z})] > 0.$$

We set $\widetilde{w} = m_\rho \widehat{w}$. Then from (3.4) we have

$$-\operatorname{div}(\Theta(z)D\widetilde{w}(z)) = 0 \quad \text{in } B_{2\rho}(\widehat{z}) \setminus \overline{B}_\rho(\widehat{z}),$$
$$\widetilde{w}|_{\partial B_\rho(\widehat{z})} = m_\rho, \qquad \widetilde{w}|_{\partial B_{2\rho}(\widehat{z})} = 0.$$

The weak comparison principle (see Pucci and Serrin [19]), implies $\widetilde{w} \leq w$ in $B_{2\rho}(\widehat{z}) \setminus \overline{B}_\rho(\widehat{z})$. Moreover, $\widetilde{w}(z_0) = w(z_0) = 0$. Hence

$$\frac{\partial w}{\partial n}(z_0) \leq \frac{\partial \widetilde{w}}{\partial n}(z_0) = m_\rho \frac{\partial \widehat{w}}{\partial n}(z_0) < 0 \quad (\text{see (3.4)}),$$
$$\Rightarrow w = v - u \in \operatorname{int} C_+. \qquad \square$$

Finally by $\widehat{\lambda}_1$ we denote the first eigenvalue of the Dirichlet p- Laplacian. We know (see, for example, Gasinski and Papageorgiou [12]) that $\widehat{\lambda}_1 > 0$ and

$$\widehat{\lambda}_1 = \inf\left[\frac{||Du||_p^p}{||u||_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right]. \tag{2.9}$$

3. Positive solutions

In this section we prove the bifurcation-type theorem describing the dependence of the positive solutions of $(\mathrm{P})_\lambda$ on the parameter $\lambda > 0$.

The hypotheses on the reaction $f(z,x)$ of $(\mathrm{P})_\lambda$, are the following:

H(f) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that $f(z,0) = 0$, for almost all $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $\alpha_\rho \in L^\infty(\Omega)_+$ such that

$$f(z,x) \leq \alpha_\rho(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\rho];$$

(ii) $\displaystyle\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iii) $\displaystyle\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for almost all $z \in \Omega$, $x \to f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$;

(v) $f(z,x) > 0$ for almost all $z \in \Omega$ and all $x > 0$.

Remark 3.1. Since we are looking for positive solutions and the hypotheses concern only the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, we may (and will) assume that

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega \text{ and all } x \le 0.$$

Hypothesis H(f)(ii) implies that for aalmost all $z \in \Omega$, $f(z,\,\cdot\,)$ is strictly $(p-1)$-sublinear near $+\infty$.

Example 3.2. Let

$$g(x) = \begin{cases} x^{r-1} & \text{if } x \in [0,1], \\ x^{q-1} & \text{if } x > 1, \end{cases}$$

with $1 < q < p < r < \infty$, $\alpha \in L^\infty(\Omega)_+$, $\alpha(z) > 0$ for almost all $z \in \Omega$ and let $f(z,x) = \alpha(z)g(x)$. Then $f(z,x)$ satisfies hypotheses H(f).

Let $\mathcal{S} = \{\lambda > 0 : \text{problem } (\mathrm{P})_\lambda \text{ has a nontrivial positive solution}\}$ and let $S(\lambda)$ be the corresponding solution set of $(\mathrm{P})_\lambda$. We set $\lambda_* = \inf \mathcal{S}$ (if $\mathcal{S} = \emptyset$, then $\lambda_* = +\infty$).

Proposition 3.3. *If hypotheses* H(a), H(f) *hold, then*

$$S(\lambda) \subseteq \operatorname{int} C_+ \quad \textit{and} \quad \lambda_* > 0.$$

Proof. Suppose that $\mathcal{S} \ne \emptyset$ and let $\lambda \in \mathcal{S}$. Then we can find $u \in W_0^{1,p}(\Omega)$, $u \ge 0$, $u \ne 0$ such that

$$-\operatorname{div} a(Du(z)) = \lambda f(z,u(z)) \quad \text{in } \Omega, \quad u\,|_{\partial\Omega} = 0.$$

From Ladyzhenskaya and Ural'tseva [14, p. 286], we have that $u \in L^\infty(\Omega)$. Then invoking the regularity result of Lieberman [17, p. 320], we have that $u \in C_+ \backslash \{0\}$. Let $\rho = ||u||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis H(f)(iv). We have

$$\begin{aligned} -\operatorname{div} a(Du(z)) \, + \lambda\xi_\rho u(z)^{p-1} &= \lambda f(z,u(z)) + \lambda\xi_\rho u(z)^{p-1} \ge 0 && \text{a.e. in } \Omega, \\ \Rightarrow \operatorname{div} a(Du(z)) &\le \lambda\xi_\rho u(z)^{p-1} && \text{a.e. in } \Omega, \\ \Rightarrow u \in \operatorname{int} C_+ & \quad \text{(see Pucci–Serrin [19, p. 120]).} \end{aligned}$$

So, we have proved that $S(\lambda) \subseteq \operatorname{int} C_+$.

Hypotheses $H(f)$(i), (ii) imply that we can find $c_6 > 0$ such that

$$f(z,x) \le c_6 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.1}$$

Let $\lambda_0 < c_1\widehat{\lambda}_1/((p-1)c_6)$ (see (2.2)) and $\eta \in (0,\lambda_0]$. Suppose that $\eta \in \mathcal{S}$. Then by virtue of the first part of the proof, we can find $u_\eta \in S(\eta) \subseteq \operatorname{int} C_+$. We have

$$\begin{aligned} A(u_\eta) = \eta N_f(u_\eta), \ \Rightarrow \ \frac{c_1}{p-1}||Du_\eta||_p^p &\le \int_\Omega \eta f(z,u_\eta)u_\eta \, dz \quad \text{(see Lemma 2.2)} \\ &\le \eta c_6 ||u_\eta||_p^p < \frac{c_1}{p-1}\widehat{\lambda}_1 ||u_\eta||_p^p, \end{aligned}$$

which contradicts (3.5). Therefore $\eta \notin \mathcal{S}$ and so $\lambda_* \geq \lambda_0 > 0$. $\square$

For $\lambda > 0$, let $\varphi_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem $(\mathrm{P})_\lambda$ defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du)\, dz - \lambda \int_\Omega F(z,u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $F(z,x) = \int_0^x f(z,s)\, ds$. Clearly, $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

Proposition 3.4. *If hypotheses* H(a), H(f) *hold, then* $\mathcal{S} \neq \emptyset$.

Proof. Hypotheses H(f)(i), (ii), imply that given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{\varepsilon}{p} x^p + c_7 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.2}$$

Therefore for $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \varphi_\lambda(u) &\geq \frac{c_1}{p(p-1)} ||Du||_p^p - \frac{\varepsilon}{p} ||u^+||_p^p - c_7 |\Omega|_N \quad \text{(see Corollary 2 and (3.7))} \\ &\geq \frac{1}{p}\left[\frac{c_1}{p-1} - \frac{\varepsilon}{\widehat{\lambda}_1}\right] ||u||^p - c_7|\Omega|_N \qquad \text{(see (3.5)).} \end{aligned}$$

Choosing $\varepsilon \in (0, \widehat{\lambda}_1 c_1/(p-1))$, we see that φ_λ is coercive. Also, exploiting the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, we check that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(\widehat{u}) = \inf[\varphi_\lambda(u) : u \in W_0^{1,p}(\Omega)]. \tag{3.3}$$

Let $L \colon L^p(\Omega) \to \mathbb{R}$ be the integral functional defined by

$$L(v) = \int_\Omega F(z, v(z))\, dz \quad \text{for all } v \in L^p(\Omega).$$

By virtue of hypothesis H(f)(v) we see that for every $v \in L^p(\Omega)$ such that $v \geq 0$ and $v \neq 0$, we have that $L(v) > 0$. Since the space $W_0^{1,p}(\Omega)$ is dense in $L^p(\Omega)$, we can find $\widehat{v} \in W_0^{1,p}(\Omega), \widehat{v} \geq 0$ such that $L(\widehat{v}) > 0$. Then we can choose $\lambda > 0$ large such that

$$\lambda L(\widehat{v}) > \int_\Omega G(D\widehat{v})\, dz \ \Rightarrow\ \varphi_\lambda(\widehat{v}) < 0 \ \Rightarrow\ \varphi_\lambda(\widehat{u}) < 0 = \varphi_\lambda(0)$$

(see (3.8)), hence $\widehat{u} \neq 0$. From (3.8), we have

$$\varphi_\lambda'(\widehat{u}) = 0 \ \Rightarrow\ A(\widehat{u}) = \lambda N_f(\widehat{u}). \tag{3.4}$$

Acting on (3.4) with $-\widehat{u}^- \in W_0^{1,p}(\Omega)$ and using Lemma 2.2, we obtain $\widehat{u} \geq 0$, $\widehat{u} \neq 0$. Therefore $\widehat{u} \in S(\lambda) \subseteq \operatorname{int} C_+$ for $\lambda > 0$ large. Hence $\mathcal{S} \neq \emptyset$. $\square$

Proposition 3.5. *If hypotheses* H(a), H(f) *hold and* $\lambda \in \mathcal{S}$, *then*

$$[\lambda, +\infty) \subseteq \mathcal{S}.$$

Proof. Let $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 3.3). Also let $\theta > \lambda$ and consider the following Caratheodory function:

$$h_\theta(z,x) = \begin{cases} \theta f(z,u_\lambda(z)) & \text{if } x \le u_\lambda(z), \\ \theta f(z,x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.5}$$

We set

$$H_\theta(z,x) = \int_0^x h_\theta(z,s)\,ds$$

and then introduce the C^1-functional $\psi_\theta \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\theta(u) = \int_\Omega G(Du)\,dz - \int_\Omega H_\theta(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As we did for the functional φ_λ in the proof of Proposition 3.4, we show that ψ_θ is coercive and sequentially weakly lower semicontinuous. So, we can find $u_\theta \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \psi_\theta(u_\theta) = \inf[\psi_\theta(u) : u \in W_0^{1,p}(\Omega)] \;&\Rightarrow\; \psi_\theta'(u_\theta) = 0 \\ &\Rightarrow\; A(u_\theta) = N_{h_\theta}(u_\theta). \end{aligned} \tag{3.6}$$

On (3.6) we act with $(u_\lambda - u_\theta)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \langle A(u_\theta), (u_\lambda - u_\theta)^+\rangle &= \int_\Omega h_\theta(z,u_\theta)(u_\lambda - u_\theta)^+\,dz \\ &= \int_\Omega \theta f(z,u_\lambda)(u_\lambda - u_\theta)^+\,dz \quad \text{(see (3.5))} \\ &\ge \int_\Omega \lambda f(z,u_\lambda)(u_\lambda - u_\theta)^+\,dz \quad (\text{since } \lambda < \theta,\ f \ge 0) \\ &= \langle A(u_\lambda), (u_\lambda - u_\theta)^+\rangle \\ &\Rightarrow \int_{\{u_\lambda > u_\theta\}} (a(Du_\lambda) - a(Du_\theta), Du_\lambda - Du_\theta)_{\mathbb{R}^N}\,dz \le 0 \\ &\Rightarrow |\{u_\lambda > u_\theta\}|_N = 0 \end{aligned}$$

(see Lemma 2.2), hence $u_\lambda \le u_\theta$. Then from (3.5) and (3.6) we have

$$\begin{aligned} A(u_\theta) = \theta N_f(u_\theta) \;&\Rightarrow\; u_\theta \in S(\theta) \subseteq \operatorname{int} C_+ \quad \text{and so } \theta \in \mathcal{S}, \\ &\Rightarrow\; [\lambda, +\infty) \subseteq \mathcal{S}. \end{aligned}$$

□

From this proposition it follows that $(\lambda_*, +\infty) \subseteq \mathcal{S}$.

Proposition 3.6. *If hypotheses* H(a), H(f) *hold and* $\lambda > \lambda^*$, *then problem* $(\mathrm{P})_\lambda$ *has at least two nontrivial positive solutions* $u_0, \widehat{u} \in \operatorname{int} C_+$.

Proof. We know that $(\lambda_*, +\infty) \subseteq \mathcal{S}$. Let $\lambda_* < \mu < \lambda < \theta$. We can find $u_\mu \in S(\mu) \subseteq \operatorname{int} C_+$ and $u_\theta \in S(\theta) \subseteq \operatorname{int} C_+$ (see Proposition 3.3) and we can have $u_\mu \le u_\theta$ (see the proof of Proposition 3.5).

We introduce the following Caratheodory function

$$\gamma_\lambda(z,x) = \begin{cases} \lambda f(z,u_\mu(z)) & \text{if } x < u_\mu(z), \\ \lambda f(z,x) & \text{if } u_\mu(z) \le x \le u_\theta(z), \\ \lambda f(z,u_\theta(z)) & \text{if } u_\theta(z) < x. \end{cases} \tag{3.7}$$

Let

$$\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)\,ds$$

and consider the C^1-functional $\sigma_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \int_\Omega G(Du)\,dz - \int_\Omega \Gamma_\lambda(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly σ_λ is coercive (see (3.7)) and sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \sigma_\lambda(u_0) = \inf[\sigma_\lambda(u) : u \in W_0^{1,p}(\Omega)] & \Rightarrow \sigma'_\lambda(u_0) = 0, \\ & \Rightarrow A(u_0) = N_{\gamma_\lambda}(u_0). \end{aligned} \tag{3.8}$$

Acting on (3.8) first with $(u_\mu - u_0)^+ \in W_0^{1,p}(\Omega)$ and then with $(u_0 - u_\theta)^+ \in W_0^{1,p}(\Omega)$ we show that

$$\begin{aligned} u_0 \in [u_\mu, u_\theta] = \{u \in W_0^{1,p}(\Omega) : u_\mu(z) \le u(z) \le u_\theta(z) \text{ a.e. in } \Omega\} \\ \Rightarrow \ u_0 \in S(\lambda) \subseteq \operatorname{int} C_+ \end{aligned}$$

(see (3.7) and (3.8)).

Let $\rho = ||u_\theta||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis H(f)(iv). Then

$$\begin{aligned} -\operatorname{div} a(Du_\mu(z)) + \mu\xi_\rho u_\mu(z)^{p-1} & = \mu f(z,u_\mu(z)) + \mu\xi_\rho u_\mu(z)^{p-1} \\ & \le \lambda f(z,u_0(z)) + \lambda\xi_\rho u_0(z)^{p-1} \end{aligned}$$

(see H(f)(iv) and recall $u_\mu \le u_0$, $\mu < \lambda$)

$$\begin{aligned} & = -\operatorname{div} a(Du_0(z)) + \lambda\xi_\rho u_0(z)^{p-1} \quad \text{a.e. in } \Omega, \\ & \Rightarrow u_0 - u_\mu \in \operatorname{int} C_+ \end{aligned}$$

(see Proposition 2.5).

In a similar fashion, we show that $u_\theta - u_0 \in \operatorname{int} C_+$. So, we have proved that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[u_\mu, u_\theta]. \tag{3.9}$$

From (3.5) and (3.7) it follows that

$$\begin{aligned} \varphi_\lambda |_{[u_\mu,u_\theta]} & = \sigma_\lambda |_{[u_\mu,u_\theta]} + \beta^*_\lambda \quad \text{with } \beta^*_\lambda \in \mathbb{R}, \\ & \Rightarrow \ u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda \qquad \text{(see (3.9))}, \\ & \Rightarrow \ u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \quad \text{(see Proposition 2.3)}. \end{aligned}$$

Hypothesis H(f)(iii) implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{\varepsilon}{p} x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{3.10}$$

Then, for $u \in C_0^1(\overline{\Omega})$ with $||u||_{C_0^1(\overline{\Omega})} \le \delta$, we have

$$\varphi_\lambda(u) \ge \frac{1}{p}\left[\frac{c_1}{p-1} - \frac{\lambda\varepsilon}{\widehat{\lambda}_1}\right]||u||^p$$

(see Corollary 2 and (3.5), (3.10)). Choosing $\varepsilon \in (0, c_1\widehat{\lambda}_1/((p-1)\lambda))$, we see that $u = 0$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_λ, hence $u = 0$ is a local $W_0^{1,p}(\Omega)$-minimizer of φ_λ (see Proposition 2.3).

Therefore we have two local minimizers 0, u_0 of φ_λ. Without any loss of generality we may assume that $\varphi_\lambda(0) = 0 \le \varphi_\lambda(u_0)$ (the analysis is similar if the opposite inequality holds). As in Aizicovici, Papageorgiou and Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0,1)$ small such that $||u_0|| > \rho$ and

$$\varphi_\lambda(0) = 0 \le \varphi_\lambda(u_0) < \inf[\varphi_\lambda(u) : ||u - u_0|| = \rho] = \eta_\rho^\lambda. \tag{3.11}$$

Since φ_λ is coercive, it satisfies the Palais–Smale condition. This fact together with (3.11) permit the application of the mountain pass theorem (see, for example, Gasinski and Papageorgiou [12, p. 648]). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\eta_\rho^\lambda \le \varphi_\lambda(\widehat{u}), \tag{3.12}$$

$$\varphi'_\lambda(\widehat{u}) = 0. \tag{3.13}$$

From (3.11), (3.12) we have $\widehat{u} \notin \{0, u_0\}$. From (3.13) we have

$$\widehat{u} \in S(\lambda) \subseteq \operatorname{int} C_+. \qquad \square$$

Proposition 3.7. *If hypotheses* H(a), H(f) *hold, then* $\lambda_* \in \mathcal{S}$.

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{S}$ be a sequence such that

$$\lambda_n > \lambda_* \quad \text{for all } n \ge 1 \text{ and } \lambda_n \downarrow \lambda_* \text{ as } n \to \infty.$$

We can find $u_n \in S(\lambda_n) \subseteq \operatorname{int} C_+$ such that

$$A(u_n) = \lambda_n N_f(u_n) \quad \text{for all } n \ge 1. \tag{3.14}$$

From the proof of Proposition 3.5, we know that we can have

$$u_n \le u_1 \quad \text{for all } n \ge 1. \tag{3.15}$$

From (3.6) we know that

$$f(z,x) \le c_6 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.16}$$

From (3.14)–(3.16), via Lemma 2.2, we infer that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^p(\Omega). \tag{3.17}$$

On (3.14) we act with $u_n - u_* \in W_0^{1,p}(\Omega)$. Passing to the limit as $n \to \infty$ and using (3.17), we obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0 \;\Rightarrow\; u_n \to u_* \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.4). So, if in (3.14) we pass to the limit as $n \to \infty$, then

$$A(u_*) = \lambda_* N_f(u_*) \;\Rightarrow\; u_* \in C_+.$$

We need to show that $u_* \neq 0$. From (3.14) we have

$$-\mathrm{div}\, a(Du_n(z)) = \lambda_n f(z, u_n(z)) \quad \text{a.e. in } \Omega, \quad u_n\,|_{\partial\Omega} = 0.$$

From Ladyzhenskaya and Ural'tseva [14, p. 286], we know that we can find $M_1 > 0$ such that

$$||u_n||_\infty \leq M_1 \quad \text{for all } n \geq 1.$$

Then invoking the regularity result of Lieberman[17, p. 320], we can find $\beta \in (0, 1)$ and $M_2 > 0$ such that

$$u_n \in C_0^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad ||u_n||_{C_0^{1,\beta}(\overline{\Omega})} \leq M_1 \quad \text{for all } n \geq 1.$$

Since $C_0^{1,\beta}(\overline{\Omega})$ is embedded compactly in $C_0^1(\overline{\Omega})$, we may assume that $u_n \to u_*$ in $C_0^1(\overline{\Omega})$. Suppose that $u_* = 0$. Then

$$u_n \to 0 \quad \text{in } C_0^1(\overline{\Omega}). \tag{3.18}$$

Hypothesis H(f)(iii) implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$f(z, x) \leq \varepsilon x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta]. \tag{3.19}$$

From (3.18) we know that we can find $n_0 \geq 1$ such that

$$\begin{aligned} u_n(z) &\in [0,\delta] && \text{for all } z \in \overline{\Omega}, \text{ all } n \geq n_0, \\ &\Rightarrow\; -\mathrm{div}\, a(Du_n(z)) \leq \lambda_n \varepsilon u_n(z)^{p-1} && \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0, \end{aligned}$$

(see (3.19))

$$\Rightarrow\; \frac{c_1}{p-1} ||Du_n||_p^p \leq \lambda_n \varepsilon ||u_n||_p^p \leq \frac{\lambda_n}{\widehat{\lambda}_1} \varepsilon ||Du_n||_p^p \quad \text{for all } n \geq n_0,$$

(see Lemma 2.2 and (3.5))

$$\begin{aligned} &\Rightarrow\; \frac{c_1 \widehat{\lambda}_1}{(p-1)\varepsilon} \leq \lambda_n && \text{for all } n \geq n_0, \\ &\Rightarrow\; \frac{c_1 \widehat{\lambda}_1}{(p-1)\varepsilon} \leq \lambda_*. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \downarrow 0$ and reach a contradiction. Hence

$$u_* \neq 0 \Rightarrow \lambda_* \in \mathcal{S}. \qquad \square$$

So, summarizing the situation for problem $(\mathrm{P})_\lambda$, we can state the following bifurcation-type result.

Theorem 3.8. *If hypotheses* H(a), H(f) *hold, then there exists* $\lambda_* > 0$ *such that*:

(a) *for every* $\lambda > \lambda_*$ *problem* $(\mathrm{P})_\lambda$ *has at least two nontrivial positive solutions* $u_0, \widehat{u} \in \operatorname{int} C_+$;
(b) *for* $\lambda = \lambda_*$ *problem* $(\mathrm{P})_\lambda$ *has at least one positive solution* $u_* \in \operatorname{int} C_+$;
(c) *for* $\lambda \in (0, \lambda_*)$ *problem* $(\mathrm{P})_\lambda$ *has no nontrivial positive solution.*

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Manuscript received July 2012, 19

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