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A NOTE ON STABLE SHEAVES ON ENRIQUES SURFACES

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Abstract. We shall give a necessary and sufficient condition for the existence of stable sheaves on Enriques surfaces based on results of Kim, Yoshioka, Hauzer and Nuer. For unnodal Enriques surfaces, we also study the relation of virtual Hodge "polynomial" of the moduli stacks.

1. Introduction. Studies of moduli spaces of stable sheaves on Enriques surfaces were started by a series of works of Kim [5], [6], [7], [8], [9]. In particular, he studied exceptional bundles and the singular locus of the moduli spaces. Recently the type of singularities are investigated by Yamada [17]. For the topological properties of the moduli spaces, the author [19] computed the Hodge polynomials of the moduli spaces if the rank is odd. In particular, the condition for the non-emptiness of the moduli spaces are known. For the even rank case, by extending our arguments, Hauzer [4] related the virtual Hodge "polynomial" of the moduli spaces to those for rank 2 or 4. Then Nuer [12] gave the condition for the non-emptiness by studying the non-emptiness for rank 2 and 4 cases. The main purpose of this note is to give another proof of his result on the non-emptiness.

THEOREM 1.1. *Let X be an unnodal Enriques surface over $\mathbb{C}$. For $r, s \in \mathbb{Z}$ and $L \in \mathrm{NS}(X)$ such that $r-s$ is even, let $\mathcal{M}_H(r, L, -\frac{s}{2})$ be the stack of semi-stable sheaves E of rank $r > 0$, $\det E = L$ and $\chi(E) = \frac{r-s}{2}$, where the polarization is H. Assume that $\gcd(r, c_1(L), \frac{r-s}{2}) = 1$, i.e., the Mukai vector is primitive. Then $\mathcal{M}_H(r, L, -\frac{s}{2}) \neq \emptyset$ for a general H if and only if*

(i) $\gcd(r, c_1(L), s) = 1$ *and* $(c_1(L)^2) + rs \geq -1$ *or*
(ii) $\gcd(r, c_1(L), s) = 2$ *and* $(c_1(L)^2) + rs \geq 2$ *or*
(iii) $\gcd(r, c_1(L), s) = 2$, $(c_1(L)^2) + rs = 0$ *and* $L \equiv \frac{r}{2}K_X \mod 2$.

If $r = 0$, then by assuming L to be effective, the same claim holds.

Since v is primitive and H is general, semi-stability implies stability.

In order to explain the difference of the proofs, we first mention the results in [19] and [4]. In [19], we introduced the virtual Hodge "polynomial" $e(\mathcal{M}_H(r, L, -\frac{s}{2}))$ of the moduli stacks, which is an extension of the virtual Hodge polynomial of an algebraic set and showed that it is preserved under a special kind of Fourier-Mukai transform. As an application, we showed that $e(\mathcal{M}_H(r, L, -\frac{s}{2}))$ is the same as $e(\mathcal{M}_H(1, 0, \frac{1}{2} - n))$ if r is odd, where $2n =$

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$(c_1(L)^2) + rs + 1$ [19, Thm. 4.6]. In particular we get the condition $(c_1(L)^2) + rs \geq -1$ for the non-emptiness. Hauzer [4] generalized our method and showed that $e(\mathcal{M}_H(r, L, -\frac{s}{2}))$ is the same as $e(\mathcal{M}_H(r', L', -\frac{s'}{2}))$ where $r' = 2, 4$ and $(c_1(L')^2) + r's' = (c_1(L)^2) + rs$. For the rank 2 case, the condition of non-emptiness follows by Kim's results [9]. Thus the remaining problem is to treat the rank 4 case.

For this problem, Nuer [12, Thm. 5.1] constructed μ-stable vector bundles of rank 4 by Serre construction, and got the condition for the non-emptiness. On the other hand, we shall reduce the rank 4 case to the rank 2 case by improving Hauzer's argument (Theorem 2.6). Combining Kim's results [9], Theorem 1.1 follows. For convenience sake, we also give another argument for the rank 2 case using a relative Fourier-Mukai transform associated to an elliptic fibration. Replacing virtual Hodge "polynomial" by numbers of $\mathbb{F}_q$-rational points, our result also holds for unnodal Enriques surfaces over an algebraically closed field of characteristic $p \neq 2$. As a corollary of Theorem 1.1, by adding a deformation argument, we shall treat the nodal case in Section 3.

Finally I would like to remark another approach in Appendix. For our argument, main tool is a special kind of Fourier-Mukai transforms. For the case of K3 surfaces, Toda [16] proved a certain counting invariant of the moduli stack of Bridgeland semi-stable objects are invariant under Fourier-Mukai transforms. Since Gieseker stability corresponds to the large volume limit of Bridgeland stability, it is possible to get Theorem 1.1 by a more sophisticated method, i.e., Bridgeland theory of stability conditions [2]. For a more general treatment, we recommend a reference [13].

2. Proof of Theorem 1.1.

2.1. Notation and some tools. We prepare several notation and results which will be used.

The Mukai vector $v(x)$ of $x \in K(X)$ is defined as an element of $H^*(X, \mathbb{Q})$:

$$
\begin{aligned}
v(x) &:= \mathrm{ch}(x)\sqrt{\mathrm{td}_X} \\
&= \mathrm{rk}(x) + c_1(x) + \left(\frac{\mathrm{rk}(x)}{2}\varrho_X + \mathrm{ch}_2(x)\right) \in H^*(X, \mathbb{Q}),
\end{aligned} \tag{2.1}
$$

where ϱ_X is the fundamental class of X. We also introduce Mukai's pairing on $H^*(X, \mathbb{Q})$ by $\langle x, y \rangle := -\int_X x^\vee \wedge y$. Then we have an isomorphism of lattices:

$$
(v(K(X)), \langle\ ,\ \rangle) \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus E_8(-1)\,. \tag{2.2}
$$

DEFINITION 2.1. We call an element of $v(K(X))$ by the Mukai vector. A Mukai vector v is primitive, if v is primitive as an element of $v(K(X))$.

We denote the torsion free quotient of $\mathrm{NS}(X)$ by $\mathrm{NS}_{\mathrm{f}}(X)$, that is, $\mathrm{NS}_{\mathrm{f}}(X) = \mathrm{NS}(X)/\mathbb{Z}K_X$.

LEMMA 2.2. *Let $v = (r, c_1, -\frac{s}{2})$ ($r, s \in \mathbb{Z}$, $2 \mid r - s$, $c_1 \in \mathrm{NS}_{\mathrm{f}}(X)$) be a Mukai vector.*

(1) *v is primitive if and only if* $\gcd(r, c_1, \frac{r-s}{2}) = 1$.

(2) *Assume that v is primitive. We set $\ell := \gcd(r, c_1, s)$. Then $\ell = 1, 2$.*

(a) *If* $\ell = 1$, *then* $\gcd(r, c_1, 2) = 1$.

(b) *If* $\ell = 2$, *then* $2 \mid r$, $2 \mid c_1$, $2 \mid s$ *and* $r + s \equiv 2 \mod 4$.

PROOF. (1) For $E = r\mathcal{O}_X + F \in K(X)$ with $\operatorname{rk} F = 0$, $v(E) = (r, 0, \frac{r}{2}) + (0, D, t)$, where $D \in \mathrm{NS_f}(X)$ and $t \in \mathbb{Z}$. Then $v(E)$ is primitive if and only if $\gcd(r, D, t) = 1$. If $v = v(E)$, then $c_1 = D$ and $t + \frac{r}{2} = -\frac{s}{2}$. Hence $\gcd(r, c_1, \frac{r-s}{2}) = \gcd(r, D, t)$, which shows the claim.

(2) It is [4, Lem. 2.5]. For convenience sake, we give a proof. Since $s = r + 2\frac{s-r}{2}$, $\ell = 1, 2$. If $\ell = 1$, then $\gcd(r, c_1, 2) = 1$. If $\ell = 2$, then $2 \mid r$, $2 \mid c_1$. Since $\gcd(r, c_1, \frac{s-r}{2}) = 1$, $r + s \equiv 2 \mod 4$. □

For a variety Y over $\mathbb{C}$, the cohomology with compact support $H_c^*(Y, \mathbb{Q})$ has a natural mixed Hodge structure. Let $e^{p,q}(Y) := \sum_k (-1)^k h^{p,q}(H_c^k(Y))$ be the virtual Hodge number and $e(Y) := \sum_{p,q} e^{p,q}(Y) x^p y^q$ the virtual Hodge polynomial of Y.

For $\alpha \in \mathrm{NS}(X)_{\mathbb{Q}}$, a torsion free sheaf E is α-twisted semi-stable with respect to H, if

$$\frac{\chi(F(-\alpha + nH))}{\operatorname{rk} F} \leq \frac{\chi(E(-\alpha + nH))}{\operatorname{rk} E} \quad (n \gg 0) \tag{2.3}$$

for all subsheaf F of E [10]. $\mathcal{M}_H^\alpha(v)$ denotes the moduli stack of α-twisted semi-stable sheaves E with $v(E) = v$, where H is the polarization. (H, α) is general with respect to v, if equality in (2.3) implies

$$\frac{v(F)}{\operatorname{rk} F} = \frac{v(E)}{\operatorname{rk} E}.$$

In particular, if v is primitive, then $\mathcal{M}_H^\alpha(v)$ consists of α-twisted stable objects for a general pair (H, α). If $\alpha = 0$, then we write $\mathcal{M}_H(v)$. Then $\mathcal{M}_H^\alpha(v)$ is described as a quotient stack $[Q^{ss}/GL(N)]$, where Q^{ss} is a suitable open subscheme of $\mathrm{Quot}_{\mathcal{O}_X^{\oplus N}/X}$. We define the virtual Hodge "polynomial" of $\mathcal{M}_H^\alpha(v)$ by

$$e(\mathcal{M}_H^\alpha(v)) = e(Q^{ss})/e(GL(N)) \in \mathbb{Q}(x, y). \tag{2.4}$$

It is easy to see that $e(Q^{ss})/e(GL(N))$ does not depend on the choice of Q^{ss}. The following was essentially proved in [18, Sect. 3.2] (see also [20, Sect. 2.2]).

PROPOSITION 2.3. *Let X be a surface such that K_X is numerically trivial. Let (H, α) be a pair of ample divisor H and a $\mathbb{Q}$-divisor α. Then $e(\mathcal{M}_H^\alpha(v))$ does not depend on the choice of H and α, if (H, α) is general with respect to v.*

By using a special kind of Fourier-Mukai transform called (-1)-reflection and using Proposition 2.3, we get the following result.

PROPOSITION 2.4 ([19, Prop. 4.5]). *Let X be an unnodal Enriques surface. Assume that $r, s > 0$. Then*

(1)

$$e\left(\mathcal{M}_H^\alpha\left(r, c_1, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H^\alpha\left(s, -c_1, -\frac{r}{2}\right)\right)$$

for a general (H,α), if $(c_1^2)<0$, i.e., $\langle v^2\rangle < rs$, where $v=(r,c_1,-\frac{s}{2})$. In particular, if $r>\langle v^2\rangle$, then we get our claim.

(2) *If we specify the first Chern class as an element of* $\operatorname{Pic}(X)\cong \operatorname{NS}(X)$*, then we also have*

$$e\left(\mathcal{M}_H^\alpha\left(r, L+\frac{r}{2}K_X, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H^\alpha\left(s, -\left(L+\frac{s}{2}K_X\right), -\frac{r}{2}\right)\right)$$

for a general (H,α), if $(c_1(L)^2)<0$, i.e., $\langle v^2\rangle < rs$, where $v=(r,c_1(L),-\frac{s}{2})$.

REMARK 2.5. (1) For the proof of Proposition 2.4 (2), we use the description of the (-1)-reflection as a Fourier-Mukai transform (see Appendix). Then the first Chern class $L+\frac{r}{2}K_X$ is replaced by $-[(L+\frac{r}{2}K_X)+\langle v, v(K_X)\rangle K_X] = -(L+\frac{s}{2}K_X)$.
(2) The same claim also holds for nodal case (see Appendix).

2.2. Reduction to the rank 2 case. From Subsection 2.2 to Subsection 2.5, we assume that X is an unnodal Enriques surface and r is even (and hence s is also even). We also assume that $\alpha=0$, that is, we consider the moduli stack of ordinary Gieseker semi-stable sheaves $\mathcal{M}_H(v)$. We shall prove the following result in this subsection.

THEOREM 2.6. *Let $v=(r,c_1,-\frac{s}{2})$ be a primitive Mukai vector such that $r>0$ is even.*

(1) *If* $\gcd(r,c_1,s)=1$*, then $e(\mathcal{M}_H(r,c_1,-\frac{s}{2}))=e(\mathcal{M}_H(2,\xi,-\frac{s'}{2}))$ for a general H, where ξ is a primitive element of* $\operatorname{NS}_{\mathrm{f}}(X)$ *and $(\xi^2)+2s'=(c_1^2)+rs$.*
(2) *If* $\gcd(r,c_1,s)=2$*, then $e(\mathcal{M}_H(r,c_1,-\frac{s}{2}))=e(\mathcal{M}_H(2,0,-\frac{s'}{2}))$ for a general H, where $2s'=(c_1^2)+rs$.*

For the proof of this result, we shall slightly improve Hauzer's argument. Let $\mathbb{Z}\sigma+\mathbb{Z}f$ be a hyperbolic lattice in $\operatorname{NS}(X)$:

$$(\sigma^2)=(f^2)=1,\ (\sigma,f)=1\,.$$

The main difference of [19] and [4] is the case $\mathcal{M}_H(r,c_1,-\frac{s}{2})$ such that r is even and $c_1=\frac{r}{2}bf+\frac{r}{2}b'\sigma+\xi$, $b,b'=0,1$, $\xi\in E_8(-1)$. In order to treat this case, we shall modify the argument in [4]. For a primitive Mukai vector $(r,\frac{r}{2}bf+\xi,-\frac{s}{2})$ ($b=0,-1,1,\xi\in E_8(-1)$), [4, Cor. 2.6] implies that $\gcd(r,\xi,s)=1,2$. Indeed $1=\gcd(r,\frac{r}{2}bf+\xi,\frac{r-s}{2})=\gcd(\frac{r}{2},\frac{s}{2},\xi)$ implies $\gcd(r,\xi,s)=1,2$.

LEMMA 2.7. *For a primitive Mukai vector $v=(r,\frac{r}{2}bf+\xi,-\frac{s}{2})$ ($b=0,-1,1,\xi\in E_8(-1)$), we set $l:=\gcd(r,\xi,s)$.*

(1) *$e(\mathcal{M}_H(r,\frac{r}{2}bf+\xi,-\frac{s}{2}))=e(\mathcal{M}_H(r',\frac{r}{2}bf+\xi',-\frac{s'}{2}))$ for a general H, where $r'\equiv r \mod 2l$, $s'\equiv s \mod 2l$, $l=\gcd(r',\xi',s')$, $\xi'/l\in E_8(-1)$ is primitive and $r's'\geq r'>\langle v^2\rangle$.*
(2) *$e(\mathcal{M}_H(r,\frac{r}{2}bf+\xi,-\frac{s}{2}))=e(\mathcal{M}_H(s'',-(\frac{r}{2}bf+\xi''),-\frac{r'}{2}))$ for a general H, where $r'\equiv r \mod 2l$, $s''\equiv s \mod 2l$, $l=\gcd(s'',\xi'',r')$, $\xi''/l\in E_8(-1)$ is primitive and $r's''\geq s''>\langle v^2\rangle$.*

PROOF. We first note that the choice of H is not important by Proposition 2.3. So we do not explain about the choice of H. (1) We set $p := (r, \xi)$. For $v = (r, \frac{r}{2}bf + \xi, -\frac{s}{2})$, we take $D \in E_8(-1)$ such that $ve^D = (r, \frac{r}{2}bf + \xi_1, -\frac{s'}{2})$ satisfies ξ_1/p is primitive and $s' > \langle v^2 \rangle$. Since $s' = s - 2(\xi, D) - r(D^2)$, $s' \equiv s \mod 2l$. By Proposition 2.4, $e(\mathcal{M}_H(v)) = e(\mathcal{M}_H(s', -(\frac{r}{2}bf + \xi_1), -\frac{r}{2}))$. Since $l = (s', p)$, we take $D_1 \in E_8(-1)$ such that $(s', -(\frac{r}{2}bf + \xi_1), -\frac{r}{2})e^{D_1} = (s', -(\frac{r}{2}bf + \xi'), -\frac{r'}{2})$ satisfies ξ'/l is primitive and $r' > \langle v^2 \rangle$. We also have $r' = r + 2(\xi_1, D_1) - s'(D_1^2) \equiv r \mod 2l$. Applying Proposition 2.4, we have

$$e\left(\mathcal{M}_H\left(s', -\left(\frac{r}{2}bf + \xi_1\right), -\frac{r}{2}\right)\right) = e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right).$$

(2) For $(r', \frac{r}{2}bf + \xi', -\frac{s'}{2})$ in (1), we take $D_2 \in E_8(-1)$ such that $(r', \frac{r}{2}bf + \xi'', -\frac{s''}{2}) = (r', \frac{r}{2}bf + \xi', -\frac{s'}{2})e^{D_2}$ satisfies $\xi''/l \in E_8(-1)$ is primitive, $s'' > \langle v^2 \rangle$. Then we have

$$e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right) = e\left(\mathcal{M}_H\left(s'', -\left(\frac{r}{2}bf + \xi''\right), -\frac{r'}{2}\right)\right)$$

by Proposition 2.4. □

LEMMA 2.8. *For a primitive Mukai vector $v = (r, \frac{r}{2}bf + \xi, -\frac{s}{2})$ ($b = 0, -1, 1, \xi \in E_8(-1)$), there exist some zeta and t such that*

$$e\left(\mathcal{M}_H\left(r, \frac{r}{2}bf + \xi, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(2, \zeta, -\frac{t}{2}\right)\right)$$

for a general H.

PROOF. (1) We first assume that $r \equiv 0 \mod 4$ and $s \equiv 2 \mod 4$. By Lemma 2.7, we have

$$e\left(\mathcal{M}_H\left(r, \frac{r}{2}bf + \xi, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right)$$

for a general H, where $r' \equiv 0 \mod 2l$, $s' \equiv 2 \mod 2l$, $\xi'/l \in E_8(-1)$ is primitive and $r' > \langle v^2 \rangle$. For $\eta \in E_8(-1)$, we set $D := \sigma - \frac{(\eta^2)}{2}f + \eta$. Then $(D^2) = 0$. Since $r \equiv 0 \mod 2l$, we can choose η such that

$$s' - rb - 2 = 2(\xi', \eta). \tag{2.5}$$

Then $(\frac{r}{2}bf + \xi', D) = \frac{r}{2}b + (\xi', \eta) = \frac{s'}{2} - 1$ and

$$\begin{aligned}\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)e^D &= \left(r', \frac{r}{2}bf + \xi' + r'D, -\frac{s' - 2\left(\frac{r}{2}bf + \xi', D\right)}{2}\right)\\ &= \left(r', \frac{r}{2}bf + \xi' + r'D, -1\right).\end{aligned} \tag{2.6}$$

Hence

$$e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right) = e\left(\mathcal{M}_H\left(2, \zeta, -\frac{r'}{2}\right)\right)$$

for a general H, where $\zeta = -(\frac{r}{2}bf + \xi' + r'D)$.

(2) We next assume that $r \equiv 2 \mod 4$. If $b = 0$ and $l = 2$, then by using Lemma 2.7 (2), we have

$$e\left(\mathcal{M}_H\left(r, \xi, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(s'', -\xi'', -\frac{r'}{2}\right)\right)$$

for a general H. Since $r' \equiv 2 \mod 2l$, it is reduced to the case (1).

Assume that $b = \pm 1$ or $l = 1$. By Lemma 2.7 (1), we have

$$e\left(\mathcal{M}_H\left(r, \frac{r}{2}bf + \xi, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right)$$

for a general H, where $r' \equiv 2 \mod 2l$, ξ'/l is primitive and $r' > \langle v^2 \rangle$. Since ξ'/l is primitive and $\frac{r}{2}$ is odd, we take $\eta \in E_8(-1)$ such that $\frac{r}{2}b + (\xi', \eta) = 1$. We set $D := \sigma - \frac{(\eta^2)}{2} f + \eta$. Then $(D, \frac{r}{2}bf + \xi') = \frac{r}{2}b + (\xi', \eta) = 1$. Hence

$$\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right) e^{\left(\frac{s'}{2}-1\right)D} = \left(r', \frac{r}{2}bf + \xi' + r'\left(\frac{s'}{2} - 1\right) D, -1\right).$$

Applying Proposition 2.4, we get

$$e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right) = e\left(\mathcal{M}_H\left(2, \zeta, -\frac{r'}{2}\right)\right)$$

for a general H, where $\zeta = -(\frac{r}{2}bf + \xi' + r'(\frac{s'}{2} - 1)D)$.

(3) Finally we assume that $r \equiv 0 \mod 4$ and $s \equiv 0 \mod 4$. If $l = 2$, then $2 \mid (\frac{r}{2}bf + \xi)$. By Lemma 2.2 (2), v is not primitive. Hence $l = 1$. By using Lemma 2.7 (1) again, we have

$$e\left(\mathcal{M}_H\left(r, \frac{r}{2}bf + \xi, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(r', \frac{r}{2}bf + \xi', -\frac{s'}{2}\right)\right)$$

for a general H, where ξ' is primitive and $r' > \langle v^2 \rangle$. Since we can take $\eta \in E_8(-1)$ with $\frac{r}{2}b + (\xi', \eta) = 1$, as in the case (2), we get the claim. □

We shall next treat the general case. We use induction on r. We set $c_1 := d_1\sigma + d_2 f + \xi$, $\xi \in E_8(-1)$. Replacing v by $v\exp(k\sigma)$, we may assume that $-\frac{r}{2} < d_1 \leq \frac{r}{2}$. We first assume that $d_1 \neq 0, \frac{r}{2}$. We note that $(c_1, f) = d_1$. Replacing v by $v\exp(\eta)$, $\eta \in E_8(-1)$, we may assume that $s > \langle v^2 \rangle$. Then by Proposition 2.4, $e(\mathcal{M}_H(v)) = e(\mathcal{M}_H(s, -c_1, -\frac{r}{2}))$ for a general H. We take an integer k such that $0 < r + 2d_1k \leq 2|d_1| < r$. Then $v\exp(kf) = (s, (-c_1 + skf), -\frac{r'}{2})$, where $r' = r + 2d_1k$. Since $s > \langle v^2 \rangle$, Proposition 2.4, implies that $e(\mathcal{M}_H(s, (-c_1 + skf), -\frac{r'}{2})) = e(\mathcal{M}_H(r', (c_1 - skf), -\frac{s}{2}))$ for a general H. By induction hypothesis, we get our claim.

If $d_1 = 0, \frac{r}{2}$, then we may assume that $-\frac{r}{2} < d_2 \leq \frac{r}{2}$. If $d_2 \neq 0, \frac{r}{2}$, then we can apply the same argument and get our claim. If $(d_1, d_2) = (0, 0), (\frac{r}{2}, 0), (0, \frac{r}{2})$, then the claim follows from Lemma 2.8.

Assume that $(d_1, d_2) = (\frac{r}{2}, \frac{r}{2})$. We may assume that $\xi = k\xi'$, ξ' is primitive and $0 \leq k \leq \frac{r}{2}$.

For $\eta \in E_8(-1)$, we set $\sigma' := \sigma - \frac{(\eta^2)}{2} f + \eta$. Then σ' and f spans a hyperbolic lattice and

$$\begin{aligned} \left(\frac{r}{2}(\sigma + f) + \xi, f\right) &= \frac{r}{2} \\ \left(\frac{r}{2}(\sigma + f) + \xi, \sigma'\right) &= \frac{r}{2}\left(1 - \frac{(\eta^2)}{2}\right) + (\xi, \eta). \end{aligned} \tag{2.7}$$

Replacing η by $-\eta$ if necessary, we can take η such that

$$(\xi', \eta) = \begin{cases} -1, & 2 \mid (\eta^2)/2 \\ 1, & 2 \nmid (\eta^2)/2. \end{cases} \tag{2.8}$$

$$\frac{r}{2}\left(1 - \frac{(\eta^2)}{2}\right) + (\xi, \eta) \equiv \begin{cases} \frac{r}{2} - k \mod r & 2 \mid (\eta^2)/2 \\ k \mod r & 2 \nmid (\eta^2)/2. \end{cases} \tag{2.9}$$

If $k \neq \frac{r}{2}, 0$, then we can reduce to the case where $|d_1| < \frac{r}{2}$. If $k = 0$, then choosing η with $(\eta^2) = -2$, we can reduced to the case $d_1 = 0$. If $k = \frac{r}{2}$, then we choose η satisfying $((\xi' - \eta)^2) \equiv ({\xi'}^2) + 2 \mod 4$. Then

$$\frac{r}{2}\left(1 - \frac{(\eta^2)}{2}\right) + \frac{r}{2}(\xi', \eta) \equiv 0 \mod r. \tag{2.10}$$

Hence we can also reduce to the case where $d_1 = 0$. Therefore Theorem 2.6 holds. $\square$

REMARK 2.9. In [4], Hauzer takes a hyperbolic lattice spanned by σ and $\sigma + f + e_1$, where $e_1 \in E_8(-1)$ is a (-2)-vector. Then $c_1 = (r + (\xi, e_1))\sigma' + (r/2)f' + \xi'$.

By Theorem 2.6, Theorem 1.1 for $r > 0$ is reduced to the following claim.

PROPOSITION 2.10 (Kim [9]). *Assume that $v := (2, c_1(L), -\frac{s}{2})$ is primitive. Then $\mathcal{M}_H(2, L, -\frac{s}{2}) \neq \emptyset$ if and only if*

(i) $\gcd(2, c_1(L)) = 1$ *and* $\langle v^2 \rangle \geq -1$ *or*
(ii) $\gcd(2, c_1(L)) = 2$ *and* $\langle v^2 \rangle \geq 2$ *or*
(iii) $\gcd(2, c_1(L)) = 2$, $\langle v^2 \rangle = 0$ *and* $L \equiv K_X \mod 2$.

For the case of Proposition 2.10 (iii), by using Proposition 2.4 (2), we have Theorem 1.1 (iii). In the next subsection, we shall give another proof of Kim's result.

2.3. Relative Fourier-Mukai transform. For $G \in K(X)$ with $\operatorname{rk} G > 0$, we define G-twisted semi-stability replacing the Hilbert polynomial $\chi(E(nH))$ by the G-twisted Hilbert polynomial $\chi(G^{\vee} \otimes E(nH))$. $M_H^G(r, L, -\frac{s}{2})$ denotes the moduli scheme of G-twisted semi-stable sheaves E with $v(E) = (r, c_1(L), -\frac{s}{2})$ and $\det E = L$. If $G = \mathcal{O}_X$, then we also denote $M_H^G(r, L, -\frac{s}{2})$ by $M_H(r, L, -\frac{s}{2})$. The G-twisted semi-stability is the same as the α-twisted semi-stability, where $\alpha = c_1(G)/\operatorname{rk} G$.

We have an elliptic fibration $X \to \mathbb{P}^1$ such that $2f$ is the divisor class of a fiber. Let G_1 be a locally free sheaf on X such that $v(G_1) = v(\mathcal{O}_X) + v(\mathcal{O}_X(\sigma)) + (0, 0, k)$. We set $Y := M_{H+nf}^{G_1}(0, 2f, 1)$, where H is an ample divisor on X and $n \geq 0$. Then $\chi(G_1, E) = -\langle v(G_1), v(E)\rangle = 0$ for $E \in M_{H+nf}^{G_1}(0, 2f, 1)$.

LEMMA 2.11. *Y consists of G_1-twisted stable sheaves.*

PROOF. If $E \in M_{H+nf}^{G_1}(0, 2f, 1)$ is properly G_1-twisted semi-stable, then there is a proper subsheaf E_1 of E such that $\chi(G_1, E_1) = 0$ and E/E_1 is also purely 1-dimensional. We set $v(E_1) = (0, \xi_1, a)$, $a \in \mathbb{Z}$. Then $(\xi_1, c_1(G_1)) = 2a \in 2\mathbb{Z}$. Since $(c_1(E_1), c_1(G_1))$, $(c_1(E/E_1), c_1(G_1)) \geq 0$ and $(c_1(E), c_1(G_1)) = 2$, $(c_1(E_1), c_1(G_1)) = 0$ or $(c_1(E/E_1), c_1(G_1)) = 0$. If every singular fiber is irreducible, then $(c_1(E_1), c_1(G_1)) > 0$ and $(c_1(E/E_1), c_1(G_1)) > 0$. Therefore Y consists of G_1-twisted stable sheaves. $\square$

By [1], Y is a smooth projective surface which is a compactification of $\operatorname{Pic}^1_{X/C}$. Hence $Y \cong X$. Let $\mathcal{E}$ be a universal family. Let $\Psi : \mathbf{D}(X) \to \mathbf{D}(Y)$ be a contravariant Fourier-Mukai transform defined by

$$\Psi(E) := \mathbf{R}\operatorname{Hom}_{p_Y}(p_X^*(E), \mathcal{E}), \tag{2.11}$$

where p_X and p_Y are the projections from $X \times Y$ to X and Y respectively.

Let L_1 be a line bundle on $C \in |H|$ and set $G_2 := \Psi(L_1)[1]$ (see the above of [22, Lem. 3.2.3]). We also set $\widehat{H} := -c_1(\Psi(G_1))$ ([22, Lem. 3.2.1]).

PROPOSITION 2.12 ([22, Prop. 3.4.5]). *Assume that $(c_1(L), f) = \frac{r}{2} \in \mathbb{Z}$ and $\chi(E, L_1) < 0$. Ψ induces an isomorphism*

$$\mathcal{M}_{H+nf}^{G_1}\left(r, L, -\frac{s}{2}\right) \cong \mathcal{M}_{\widehat{H}+nf}^{G_2}\left(0, D, -\frac{s'}{2}\right)$$

for $n \gg 0$, where D is an effective divisor such that $(D^2) = (c_1(L)^2) + rs$ and $(D, 2f) = r$.

REMARK 2.13. Replacing E by $E(mf)$ ($m \gg 0$), $\chi(E, L_1) < 0$ holds.

REMARK 2.14. Although G_1 is fixed, H is not fixed. So we can change H to be general.

COROLLARY 2.15. *Assume that $2 \nmid c_1(L)$. Then $\mathcal{M}_{H+nf}^{G_1}(2, L, -\frac{s}{2}) \neq \emptyset$ if and only if $(c_1(L)^2) + 2s \geq 0$.*

PROOF. Let D be the divisor in Proposition 2.12. Since $(D^2) = (c_1(L)^2) + 2s$, we shall prove that the condition is $(D^2) \geq 0$. Obviously the condition is necessary. Conversely assume that $(D^2) \geq 0$. Since $Y \cong X$ is unnodal, $|D|$ contains a reduced and irreducible curve C by [3, Thm. 3.2.1], where we also use $(D, f) = 1$ if $(D^2) = 0$. Then a line bundle F on C with $\chi(F) = -\frac{s'}{2}$ is a member of $\mathcal{M}_{\widehat{H}+nf}^{G_2}(0, D, -\frac{s'}{2})$. □

2.4. Rank 2 case.

PROPOSITION 2.16. *Assume that $2 \nmid c_1(L)$ is primitive. Then $\mathcal{M}_H(2, L, -\frac{s}{2}) \neq \emptyset$ for a general H if and only if $(c_1(L)^2) + 2s \geq 0$.*

PROOF. If $2 \nmid (c_1(L), f)$ or $2 \nmid (c_1(L), \sigma)$, then the claim follows from Corollary 2.15. Otherwise we may assume that $c_1(L) \in E_8(-1)$ and $c_1(L)$ is primitive. Then there is $\eta \in E_8(-1)$ with $(c_1(L), \eta) = 1$. We set $\sigma' := \sigma - \frac{(\eta^2)}{2} f + \eta$. Then $\mathbb{Z}\sigma' + \mathbb{Z}f$ spans a hyperbolic lattice and $(\sigma', c_1(L)) = 1$. Since X is unnodal and f is effective, σ' is effective and $2\sigma'$ defines an elliptic fibration. Therefore the claim also holds for this case. □

PROPOSITION 2.17. *Assume that $2 \mid c_1(L)$. Then $\mathcal{M}_H(2, L, -\frac{s}{2}) \neq \emptyset$ if and only if*

(i) $(c_1(L)^2) + 2s > 0$ *or*
(ii) $(c_1(L)^2) + 2s = 0$ *and* $L \equiv K_X \mod 2$.

PROOF. We may assume that $L = 0, K_X$. If there is a stable sheaf E, then $E \cong E(K_X)$ and $(c_1(L)^2)+2s \geq -2$, or $E \not\cong E(K_X)$ and $(c_1(L)^2)+2s \geq -1$. Since $4 \mid s$, $(c_1(L)^2)+2s = 2s \geq 0$.

Assuming $(c_1(L)^2)+2s > 0$, we first prove $\mathcal{M}_H(2, L, -\frac{s}{2}) \neq \emptyset$ for a general H. We set $k := \frac{s}{4} > 0$. Then $E_1 \oplus E_2$ with $v(E_1) = (1, 0, -k - \frac{1}{2})$, $v(E_2) = (1, 0, -k + \frac{1}{2})$ belongs to the moduli stack $\mathcal{M}_H(2, L, -\frac{s}{2})^{\mu\text{-}ss}$ of μ-semi-stable sheaves. Let $\mathcal{F}(v_1, v_2)$ be the substack of $\mathcal{M}_H(2, L, -\frac{s}{2})^{\mu\text{-}ss}$ consisting of E whose Harder-Narasimhan filtration $0 \subset F_1 \subset F_2 = E$ satisfies $v(F_1) = v_1$ and $v(F/F_1) = v_2$. Then

$$\begin{aligned} \dim \mathcal{F}(v_1, v_2) =& \langle v_1, v_2 \rangle + \dim \mathcal{M}_H(v_1) + \dim \mathcal{M}_H(v_2) \\ =& \langle v^2 \rangle - \langle v_1, v_2 \rangle . \end{aligned} \tag{2.12}$$

We set $v_1 = (1, \xi_1, -\frac{s_1}{2})$, $v_2 = (1, \xi_2, -\frac{s_2}{2})$. Then ξ_1 and ξ_2 are numerically trivial, $s_1 < s_2$ and $s_1 + s_2 = s$. Then $\langle v_1, v_2 \rangle = \frac{s_1+s_2}{2} = \frac{s}{2} > 0$. By the deformation theory, each irreducible component $\mathcal{M}$ of $\mathcal{M}_H(v)^{\mu\text{-}ss}$ satisfies $\dim \mathcal{M} \geq \langle v^2 \rangle$. Hence there is a stable sheaf.

We next treat the case where $(c_1(L)^2) + 2s = 0$. By [19], $M_H(2, K_X, 0) \cong X$ and $E(K_X) \cong E$ for all $E \in M_H(2, K_X, 0)$. Moreover there is a universal family which defines a Fourier-Mukai transform. Then for a stable sheaf E with $v(E) \equiv v \mod K_X$, we see that $E \in M_H(2, K_X, 0)$. In particular, $M_H(2, 0, 0) = \emptyset$. □

Therefore Proposition 2.10 holds by Proposition 2.16, 2.17, and we complete the proof of Theorem 1.1 for $r > 0$.

REMARK 2.18. Nuer constructed μ-stable vector bundles of rank 4 in [12, Thm. 5.1]. This reuslt ([12, Thm. 5.1]) does not follow from our method.

2.5. Rank 0 case. We shall prove Theorem 1.1 for $r = 0$. We first note that if $\mathcal{M}_H(0, L, -\frac{s}{2}) \neq \emptyset$, then L is effective. For the proof of Theorem 1.1, we use Proposition 2.12. By choosing a suitable elliptic fibration, we may assume that $(c_1(L), f) > 0$. Then we have

$$e\left(\mathcal{M}_H\left(0, L, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(r, L', -\frac{s'}{2}\right)\right),$$

where $(c_1(L'), 2f) = r$. Then the case of $r = 0$ is reduced to the case of $r > 0$ at least for $\gcd(c_1(L), s) = 1$ or $(c_1(L)^2) > 0$. Assume that $\gcd(c_1(L), s) = 2$ and $(c_1(L)^2) = 0$. Then $\mathcal{M}_H(0, L, -\frac{s}{2}) = \emptyset$ or $\mathcal{M}_H(0, L + K_X, -\frac{s}{2}) = \emptyset$. If $L \equiv 0 \mod 2$, then there is $\frac{r}{2}C \in |L|$ such that C is a smooth fiber of the elliptic fibration, and a stable vector bundle F of rank $\frac{r}{2}$ and $\chi(F) = -\frac{s}{2}$ on C is a member of $\mathcal{M}_H(0, L, -\frac{s}{2})$. Hence $\mathcal{M}_H(0, L, -\frac{s}{2}) \neq \emptyset$ if and only if $L \equiv 0 \mod 2$ as we claimed in Theorem 1.1.

REMARK 2.19. It is easy to see that [21, Thm. 1.7] holds for Enriques surfaces. Indeed a similar claim to [21, Prop. 2.7] (see Appendix) holds and [21, Prop. 2.8, Prop. 2.11] hold if we modify the number N in the claims suitably.

Then Theorem 1.1 for $r = 0$ can also be reduced to the claim for $r > 0$.

REMARK 2.20. Since X is unnodal, effectivity implies $(c_1(L)^2) \geq 0$ and $(c_1(L), H) > 0$. Conversely if $(c_1(L)^2) \geq 0$ and $(c_1(L), H) > 0$, then L is effective by the Riemann-Roch theorem.

3. A nodal case. We shall treat the nodal case by adding a deformation argument and results of Kim [5] and [8].

THEOREM 3.1. *Let X be a nodal Enriques surface over $\mathbb{C}$. We take $r, s \in \mathbb{Z}$ $(r > 0)$ and $L \in \mathrm{NS}(X)$ such that $r - s$ is even. Assume that $\gcd(r, c_1(L), \frac{r-s}{2}) = 1$, i.e., the Mukai vector is primitive. Then $\mathcal{M}_H(r, L, -\frac{s}{2}) \neq \emptyset$ for a general H if and only if*

(i) $\gcd(r, c_1(L), s) = 1$ *and* $(c_1(L)^2) + rs \geq -1$ *or*
(ii) $\gcd(r, c_1(L), s) = 2$ *and* $(c_1(L)^2) + rs \geq 2$ *or*
(iii) $\gcd(r, c_1(L), s) = 2$, $(c_1(L)^2) + rs = 0$ *and* $L \equiv \frac{r}{2}K_X \mod 2$ *or*
(iv) $(c_1(L)^2) + rs = -2$, $L \equiv D + \frac{r}{2}K_X \mod 2$, *where D is a nodal cycle, i.e., D is effective, $(D^2) = -2$ and $|D + K_X| = \emptyset$.*

REMARK 3.2. If $(c_1(L), H') > 0$ for an ample divisor H', then the same claim holds for $r = 0$.

Obviously $(c_1(L)^2) + rs \geq -2$ is necessary for the non-emptyness of the moduli stack. We first assume that $(c_1(L)^2) + rs \geq -1$. In his case, the existence is a consequence of Theorem 1.1. Let (X, H) be an Enriques surface X and an ample divisor H on X. By [3, Prop. 1.4.1], $H^1(X, T_X) \cong \mathbb{C}^{\oplus 10}$ and $H^2(X, T_X) = 0$. We also have $H^2(X, \mathcal{O}_X) = 0$. Hence a polarized deformation of the pair (X, H) is unobstructed. Let $(\mathcal{X}, \mathcal{H}) \to S$ be a general deformation of (X, H) such that a general member is not nodal and $(\mathcal{X}_0, \mathcal{H}_0) = (X, H)$ $(0 \in S)$. Then we have a family of moduli spaces of semi-stable sheaves $f : M_{(\mathcal{X},\mathcal{H})}(v) \to S$.

Under the assumption (i), (ii), (iii) in Theorem 1.1, $M_{(\mathcal{X},\mathcal{H})}(v)_s \neq \emptyset$ for unnodal $\mathcal{X}_s$. Hence f is dominant. By the projectivity of f, $\operatorname{im} f = S$. Hence $M_{(\mathcal{X},\mathcal{H})}(v)_s \neq \emptyset$ for all s.

PROPOSITION 3.3. *Let X be an Enriques surface. Under the conditions* (i), (ii), (iii) *of Theorem* 1.1, *$\mathcal{M}_H(r, L, -\frac{s}{2}) \neq \emptyset$ for a general H.*

If $\gcd(r, c_1(L), a) = 2$, $(c_1(L)^2)+rs = 0$ and $L \not\equiv \frac{r}{2}K_X \mod 2$, then $\mathcal{M}_H(r, L, -\frac{s}{2}) = \emptyset$. Indeed since $M_H(r, L + K_X, -\frac{s}{2}) (\neq \emptyset)$ is an Enriques surface for a general H and the universal family induces a Fourier-Mukai transform, we see that every stable sheaf E with $v(E) = (r, c_1(L), -\frac{s}{2})$ belongs to $M_H(r, L + K_X, -\frac{s}{2})$. Therefore Theorem 3.1 holds if $(c_1(L)^2) + rs \geq -1$.

REMARK 3.4. If r is odd and H is general, then $\operatorname{Ext}^2(E, E) = 0$ for $E \in \mathcal{M}_H(r, c_1, -\frac{s}{2})$. In this case, f is a smooth morphism in a neighborhood of 0.

We treat the remaining case, i.e., $(c_1(L)^2)+rs = -2$. This case is completely studied by Kim in [5] and [8]. For completeness of the proof, we add an outline of the proof in [8]. Let $\pi : \widetilde{X} \to X$ be the universal cover of X. $\widetilde{X}$ is a K3 surface. We need the following elementary fact.

LEMMA 3.5. *For a locally free sheaf F of rank r on $\widetilde{X}$,*

$$\det \pi_*(F) \cong \det(\pi_*(\det F))((r-1)K_X)\,.$$

PROOF. Let H be an ample divisor on X. Since $\pi^*(H)$ is ample, we have an exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-n\pi^*(H))^{\oplus(r-1)} \to F \to I_Z(D) \to 0\,, \tag{3.1}$$

where D is a divisor, Z is a 0-dimensional subscheme of $\widetilde{X}$ and n is sufficiently large. Since $\pi_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_X \oplus \mathcal{O}_X(K_X)$ and $\mathcal{O}_{\widetilde{X}}(D - (r-1)n\pi^*(H)) = \det F$, we get the claim. □

We also need the following result of Kim [8, Thm. 1].

LEMMA 3.6. *Assume that $r \in 2\mathbb{Z}_{>0}$, $a \in \mathbb{Z}$ and $L \in \operatorname{NS}(X)$ satisfy $(c_1(L)^2) - 2ra = -2$. Then $\mathcal{M}_H(r, L, a) \neq \emptyset$ for a general H if and only if $\mathcal{M}_H(2, L - (\frac{r}{2} - 1)K_X, \frac{ra}{2}) \neq \emptyset$.*

PROOF. Since the formulation of the claim is slightly different from [8, Thm. 1], we write the proof. We set $v := (r, c_1(L), a)$. Since $\gcd(r, c_1(L)) = 1$, there is an ample divisor H with $\gcd(r, (c_1(L), H)) = 1$. Indeed we first take a divisor η with $\gcd(r, (c_1(L), \eta)) = 1$. Then we have an ample divisor $H = \eta + r\lambda$, $\lambda \in \operatorname{Amp}(X)$, which satisfies the claim. We may prove the claim for this polarization.

For $E \in \mathcal{M}_H(r, L, a)$, we have $E \cong E(K_X)$. By the proof of [15, Lem. 1.12], there is a simple vector bundle F such that $E = \pi_*(F)$. Since E is rigid, F is also rigid (see the proof of [8, Thm. 1]). By the stability of E, F is stable with respect to $\pi^*(H)$. We have $\pi^*(E) \cong F \oplus \iota^*(F)$. We set $C := \det(F)$. Then $v(F) = (\frac{r}{2}, C, a)$ and $C + \iota^*(C) = \pi^*(L)$. We see that $(C^2) = (C, \iota^*(C)) - 2$ and $(C^2) - ra = -2$. We set $E' := \pi_*(\mathcal{O}_{\widetilde{X}}(C))$. Since $\pi^*(E) \cong \mathcal{O}_{\widetilde{X}}(C) \oplus \mathcal{O}_{\widetilde{X}}(\iota^*(C))$, we see that $v(E') = (2, c_1(L), \frac{(C^2)}{2} + 1) = (2, c_1(L), \frac{ra}{2})$.

By Lemma 3.5,

$$\det E' = (\det E)\left(-\left(\frac{r}{2}-1\right)K_X\right) = \mathcal{O}_X\left(L-\left(\frac{r}{2}-1\right)K_X\right).$$

Obviously E' is semi-stable with respect to H. Since $\gcd(r, (c_1(E'), H)) = 1$, it is μ-stable. Therefore $\mathcal{M}_H(2, L-(\frac{r}{2}-1)K_X, \frac{rs}{2}) \neq \emptyset$.

Conversely for $E' \in \mathcal{M}_H(2, L-(\frac{r}{2}-1)K_X, \frac{ra}{2})$, there is a divisor C with $\pi_*(\mathcal{O}_{\widetilde{X}}(C)) = E'$. Since $\pi^*(E') \cong \mathcal{O}_{\widetilde{X}}(C) \oplus \mathcal{O}_{\widetilde{X}}(\iota^*(C))$, we see that $(C^2) = (C, \iota^*(C)) - 2$ and $(C^2)+2 = ra$. For $u := (\frac{r}{2}, C, a)$, we have $\langle u^2 \rangle = -2$. By $\gcd(r, (c_1(L), H)) = 1$ and $\pi^*(L) = C + \iota^*(C)$, $\gcd(r, (C, \pi^*(H))) = 1$. Let F be a μ-stable locally free sheaf such that $v(F) = u$ with respect to $\pi^*(H)$. Then $E := \pi_*(F)$ is a μ-stable locally free sheaf with $v(\pi_*(F)) = v$. By Lemma 3.5, $\det E = L$. Therefore $\mathcal{M}_H(r, L, a) \neq \emptyset$. □

PROPOSITION 3.7. *Assume that $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}$ and $L \in \mathrm{NS}(X)$ satisfy $r \equiv s \mod 2$ and $(c_1(L)^2) + rs = -2$. If $r = 0$, then we further assume that $(c_1(L), H') > 0$ for an ample divisor H' on X. Then $\mathcal{M}_H(r, L, -\frac{s}{2}) \neq \emptyset$ for a general H if and only if $L = D + 2A + \frac{r}{2}K_X$, where D is a nodal cycle and $A \in \mathrm{NS}(X)$.*

PROOF. If $r > 0$, then the claim is a consequence of Lemma 3.6 and [5, Thm. 3.4]. If $r = 0$, then the claim is a consequence of Remark 2.19 (see also Corollary 4.5). □

4. Appendix. Let X be any Enriques surface and H be an ample divisor on X. For $\omega = tH$, $t > 0$, let $Z_{(0,\omega)} : \mathbf{D}(X) \to \mathbb{C}$ be a stability function defined by

$$Z_{(0,\omega)}(E) := \langle e^{\omega\sqrt{-1}}, v(E)\rangle, \quad E \in \mathbf{D}(E). \tag{4.1}$$

Let $\mathfrak{T}_{(0,\omega)}$ be the full subcategory of $\mathrm{Coh}(X)$ generated by torsion sheaves and torsion free stable sheaves E with $Z_{(0,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Let $\mathfrak{F}_{(0,\omega)}$ be the full subcategory of $\mathrm{Coh}(X)$ generated by torsion free stable sheaves E with $-Z_{(0,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Let $\mathfrak{A}_{(0,\omega)} (\subset \mathbf{D}(X))$ be the category generated by $\mathfrak{T}_{(0,\omega)}$ and $\mathfrak{F}_{(0,\omega)}[1]$. If $(\omega^2) \neq 1$, then $\sigma(0,\omega) := (\mathfrak{A}_{(0,\omega)}, Z_{(0,\omega)})$ is a stability condition. $\mathfrak{A}_{(0,\omega)}$ is constant on $(\omega^2) \neq 1$. We set

DEFINITION 4.1. (1) For $(\omega^2) > 1$, we set $\mathfrak{T}^\mu := \mathfrak{T}_{(0,\omega)}$, $\mathfrak{F}^\mu := \mathfrak{F}_{(0,\omega)}$ and $\mathfrak{A}^\mu := \mathfrak{A}_{(0,\omega)}$.
(2) For $(\omega^2) < 1$, we set $\mathfrak{T} := \mathfrak{T}_{(0,\omega)}$, $\mathfrak{F} := \mathfrak{F}_{(0,\omega)}$ and $\mathfrak{A} := \mathfrak{A}_{(0,\omega)}$.

For $E \in \mathfrak{F}^\mu$, we have an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{4.2}$$

such that

(1) E_1 is generated by $\mathcal{O}_X$ and K_X, and
(2) $E_2 \in \mathfrak{F}^\mu$ satisfies $\mathrm{Hom}(\mathcal{O}_X, E_2) = \mathrm{Hom}(\mathcal{O}_X(K_X), E_2) = 0$, i.e., $E_2 \in \mathfrak{F}$.

Since $H^1(\mathcal{O}_X(K_X)) = H^1(\mathcal{O}_X) = 0$,

$$E_1 \cong \mathcal{O}_X^{\oplus n} \oplus \mathcal{O}_X(K_X)^{\oplus m}.$$

We also have $E_1 = \mathrm{Hom}(\mathcal{O}_X, E) \otimes \mathcal{O}_X \oplus \mathrm{Hom}(\mathcal{O}_X(K_X), E) \otimes \mathcal{O}_X(K_X)$. For $E \in \mathfrak{T}$, the natural homomorphism

$$\phi : E \to \mathrm{Hom}(E, \mathcal{O}_X)^\vee \otimes \mathcal{O}_X \oplus \mathrm{Hom}(E, \mathcal{O}_X(K_X))^\vee \otimes \mathcal{O}_X(K_X)$$

is surjective and $\ker \phi \in \mathfrak{T}^\mu$.

We set

$$\mathcal{E} := \ker(\mathcal{O}_X \boxtimes \mathcal{O}_X \oplus \mathcal{O}_X(K_X)^\vee \boxtimes \mathcal{O}_X(K_X) \to \mathcal{O}_\Delta)\,.$$

As in [11], $\Phi_{X\to X}^{\mathcal{E}^\vee[1]} : \mathbf{D}(X) \to \mathbf{D}(X)$ induces an isomorphism $\mathfrak{A} \to \mathfrak{A}^\mu$ and we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\Phi_{X\to X}^{\mathcal{E}^\vee[1]}} & \mathfrak{A}^\mu \\ {\scriptstyle Z_{(0,\omega)}}\downarrow & & \downarrow{\scriptstyle Z_{(0,\omega')}} \\ \mathbb{C} & \xleftarrow[\times(\omega^2)]{} & \mathbb{C} \end{array} \tag{4.3}$$

where $\omega' = \omega/(\omega^2)$. In particular, we get the following.

PROPOSITION 4.2. *$\Phi_{X\to X}^{\mathcal{E}^\vee[1]}$ induces an isomorphism*

$$\mathcal{M}_{(0,\omega)}\left(r, \eta + \frac{r}{2}K_X, -\frac{s}{2}\right) \cong \mathcal{M}_{(0,\omega')}\left(s, \eta + \frac{s}{2}K_X, -\frac{r}{2}\right). \tag{4.4}$$

Applying Toda's argument to the wall crossing along the line $\omega = tH$, $t > 0$, we get the following result (see also the argument in [11]).

PROPOSITION 4.3 (cf. Toda [16]). (1) *If $(\omega^2) \gg 0$ and $(\eta, \omega) > 0$, then*

$$\mathcal{M}_{(0,\omega)}\left(r, \eta + \frac{r}{2}K_X, -\frac{s}{2}\right) = \mathcal{M}_\omega\left(r, \eta + \frac{r}{2}K_X, -\frac{s}{2}\right).$$

(2) *$e\left(\mathcal{M}_{(0,\omega)}\left(r, \eta + \frac{r}{2}K_X, -\frac{s}{2}\right)\right)$ is independent of a general choice of ω.*

REMARK 4.4. Wall crossing along the line $\omega = tH$ is very similar to the classical wall crossing of Gieseker semi-stability, since $\mathfrak{A}_{(0,\omega)}$ is almost the same.

COROLLARY 4.5.

$$e\left(\mathcal{M}_H\left(r, \eta + \frac{r}{2}K_X, -\frac{s}{2}\right)\right) = e\left(\mathcal{M}_H\left(s, \eta + \frac{s}{2}K_X, -\frac{r}{2}\right)\right)$$

for a general H.

We have another proof of Proposition 3.7.

PROPOSITION 4.6. *Assume that $(\eta^2) + rs = -2$. $\mathcal{M}_H(r, \eta + \frac{r}{2}K_X, -\frac{s}{2}) \neq \emptyset$ for a general H if and only if $\eta \equiv D \mod 2$, where D is a nodal cycle.*

PROOF. By the proof of Theorem 2.6, we have $e(\mathcal{M}_H(r, \eta + \frac{r}{2}K_X, -\frac{s}{2})) = e(\mathcal{M}_H(2, \eta' + K_X, -\frac{s'}{2}))$, where $\eta \equiv \eta' \mod 2$. By [5], $\mathcal{M}_H(2, \eta' + K_X, -\frac{s'}{2}) \neq \emptyset$ if and only if $\eta \equiv D \mod 2$, where D is a nodal cycle. Therefore the claim holds. $\square$

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