Tohoku Math. J.
67 (2015), 585–609

THE SPECIAL McKAY CORRESPONDENCE AND EXCEPTIONAL COLLECTIONS

AKIRA ISHII AND KAZUSHI UEDA

(Received September 13, 2013)

Abstract. We show that the derived category of coherent sheaves on the quotient stack of the affine plane by a finite small subgroup of the general linear group is obtained from the derived category of coherent sheaves on the minimal resolution by adding a semiorthogonal summand with a full exceptional collection. The proof is based on an explicit construction in the abelian case, together with the analysis of the behavior of the derived categories of coherent sheaves under root constructions.

1. Introduction. Let G be a finite small subgroup of $GL_2(\mathbb{C})$ acting on the affine plane $\mathbb{A}^2 = \operatorname{Spec} R$ with the coordinate ring $R = \mathbb{C}[x, y]$. The quotient singularity $X = \mathbb{A}^2/G = \operatorname{Spec} R^G$ has two kinds of natural resolutions: One is the minimal resolution $\tau : Y \to X$, which exists uniquely by the minimal model theory in dimension two. The other is the non-commutative ring $A = \operatorname{End}_{R^G} R$, which is a *non-commutative crepant resolution* in the sense of Van den Bergh [vdB04a, Definition 4.1].

The minimal resolution Y is crepant if and only if G is a subgroup of $SL_2(\mathbb{C})$, whereas the non-commutative resolution A is always crepant. It is known by Auslander (cf. e.g. [IT13, Theorem 4.2]) that the ring A is isomorphic to the crossed-product algebra $R \rtimes G$. It follows that the category of finitely-generated A-modules is equivalent to the category of G-equivariant coherent sheaves on $\mathbb{A}^2$, which in turn is equivalent to the category of coherent sheaves on the quotient stack $[\mathbb{A}^2/G]$;

$$\operatorname{mod} A \cong \operatorname{mod} R \rtimes G \cong \operatorname{coh}[\mathbb{A}^2/G]\,.$$

When G is a subgroup of $SL_2(\mathbb{C})$, Ito and Nakamura [IN99] constructed the commutative crepant resolution Y as the G-Hilbert scheme [Nak01] parametrizing G-invariant subschemes $Z \subset \mathbb{A}^2$ such that $H^0(\mathcal{O}_Z)$ is isomorphic to the regular representation of G as a G-module. This fine moduli interpretation comes with the universal flat family

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & \mathbb{A}^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & X\,, \end{array} \tag{1.1}$$

which allows one to define the integral functor

$$\Phi = q_* \circ p^* : D^b \operatorname{coh} Y \to D^b \operatorname{coh}[\mathbb{A}^2/G] \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 14E16; Secondary 14F05.
Key words and phrases. McKay correspondence, exceptional collections.

realizing the McKay correspondence as an equivalence of derived categories [KV00, BKR01]. This provides an example of a generalization [vdB04a, Conjecture 4.6] of a conjecture of Bondal and Orlov [BO] that any crepant resolutions of X, either commutative or non-commutative, are derived equivalent.

Even if G is not a subgroup of $SL_2(\mathbb{C})$, the Hilbert-Chow morphism τ in the diagram (1.1) is still a resolution of X, which is minimal but not crepant [Ish02]. The integral functor Φ is not an equivalence but a full and faithful embedding, and its essential image is admissible [BO, Definition 2.1] since Φ has both left and right adjoints by Lemma 2.9.

The essential image of Φ and its right orthogonal are described as follows:

PROPOSITION 1.1. *Let G be a finite subgroup of $GL_2(\mathbb{C})$ and Y be the Hilbert scheme of G-orbits in $\mathbb{A}^2$. Then the essential image of Φ is generated by $\{\mathcal{O}_{\mathbb{A}^2} \otimes \rho\}_{\rho:special}$, and its right orthogonal is generated by $\{\mathcal{O}_0 \otimes \rho\}_{\rho:non\text{-}special}$.*

Special representations are introduced by Wunram [Wun88] to extend the McKay correspondence to subgroups of $GL_2(\mathbb{C})$. We recall the basic definitions and properties of special representations in Section 2, where the proof of Proposition 1.1 is also given.

In the case of cyclic groups, we can prove the existence of a full exceptional collection in the semiorthogonal complement of the essential image of Φ:

THEOREM 1.2. *Let G be a finite small cyclic subgroup of $GL_2(\mathbb{C})$ and Y be the Hilbert scheme of G-orbits in $\mathbb{A}^2$. Then there is an exceptional collection $(E_1, \dots, E_n)$ in $D^b \operatorname{coh}[\mathbb{A}^2/G]$ and a semiorthogonal decomposition*

$$D^b \operatorname{coh}[\mathbb{A}^2/G] = \langle E_1, \dots, E_n, \Phi(D^b \operatorname{coh} Y)\rangle ,$$

where n is the number of irreducible non-special representations of G.

Theorem 1.2 is not obvious at all, since

- the set $\{\mathcal{O}_0 \otimes \rho\}_{\rho:\text{non-special}}$ rarely form an exceptional collection (cf. Example 2.11), and
- the category $D^b \operatorname{coh}[\mathbb{A}^2/G]$ does not have an exceptional object at all when G is a subgroup of $SL_2(\mathbb{C})$.

We use the abelian case to obtain a similar result in a general case by using a slightly different functor, while we expect the same result for the functor Φ.

THEOREM 1.3. *Let G be a finite small subgroup of $GL_2(\mathbb{C})$ and $Y \to \mathbb{A}^2/G$ be the minimal resolution of $\mathbb{A}^2/G$. For a suitable fully faithful functor*

$$\Phi' : D^b \operatorname{coh} Y \to D^b \operatorname{coh}[\mathbb{A}^2/G] ,$$

there is an exceptional collection $(E_1, \dots, E_n)$ in $D^b \operatorname{coh}[\mathbb{A}^2/G]$ and a semiorthogonal decomposition

$$D^b \operatorname{coh}[\mathbb{A}^2/G] = \langle E_1, \dots, E_n, \Phi'(D^b \operatorname{coh} Y)\rangle ,$$

where n is the number of irreducible non-special representations of G.

Theorem 1.3 is complementary to the works of Craw [Cra11] and Wemyss [Wem11], which describe $D^b \operatorname{coh} Y$ as the derived category of modules over the path algebra of a quiver with relations called the *special McKay quiver*. One can say that their works give a non-commutative description of the commutative non-crepant resolution, whereas Theorem 1.3 gives the relation between the commutative non-crepant resolution and the non-commutative crepant resolution.

We now give the definition of the functor Φ'. The action of G on $\mathbb{A}^2$ induces

- an action of $G_0 := G \cap SL(2, \mathbb{C})$ on $\mathbb{A}^2$, and
- an action of G/G_0 on G_0-Hilb$(\mathbb{A}^2)$.

The Hilbert-Chow morphism $Y_2 := G/G_0\text{-Hilb}(G_0\text{-Hilb}(\mathbb{A}^2)) \to \mathbb{A}^2/G$ from the iterated Hilbert scheme is a resolution of $\mathbb{A}^2/G$. The resolution $Y_2 \to \mathbb{A}^2/G$ is not necessarily minimal, and factors through the minimal resolution $Y \to \mathbb{A}^2/G$;

$$\begin{tikzcd} Y_2 \arrow[rr, "\varphi"] \arrow[dr] & & Y \arrow[dl] \\ & \mathbb{A}^2/G\,. & \end{tikzcd}$$

By embedding G into $SL_3(\mathbb{C})$ and embedding Y_2 as a divisor in G/G_0-Hilb$(G_0$-Hilb$(\mathbb{A}^3))$, one can deduce from [IINdC13, Theorem 2.7] that Y_2 can be identified with the moduli space $\mathcal{M}_\theta$ of stable G-equivariant sheaves on $\mathbb{A}^2$ for a suitable choice of a stability parameter θ. This gives a fully faithful functor

$$\Phi_2'(-) := \pi_{2*}\left(\pi_1{}^*(-) \otimes \mathcal{E}_\theta\right) : D^b \operatorname{coh} Y_2 \to D^b \operatorname{coh}[\mathbb{A}^2/G]\,,$$

where $\mathcal{E}_\theta$ is the universal family on $\mathcal{M}_\theta \times [\mathbb{A}^2/G]$. The composition

$$\Phi' := \Phi_2' \circ \varphi^* : D^b \operatorname{coh} Y \to D^b \operatorname{coh}[\mathbb{A}^2/G]$$

is a fully faithful functor.

The proof of Theorem 1.3 proceeds as follows:

1. If $G \subset GL_2(\mathbb{C})$ is a cyclic group, then special representations can be computed by continued fraction expansions [Wun87, Wun88], and we can explicitly construct an exceptional collection $E_1, \dots, E_n$ in coh$[\mathbb{A}^2/G]$ as in Theorem 3.1.[1]
2. Let G be a finite small subgroup of $GL_2(\mathbb{C})$ and put $G_0 = G \cap SL_2(\mathbb{C})$. Then G_0 is a normal subgroup of G and $A = G/G_0$ is a cyclic group. The group A acts on $Y_0 = G_0\text{-Hilb}\,\mathbb{A}^2$ and one has an equivalence

$$\Phi_0 : D^b \operatorname{coh}[Y_0/A] \xrightarrow{\sim} D^b \operatorname{coh}[\mathbb{A}^2/G] \tag{1.3}$$

[1] Kawamata pointed out that this step can also be carried out using his arguments [Kaw05, Kaw06], and has subsequently written a paper [Kaw13] which includes it as a special case.

by Theorem 4.1, which is an equivariant version of the McKay correspondence [KV00, BKR01]. Since Y_0 is a resolution of $\mathbb{A}^2/G_0$, a resolution of Y_0/A is a resolution of $\mathbb{A}^2/G$.

3. The stack $[Y_0/A]$ may have non-trivial stabilizer groups along divisors, whereas the *canonical stack* $\mathcal{Y}_1$ associated with the coarse moduli space $Y_1 := Y_0/A$ is a stack which has trivial stabilizer groups except at the singular points. There is a morphism $[Y_0/A] \to \mathcal{Y}_1$ coming from the universal property of the canonical stack, which can be regarded as an iteration of *root constructions* [AGV08, Cad07] along simple normal crossing divisors. The coarse moduli spaces of irreducible divisors with non-trivial stabilizer groups are smooth rational curves, so that one has a full and faithful functor $\Phi_1 : D^b \operatorname{coh} \mathcal{Y}_1 \to D^b \operatorname{coh}[Y_0/A]$ and a semiorthogonal decomposition

$$D^b \operatorname{coh}[Y_0/A] = \langle E_1, \dots, E_{n_1}, \Phi_1(D^b \operatorname{coh} \mathcal{Y}_1)\rangle \tag{1.4}$$

by Proposition 7.2.

4. The coarse moduli space Y_1 of $\mathcal{Y}_1$ has cyclic quotient singularities. By taking the minimal resolution of it, we obtain a resolution Y_2 of $\mathbb{A}^2/G$. This gives a full and faithful functor $\Phi_2 : D^b \operatorname{coh} Y_2 \to D^b \operatorname{coh} \mathcal{Y}_1$ and a semiorthogonal decomposition

$$D^b \operatorname{coh} \mathcal{Y}_1 = \langle E_{n_1+1}, \dots, E_{n_2}, \Phi_2(D^b \operatorname{coh} Y_2)\rangle \tag{1.5}$$

by Proposition 8.1.

5. The minimal resolution Y can be obtained from Y_2 by contracting (-1)-curves. This gives a full and faithful functor $\Phi_3 : D^b \operatorname{coh} Y \to D^b \operatorname{coh} Y_2$ and a semiorthogonal decomposition

$$D^b \operatorname{coh} Y_2 = \langle E_{n_2+1}, \dots, E_n, \Phi_3(D^b \operatorname{coh} Y)\rangle \tag{1.6}$$

by Orlov [Orl92, Theorem 4.3].

By combining the semiorthogonal decompositions from (1.3) to (1.6), one obtains Theorem 1.3. In fact, our proof of Theorem 1.3 readily gives the following global analog:

THEOREM 1.4. *Let $\mathcal{X}$ be the canonical stack associated with a surface X with at worst quotient singularities, and Y be the minimal resolution of X. Then there is a full and faithful functor*

$$\Phi : D^b \operatorname{coh} Y \to D^b \operatorname{coh} \mathcal{X}$$

and a semiorthogonal decomposition

$$D^b \operatorname{coh} \mathcal{X} = \langle E_1, \dots, E_\ell, \Phi(D^b \operatorname{coh} Y)\rangle$$

where $E_1, \dots, E_\ell$ is an exceptional collection.

This gives the relation between the derived categories of the commutative minimal resolution and a non-commutative crepant resolution for any surface X with at worst quotient singularities. As an application of Theorem 1.4, we show the existence of a full exceptional collection on a two-dimensional stack associated with an invertible polynomial in Theorem 10.2.

Root constructions appearing in Step 3 are introduced independently by Cadman [Cad07] and Abramovich, Graber and Vistoli [AGV08], and play important roles in the theory of toric stacks [BCS05, FMN10] and orbifold Gromov-Witten theory [AGV08]. As the analysis of the derived categories of root stacks in Step 3 may also be of independent interest, we state it as theorems here. The first result concerns the root stack of a line bundle:

THEOREM 1.5. *Let $\mathcal{L}$ be a line bundle on a Deligne-Mumford stack $\mathcal{X}$ and $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ be the r-th root stack for a positive integer r. Then the abelian category of coherent sheaves on $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ is the direct sum of r copies of the abelian category of coherent sheaves on $\mathcal{X}$;*

$$\operatorname{coh} \sqrt[r]{\mathcal{L}/\mathcal{X}} \cong (\operatorname{coh} \mathcal{X})^{\oplus r} \ .$$

Note that the decomposition above is not only semiorthogonal but orthogonal, and we do not need to pass to the derived categories. Theorem 1.5 enables us to generalize the results of Borisov and Hua [BH09] to the case when the N-lattice has torsion (cf. the second paragraph in [BH09, Section 2]).

The second result deals with the root stack of a line bunlde with a section:

THEOREM 1.6. *Let $\mathcal{D}$ be a smooth divisor in a smooth Deligne-Mumford stack $\mathcal{X}$ and $\mathcal{Y} = \sqrt[r]{(\mathcal{O}_{\mathcal{X}}(\mathcal{D}), 1)/\mathcal{X}}$ be the r-th root stack of the line bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ with the canonical section $1 \in H^0(\mathcal{O}_{\mathcal{X}}(\mathcal{D}))$. Then there are full and faithful functors*

$$\Phi_{\mathcal{X}} : D^b \operatorname{coh} \mathcal{X} \to D^b \operatorname{coh} \mathcal{Y},$$
$$\Phi_{\mathcal{D}} : D^b \operatorname{coh} \mathcal{D} \to D^b \operatorname{coh} \mathcal{Y}$$

and a semiorthogonal decomposition

$$D^b \operatorname{coh} \mathcal{Y} = \left\langle \Phi_{\mathcal{D}}(D^b \operatorname{coh} \mathcal{D}) \otimes \mathcal{M}^{\otimes(r-1)}, \ldots, \Phi_{\mathcal{D}}(D^b \operatorname{coh} \mathcal{D}) \otimes \mathcal{M}, \Phi_{\mathcal{X}}(D^b \operatorname{coh} \mathcal{X}) \right\rangle,$$

where $\mathcal{M}$ is the universal line bundle on $\mathcal{Y}$.

We assume that all divisors are Cartier throughout this paper. Theorem 1.6 is a root stack analog of [Orl92, Theorem 4.3], where the derived category of the blow-up is described in terms of derived categories of the original variety and the center. This shows that the root construction behaves very much like the ‘blow-up along a divisor’ as far as derived categories of coherent sheaves are concerned.

This paper is organized as follows: We recall the definition of special representations and prove Proposition 1.1 in Section 2. Steps 1 and 2 are carried out in Sections 3 and 4 respectively. Theorem 1.5 is proved in Section 5, and Theorem 1.6 is proved in Section 6. Steps 3 and 4 are carried out in Sections 7 and 8 respectively. Theorems 1.3 and 1.4 are proved in Section 9. As a corollary, we show in Section 10 that the two-dimensional Deligne-Mumford stack associated with an invertible polynomial in four variables has a full exceptional collection.

Acknowledgment. We thank Yujiro Kawamata for the remark on Step 1 above. A. I. is supported by Grant-in-Aid for Scientific Research (No. 18540034). K. U. is supported by Grant-in-Aid for Young Scientists (No. 20740037 and No. 24740043).

2. The special McKay correspondence. In this section, we recall the definition of special representations and prove Proposition 1.1. Let G be a finite small subgroup of $GL_2(\mathbb{C})$ acting on the affine plane $\mathbb{A}^2 = \operatorname{Spec} R$ and $\tau : Y \to X = \operatorname{Spec} R^G$ be the minimal resolution of the quotient singularity. First we recall the relation between full sheaves on Y and reflexive modules on X:

DEFINITION-LEMMA 2.1 (Esnault [Esn85]). *Let $\mathcal{M}$ be a sheaf on Y and $\mathcal{M}^\vee$ be its dual sheaf. Then there exists a reflexive module M on X such that $\mathcal{M} \cong \widetilde{M} := \tau^* M/\text{torsion}$ if and only if the following three conditions are satisfied:*

1. *$\mathcal{M}$ is locally-free.*
2. *$\mathcal{M}$ is generated by global sections.*
3. *$H^1((\mathcal{M})^\vee \otimes \omega_Y) = 0$.*

In this case $\mathcal{M}$ is said to be full.

Note that reflexive modules coincide with Cohen-Macaulay modules since X is a normal surface.

THEOREM 2.2 (Auslander [Aus86]). *The functor $(-)^G$ of taking G-invariant part gives an equivalence from the category of projective $R \rtimes G$-modules to the category of Cohen-Macaulay R^G-modules.*

It follows that indecomposable full sheaves on Y are in one-to-one correspondence with irreducible representations of G.

THEOREM 2.3 (Wunram [Wun88, Main Result]). *Let $E = \bigcup_{i=1}^r E_i$ be the decomposition into irreducible components of the exceptional set E. Then for every curve E_i there exists exactly one indecomposable reflexive module M_i such that the corresponding full sheaf $\widetilde{M}_i = \tau^* M_i/\text{torsion}$ satisfies the conditions $H^1((\widetilde{M})^\vee) = 0$ and*

$$c_1(\widetilde{M}_i) \cdot E_j = \delta_{ij} .$$

A full sheaf is said to be *special* if there is an index $1 \leq i \leq r$ such that $\mathcal{M} = \mathcal{M}_i$ or it is isomorphic to the structure sheaf $\mathcal{O}_Y$. The special full sheaf $\mathcal{O}_Y$ corresponds to the trivial representation and is denoted by $\mathcal{M}_0$.

We will repeatedly use the following lemma:

LEMMA 2.4. *Let $f : S \to T$ be a proper morphism of Noetherian schemes such that T is affine and all fibers of f are at most 1-dimensional. Let further $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on S such that $\mathcal{E}$ is generated by global sections and $H^1(\mathcal{F}) = 0$. Then one has $H^1(\mathcal{E} \otimes \mathcal{F}) = 0$.*

PROOF. One has a surjection $\mathcal{O}_S^{\oplus r} \to \mathcal{E}$ for some $r \in \mathbb{N}$ since $\mathcal{E}$ is generated by global sections. Let $\mathcal{I}$ be the kernel of this surjection, so that one has a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_S^{\oplus r} \to \mathcal{E} \to 0 \tag{2.1}$$

of sheaves. Since f has at most one-dimensional fibers, one has $H^2(\mathcal{F} \otimes \mathcal{I}) = 0$. By tensoring (2.1) with $\mathcal{F}$ and taking the associated long exact sequence, one obtains a surjection $H^1(\mathcal{F})^{\oplus r} \to H^1(\mathcal{E} \otimes \mathcal{F})$. This implies $H^1(\mathcal{E} \otimes \mathcal{F}) = 0$ since $H^1(\mathcal{F}) = 0$. □

We also need the following:

LEMMA 2.5 ([Esn85, Lemma 2.1]). *If $\mathcal{M}$ is a locally-free sheaf on Y such that $H^1(\mathcal{M}^\vee \otimes \omega_Y) = 0$, then $\tau_* \mathcal{M}$ is reflexive.*

Special full sheaves are characterized as follows:

THEOREM 2.6 (Wunram [Wun88, Theorem 1.2]). *An indecomposable full sheaf $\mathcal{M}$ is special if and only if $H^1(\mathcal{M}^\vee) = 0$.*

An irreducible representation ρ of G is said to be special if the corresponding full sheaf $\mathcal{M}_\rho = \tau^* \left((\rho^\vee \otimes R)^G \right)$ /torsion is special.

LEMMA 2.7. *For a special representation ρ, we have an isomorphism*

$$\Phi(\mathcal{M}_\rho^\vee) \cong \mathcal{O}_{\mathbb{A}^2} \otimes \rho$$

in $D^b(\mathrm{coh}[\mathbb{A}^2/G])$.

PROOF. Note that $p_* \mathcal{O}_{\mathcal{Z}} \cong \bigoplus_{\sigma \in \mathrm{Irrep}(G)} \mathcal{M}_\sigma \otimes \sigma$ as shown in [Ish02, Corollary 3.2]. One has

$$\pi_* \Phi(\mathcal{M}_\rho^\vee) = \pi_* q_* p^* (\mathcal{M}_\rho^\vee) \tag{2.2}$$

$$= \tau_* p_* p^* (\mathcal{M}_\rho^\vee) \tag{2.3}$$

$$= \tau_* (p_* \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{M}_\rho^\vee) \tag{2.4}$$

$$= \bigoplus_{\sigma \in \mathrm{Irrep}(G)} \tau_* (\mathcal{M}_\sigma \otimes \mathcal{M}_\rho^\vee) \otimes \sigma \,. \tag{2.5}$$

For any $\sigma \in \mathrm{Irrep}(G)$, one has $H^1(\mathcal{M}_\sigma \otimes \mathcal{M}_\rho^\vee) = 0$ since $\mathcal{M}_\sigma$ is generated by global sections (this follows from the fact that $\mathcal{M}_\sigma$ is a full sheaf), $H^1(\mathcal{M}_\rho^\vee) = 0$ (this follows from the fact that $\mathcal{M}_\rho$ is a special full sheaf), and τ has at most one-dimensional fibers. This implies that $\pi_* \Phi(\mathcal{M}_\rho^\vee)$ is a sheaf, which in turn implies that $\Phi(\mathcal{M}_\rho)$ is a sheaf since π is finite.

Since both $\mathcal{M}_\rho$ and $\mathcal{M}_\sigma$ are full sheaves, the sheaf $\mathcal{M}_\rho$ is generated by global sections and $H^1(\mathcal{M}_\sigma^\vee \otimes \omega_Y) = 0$. By Lemma 2.4, one obtains

$$H^1((\mathcal{M}_\rho^\vee \otimes \mathcal{M}_\sigma)^\vee \otimes \omega_Y) \cong H^1(\mathcal{M}_\rho \otimes (\mathcal{M}_\sigma^\vee \otimes \omega_Y)) = 0\,.$$

By Lemma 2.5, this implies that $\tau_*(\mathcal{M}_\rho^\vee \otimes \mathcal{M}_\sigma)$ is reflexive. Therefore,

$$\pi_*(\Phi(\mathcal{M}_\rho^\vee)) \cong \tau_* \Bigg(\bigoplus_{\sigma \in \mathrm{Irrep}(G)} \mathcal{M}_\rho^\vee \otimes \mathcal{M}_\sigma \otimes \sigma \Bigg)$$

is reflexive as an $\mathcal{O}_X$-module. This implies that $\Phi(\mathcal{M}_\rho^\vee)$ is reflexive, since τ is finite and surjective (cf. e.g. [BD08, Lemma 2.24]). Since the isomorphism class of a reflexive sheaf is determined by the restriction to a complement of any codimension 2 subset, the assertion follows from the isomorphism

$$\Phi(\mathcal{M}_\rho^\vee)|_{\mathbb{A}^2\setminus\{0\}} \cong \pi^*(M_\rho^\vee)|_{\mathbb{A}^2\setminus\{0\}} \cong \pi^*(((\mathcal{O}_{\mathbb{A}^2} \otimes \rho^\vee)^G)^\vee)|_{\mathbb{A}^2\setminus\{0\}} \cong (\mathcal{O}_{\mathbb{A}^2} \otimes \rho)|_{\mathcal{A}^2\setminus\{0\}}$$

of the restrictions to $\mathbb{A}^2 \setminus \{0\}$. Here, the last isomorphism follows from the fact that the action of G on $\mathbb{A}^2 \setminus \{0\}$ is free, so that one has an equivalence

$$D^b \operatorname{coh}(\mathbb{A}^2 \setminus \{0\})/G \cong D^b \operatorname{coh}^G(\mathbb{A}^2 \setminus \{0\})$$

which commutes with taking duals. □

We also need the following lemma:

LEMMA 2.8. *Let $\iota\colon U \hookrightarrow T$ be an open immersion of smooth varieties and $S \subset U$ be a closed subscheme of U whose image $\iota(S)$ is closed in T. Suppose an object $\mathcal{E} \in D^b \operatorname{coh} U$ is supported on S. Then for any object $\mathcal{F} \in D^b \operatorname{coh} T$, one has a functorial isomorphism*

$$\operatorname{Hom}(\iota_*\mathcal{E}, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{E}, \iota^*\mathcal{F})\,. \tag{2.6}$$

PROOF. First note that one has

$$\iota^*\iota_*\mathcal{E} \cong \mathcal{E}$$

since $\iota\colon U \to T$ is an open immersion. This implies that

$$\iota^*((\iota_*\mathcal{E})^\vee) \cong (\iota^*(\iota_*\mathcal{E}))^\vee \cong \mathcal{E}^\vee$$

since pull-back commutes with taking duals. It follows that the adjunction morphism

$$(\iota_*\mathcal{E})^\vee \to \iota_*\iota^*((\iota_*\mathcal{E})^\vee) \cong \iota_*(\mathcal{E}^\vee) \tag{2.7}$$

becomes an isomorphism if it is pulled-back by ι^*;

$$\iota^*((\iota_*\mathcal{E})^\vee) \xrightarrow{\sim} \iota^*(\iota_*(\mathcal{E}^\vee))\,. \tag{2.8}$$

Note that the supports of both $\iota_*(\mathcal{E}^\vee)$ and $(\iota_*\mathcal{E})^\vee$ are contained in $\iota(S)$, since $\operatorname{supp}(\mathcal{E}) \subset S$ and $\iota(S)$ is closed in T. It follows that $(\iota_*\mathcal{E})^\vee \cong \iota_*(\mathcal{E}^\vee)$, and one has

$$\begin{aligned}
\operatorname{Hom}(\iota_*\mathcal{E}, \mathcal{F}) &\cong H^0((\iota_*\mathcal{E})^\vee \otimes \mathcal{F}) \\
&\cong H^0(\iota_*(\mathcal{E}^\vee) \otimes \mathcal{F}) \\
&\cong H^0(\iota_*(\mathcal{E}^\vee \otimes \iota^*\mathcal{F})) \\
&\cong H^0(\mathcal{E}^\vee \otimes \iota^*\mathcal{F}) \\
&\cong \operatorname{Hom}(\mathcal{E}, \iota^*\mathcal{F})\,.
\end{aligned}$$

□

Now we prove the following:

LEMMA 2.9. *Φ has left and right adjoints.*

PROOF. Φ is defined as

$$\Phi(-) = p_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(-))$$

where $p_1 : Y \times \mathbb{A}^2 \to Y$ and $p_2 : Y \times \mathbb{A}^2 \to \mathbb{A}^2$ are the projections. As in [BKR01], the problem is that p_i are not projective morphisms; p_1^* does not have a left adjoint and p_{2*} does not have a right adjoint. Note however that the restrictions of p_1 and p_2 to $\mathcal{Z}$ are projective morphisms. Take a smooth projective variety $\overline{Y}$ which contains Y as an open set. Let $i_1 : Y \times \mathbb{A}^2 \hookrightarrow Y \times \mathbb{P}^2$ and $i_2 : Y \times \mathbb{A}^2 \hookrightarrow \overline{Y} \times \mathbb{A}^2$ be open immersions, and $\overline{p}_1 : Y \times \mathbb{P}^2 \to Y$ and $\overline{p}_2 : \overline{Y} \times \mathbb{A}^2 \to \mathbb{A}^2$ be the projections. Then, by the projectivity of $\mathcal{Z}$ over Y and $\mathbb{A}^2$, $i_{1*}(\mathcal{O}_{\mathcal{Z}})$ and $i_{2*}(\mathcal{O}_{\mathcal{Z}})$ are coherent sheaves. Thus for objects $\alpha \in D^b(\operatorname{coh} Y)$ and $\beta \in D^b(\operatorname{coh}[\mathbb{A}^2/G])$, $i_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\alpha)) \in D^b(\operatorname{coh} \overline{Y} \times [\mathbb{A}^2/G])$ and we can apply the Grothendieck duality for $\overline{p}_2$ to obtain

$$\operatorname{Hom}(\Phi(\alpha), \beta) = \operatorname{Hom}(\overline{p}_{2*} i_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\alpha)), \beta) \cong \operatorname{Hom}(i_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\alpha)), \overline{p}_2^* \beta \otimes \omega_{\overline{Y}}[2]) .$$

Since the support of $\mathcal{O}_{\mathcal{Z}}$ (and hence that of $\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\alpha)$) is closed in $\overline{Y} \times \mathbb{A}^2$, the last group is isomorphic to

$$\operatorname{Hom}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^*(\alpha), p_2^* \beta \otimes \omega_Y[2]) \cong \operatorname{Hom}(\alpha, p_{1*}(p_2^* \beta \otimes \omega_Y \otimes \mathcal{O}_{\mathcal{Z}}^{\vee}[2])^G)$$

by Lemma 2.8. This proves that Φ has a right adjoint. A left adjoint is obtained in a similar way by applying the Grothendieck duality for $\overline{p}_1$. □

Special full sheaves generate the derived category of coherent sheaves on Y:

THEOREM 2.10 (Van den Bergh [VdB04b, Theorem B]). *The direct sum of indecomposable special full sheaves generates $D^b \operatorname{coh} Y$.*

PROOF OF PROPOSITION 1.1. It follows from Lemma 2.7 and Theorem 2.10 that the essential image of Φ is generated by $\{\mathcal{O}_{\mathbb{A}^2} \otimes \rho\}_{\rho:\text{special}}$. Moreover, for special representations ρ and σ, we have

$$\operatorname{Hom}_{D^b \operatorname{coh} Y}(\mathcal{M}_\rho^{\vee}, \mathcal{M}_\sigma^{\vee}) \cong (R \otimes \rho^{\vee} \otimes \sigma)^G \cong \operatorname{Hom}_{D^b \operatorname{coh}[\mathbb{A}^2/G]}(\mathcal{O}_{\mathbb{A}^2} \otimes \rho, \mathcal{O}_{\mathbb{A}^2} \otimes \sigma) ,$$

which implies that Φ is full and faithful.

$\{\mathcal{O}_0 \otimes \rho\}_{\rho:\text{non-special}}$ is right orthogonal to $\{\mathcal{O}_{\mathbb{A}^2} \otimes \rho\}_{\rho:\text{special}}$ since

$$\mathbb{R}\operatorname{Hom}_{[\mathbb{A}^2/G]}(\mathcal{O}_{\mathbb{A}^2} \otimes \rho, \mathcal{O}_0 \otimes \tau) \cong \begin{cases} \mathbb{C} & \rho = \tau , \\ 0 & \text{otherwise} . \end{cases}$$

Conversely, suppose an object α in $D^b \operatorname{coh}[\mathbb{A}^2/G]$ is right orthogonal to $\{\mathcal{O}_{\mathbb{A}^2} \otimes \rho\}_{\rho:\text{special}}$. Then since X is affine, the vanishing of

$$\operatorname{Hom}^i_{D^b \operatorname{coh}[\mathbb{A}^2/G]}(\mathcal{O}_{\mathbb{A}^2}, \alpha) \cong H^i(\mathbb{A}^2, \alpha)^G \cong H^i(X, \pi_* \alpha)^G$$

for any i implies that $(\pi_* \alpha)^G = 0$. Then one can see by using the isomorphism

$$[(\mathbb{A}^2 \setminus 0)/G] \cong X \setminus 0$$

that the restriction of α to $[(\mathbb{A}^2 \setminus 0)/G]$ is zero and hence that α must be supported on the origin of $\mathbb{A}^2$. Moreover, for any special representation ρ, the vanishing of

$$\operatorname{Hom}^i_{D^b \operatorname{coh}[\mathbb{A}^2/G]}(\mathcal{O}_{\mathbb{A}^2} \otimes \rho, \alpha) \cong H^i(\rho^* \otimes \alpha)^G$$

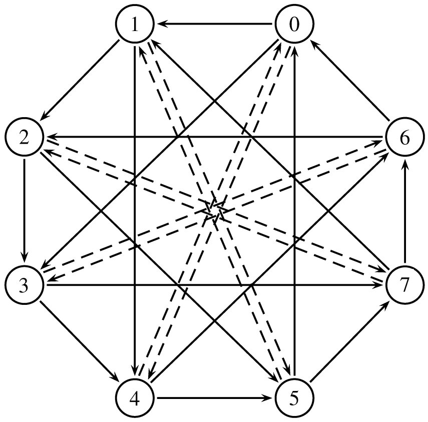

FIGURE 2.1. The McKay quiver.

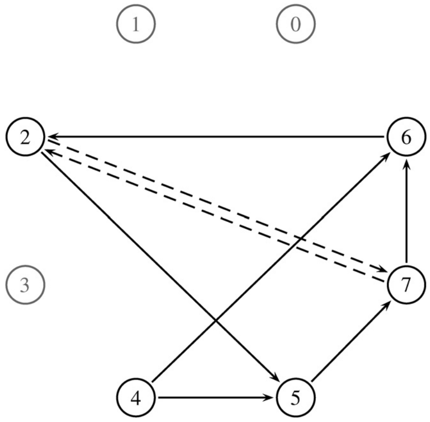

FIGURE 2.2. The non-special quiver.

implies that $\mathcal{O}_0 \otimes \rho$ does not appear in the Jordan-Hölder filtration of (cohomology sheaves of) α. Therefore, the right orthogonal complement of $\{\mathcal{O}_{\mathbb{A}^2} \otimes \rho\}_{\rho:\text{special}}$ is generated by $\{\mathcal{O}_0 \otimes \rho\}_{\rho:\text{non-special}}$. □

Let ρ_{Nat} be the two-dimensional representation of G coming from the inclusion $G \subset GL_2(\mathbb{C})$, and $a_{\mu\nu}$ be the multiplicity appearing in the irreducible decomposition

$$\mu \otimes \rho_{\text{Nat}}^{\vee} = \bigoplus_{\nu \in \text{Irrep}(G)} \nu^{\oplus a_{\mu\nu}}$$

of tensor products of in the representation ring of G. It follows from the projective resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}^2} \otimes \det \rho_{\text{Nat}}^{\vee} \longrightarrow \mathcal{O}_{\mathbb{A}^2} \otimes \rho_{\text{Nat}}^{\vee} \longrightarrow \mathcal{O}_{\mathbb{A}^2} \longrightarrow \mathcal{O}_0 \longrightarrow 0$$

that one has

$$\dim \operatorname{Hom}(\mathcal{O}_0 \otimes \mu, \mathcal{O}_0 \otimes \nu) = \delta_{\mu\nu}\,,$$
$$\dim \operatorname{Ext}^1(\mathcal{O}_0 \otimes \mu, \mathcal{O}_0 \otimes \nu) = a_{\mu\nu}\,,$$

and

$$\dim \operatorname{Ext}^2(\mathcal{O}_0 \otimes \mu, \mathcal{O}_0 \otimes \nu) = \dim \operatorname{Hom}(\mathcal{O}_0 \otimes \nu, \mathcal{O}_0 \otimes \mu \otimes \det \rho_{\text{Nat}}^{\vee}).$$

This is summarized in the *McKay quiver* of G, whose vertices are irreducible representations of G whose solid arrows from μ to ν are basis of $\operatorname{Ext}^1(\mathcal{O}_0 \otimes \mu, \mathcal{O}_0 \otimes \nu)$, and whose dashed arrows are basis of $\operatorname{Ext}^2(\mathcal{O}_0 \otimes \mu, \mathcal{O}_0 \otimes \nu)$.

EXAMPLE 2.11. As an example, consider the case $G = \langle \operatorname{diag}(\zeta, \zeta^3) \rangle$ where $\zeta = \exp(2\pi\sqrt{-1}/8)$. The McKay quiver is shown in Figure 2.1, and its full subquiver consisting of non-special vertices is shown in Figure 2.2. Here, the number i indicates the representation which sends the generator of G to ζ^{-i}. This clearly shows that the set $\{\mathcal{O}_0 \otimes \rho\}_{\rho:\text{non-special}}$ does not form an exceptional collection.

3. The case of cyclic groups. We prove the following in this section:

THEOREM 3.1. *Let A be a finite small abelian subgroup of $GL_2(\mathbb{C})$ and Y be the Hilbert scheme of A-orbits in $\mathbb{A}^2$. Then there is an exceptional collection $(E_1, \dots, E_n)$ in $D^b \operatorname{coh}[\mathbb{A}^2/A]$ and a semiorthogonal decomposition*

$$D^b \operatorname{coh}[\mathbb{A}^2/A] = \langle E_1, \dots, E_n, \Phi(D^b \operatorname{coh} Y) \rangle ,$$

where n is the number of indecomposable non-special representations of G.

To prove Theorem 3.1, we recall Wunram's description of special representations in the case of cyclic groups. For relatively prime integers $0 < q < n$, consider the cyclic small subgroup $G = \langle \frac{1}{n}(1, q) \rangle$ of $GL_2(\mathbb{C})$ generated by

$$\frac{1}{n}(1, q) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix},$$

where ζ is a primitive n-th root of unity. For $a \in \mathbb{Z}/n\mathbb{Z}$, let ρ_a denote the irreducible representation of G that sends the above generator to ζ^{-a}.

Define integers $r, b_1, \dots, b_r$ and $i_0, \dots, i_{r+1}$ as follows: Put $i_0 := n$, $i_1 := q$ and define i_{t+2}, b_{t+1} inductively by

$$i_t = b_{t+1} i_{t+1} - i_{t+2} \quad (0 < i_{t+2} < i_{t+1})$$

until we finally obtain $i_r = 1$ and $i_{r+1} = 0$. This gives a continued fraction expansion

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}$$

and $-b_t$ is the self intersection number of the t-th irreducible exceptional curve C_t in the minimal resolution Y of $\mathbb{A}^2/G$.

Special representations are described as follows:

THEOREM 3.2 (Wunram [Wun87]). *Special representations are $\rho_{i_0} = \rho_{i_{r+1}}, \rho_{i_1}, \dots, \rho_{i_r}$.*

For an integer d with $0 \le d < n$, there is a unique expression

$$d = d_1 i_1 + d_2 i_2 + \dots + d_r i_r \tag{3.1}$$

where $d_i \in \mathbb{Z}_{\ge 0}$ are non-negative integers satisfying

$$0 \le \sum_{t > t_0} d_t i_t < i_{t_0}$$

for any t_0.

LEMMA 3.3 (Wunram [Wun87, Lemma 1]). *A sequence $(d_1, \dots, d_r) \in (\mathbb{Z}_{\ge 0})^r$ is obtained from an integer $d \in [0, n-1]$ as above if and only if the following hold:*

- $0 \le d_t \le b_t - 1$ *for any t.*

- *If $d_s = b_s - 1$ and $d_t = b_t - 1$ for $s < t$, then there is l with $s < l < t$ and $d_l \le b_l - 3$.*

Let $q' \in [0, n-1]$ be the integer with $qq' \equiv 1 \mod n$. Then $\langle \frac{1}{n}(1, q)\rangle$ coincides with $\langle \frac{1}{n}(q', 1)\rangle$ as a subgroup of $GL_2(\mathbb{C})$. Introduce the dual sequence $j_0, \dots, j_{r+1}$ by $j_0 = 0$, $j_1 = 1$ and $j_t = j_{t-1}b_{t-1} - j_{t-2}$ for $t > 1$. Then one has $j_r = q'$ and $j_{r+1} = n$.

LEMMA 3.4 (Wunram [Wun87, Lemma 2]). *Let $d = d_1 i_1 + \cdots + d_r i_r$ be as in* (3.1) *and put $f = d_1 j_1 + \cdots + d_r j_r$. Then one has $0 \le f \le n-1$ and $qf \equiv d \mod n$.*

Let $R = \mathbb{C}[x, y]$ be the coordinate ring of $\mathbb{A}^2$ and put

$$R_k = R/(x, y^k)\,.$$

For an integer $d \in [0, n-1]$ with ρ_d non-special, take t with $i_{t-1} > d > i_t$. Then we define

$$E_d = R_{j_t} \otimes \rho_{d-(j_t-1)q}\,.$$

Note that the socle of E_d is $\mathcal{O}_0 \otimes \rho_d$ and one has the direct sum decomposition $E_d \cong \bigoplus_{0 \le l < j_t} \rho_{d-lq}$ as a representation of G. We show that $\{E_d \mid \rho_d\text{: non-special}\}$ is a desired exceptional collection (with respect to the order of $d \in [1, n-1]$).

We first show the following:

PROPOSITION 3.5. *The following two triangulated subcategories are equal:*

$$\langle \mathcal{O}_0 \otimes \rho \rangle_{\rho\text{:non-special}} = \langle E_d \rangle_{\rho_d\text{:non-special}}\,.$$

We introduce the following order $\preceq$ on $\mathbb{Z}/n\mathbb{Z}$: for $a, b \in \mathbb{Z}/n\mathbb{Z}$, we write $a \preceq b$ if $a' \le b'$ holds for the representatives $a', b' \in \mathbb{Z} \cap [0, n-1]$ of a, b. We also write $x \preceq y$ for $x, y \in \mathbb{Z}$ if the inequality holds for their classes in $\mathbb{Z}/n\mathbb{Z}$.

LEMMA 3.6. *If $0 < l < j_t$, then one has $i_{t-1} \preceq lq$.*

PROOF. We can write $l = d_1 j_1 + \cdots + d_{t-1} j_{t-1}$ as in (3.1) by using $\{j_t\}$ instead of $\{i_t\}$, where $(d_1, \dots, d_{t-1}, 0, \dots, 0)$ satisfies the condition in Lemma 3.3. Then we have $lq \equiv d_1 i_1 + \cdots + d_{t-1} i_{t-1} \mod n$. Since $(d_1, \dots, d_{t-1}, 0, \dots, 0)$ satisfies the condition in Lemma 3.3 and is non-zero, $d_1 i_1 + \cdots + d_{t-1} i_{t-1}$ is an integer in $[i_{t-1}, n-1]$. This implies the desired inequality. □

Note that the following hold by the definition of $\preceq$.

LEMMA 3.7. *If $b \ne 0$, $a + b \preceq a$ implies $a + b \preceq b$.*

COROLLARY 3.8. *If $i_{t-1} > d > i_t$, then we have $d \prec d - lq$ for $0 < l < j_t$.*

PROOF. Since $i_{t-1} \preceq lq$ by Lemma 3.6, we apply Lemma 3.7 for $a = lq$ and $b = d - lq$ to obtain $d \preceq d - lq$. The equality does not hold since $(n, q) = 1$. □

LEMMA 3.9. *If $i_{t-1} > d > i_t$, then ρ_{d-lq} is non-special for $0 \le l < j_t$.*

PROOF. Write $d = d_t i_t + d_{t+1} i_{t+1} + \cdots + d_r i_r$ as in (3.1) and put $f = d_t j_t + d_{t+1} j_{t+1} + \cdots + d_r j_r$. Then since ρ_d is non-special, we have $f \ge 2 j_t$.

Assume that ρ_{d-lq} is special. Then $d - lq \equiv i_s$ for some s and the above corollary implies $s < t$. Moreover, $d \equiv i_s + lq$ yields $f \equiv j_s + l$. On the other hand, since j_s and l are smaller than j_t, we see $j_s + l < 2j_t$. This contradicts $n > f \geq 2j_t$. □

PROOF OF PROPOSITION 3.5. Lemma 3.9 implies that E_d belongs to $\langle \mathcal{O}_0 \otimes \rho \rangle_{\rho:\text{non-special}}$. Moreover, note that the socle of E_d is $\mathcal{O}_0 \otimes \rho_d$. Then, for non-special ρ_f, it follows from Corollary 3.8 and the reverse induction on f with respect to $\preceq$ that $\mathcal{O}_0 \otimes \rho_f$ belongs to $\langle E_d \rangle_{\rho_d:\text{non-special}}$. □

PROPOSITION 3.10. *$\{E_d\}_{\rho_d:\text{non-special}}$ forms an exceptional collection.*

PROOF. Take $E_d, E_{d'}$ with $d' \leq d$ and suppose $i_{t-1} > d > i_t$ and $i_{t'-1} > d' > i_{t'}$. To compute $\mathrm{Ext}^i(E_d, E_{d'})$, consider the following projective resolution of E_d:

$$0 \to R \otimes \rho_{1+d+q} \xrightarrow{\begin{pmatrix} y^{j_t} \\ -x \end{pmatrix}} R \otimes \rho_{1+d+q-j_tq} \oplus R \otimes \rho_{d+q} \xrightarrow{\begin{pmatrix} x & y^{j_t} \end{pmatrix}} R \otimes \rho_{d+q-j_tq} \to E_d \to 0\,.$$

Then $\mathbb{R}\,\mathrm{Hom}_R(E_d, E_{d'})$ splits into the direct sum of

$$R_{j_{t'}} \otimes \rho_{d'-d+(j_t-j_{t'})q} \xrightarrow{\alpha} R_{j_{t'}} \otimes \rho_{d'-d-j_{t'}q}$$

and

$$R_{j_{t'}} \otimes \rho_{d'-d-1+(j_t-t_{t'})q} \xrightarrow{\beta} R_{j_{t'}} \otimes \rho_{d'-d-1-j_{t'}q}$$

where α and β are the multiplications by y^{j_t}. The degrees of terms of these complexes are determined so that $\mathrm{Hom}(E_d, E_{d'}) = (\ker\alpha)^G$, $\mathrm{Ext}^1(E_d, E_{d'}) = (\operatorname{coker}\alpha)^G \oplus (\ker\beta)^G$ and $\mathrm{Ext}^2(E_d, E_{d'}) = (\operatorname{coker}\beta)^G$.

As a representation of G, $\ker\alpha$ is the direct sum of $\rho_{d'-d+lq}$ for $0 \leq l < j_t$. Assume that $\rho_{d'-d+lq}$ is trivial, i.e., $d - d' \equiv lq$. If $l \neq 0$, then Lemma 3.6 implies $i_{t-1} \preceq lq$, which contradicts $0 \leq d' \leq d < i_{t-1}$ and $d - d' \equiv lq$. Therefore, we obtain $l = 0$ and $d = d'$. Thus $(\ker\alpha)^G = 0$ if $d \neq d'$ and it is one-dimensional if $d = d'$. $\operatorname{coker}\alpha$ is the direct sum of $\rho_{d'-d-(j_{t'}-l)q}$ for $0 \leq l < j_t$. Assume $\rho_{d'-d-(j_{t'}-l)q}$ is trivial. Then we see $d - d' + i_{t'} \equiv lq$, which again contradicts Lemma 3.6. Hence we obtain $(\operatorname{coker}\alpha)^G = 0$. In a similar way, we can show $(\ker\beta)^G = (\operatorname{coker}\beta)^G = 0$ and we are done. □

Since $\langle \mathcal{O}_0 \otimes \rho \rangle_{\rho:\text{non-special}}$ is the right orthogonal complement of the essential image of Φ, Propositions 3.5 and 3.10 imply Theorem 3.1.

4. Equivariant McKay correspondence. Let G be a finite subgroup of $GL_2(\mathbb{C})$ and put $G_0 := G \cap SL(2, \mathbb{C})$. Then G_0 is a normal subgroup of G and $A := G/G_0$ is a cyclic group, since $\det\colon GL_2(\mathbb{C}) \to \mathbb{C}^\times$ identifies it with a subgroup of $\mathbb{C}^\times$. There is a natural G-action on $Y_0 := G_0\text{-Hilb}\,\mathbb{A}^2$ such that an element $g \in G$ sends a subscheme $Z \in G_0\text{-Hilb}\,\mathbb{A}^2$ to its image $g \cdot Z$ by the action $g : \mathbb{A}^2 \to \mathbb{A}^2$. Since Z is G_0-invariant by the definition of $G_0\text{-Hilb}\,\mathbb{A}^2$, this G-action on Y_0 descends to an $A = G/G_0$-action on the scheme Y_0.

THEOREM 4.1. *There is a derived equivalence*

$$\Phi_0 : D^b \operatorname{coh}[Y_0/A] \xrightarrow{\sim} D^b \operatorname{coh}[\mathbb{A}^2/G] .$$

PROOF. The groups G and A acts naturally on Y_0, and there is a natural morphism

$$\varphi : [Y_0/G] \to [Y_0/A]$$

coming from the surjection $G \twoheadrightarrow A$ The push-forward functor

$$\varphi_* : D^b \operatorname{coh}[Y_0/G] \to D^b \operatorname{coh}[Y_0/A]$$

sends a G-equivariant cohenrent sheaf $\mathcal{E}$ on Y_0 to the G_0-invariant subsheaf $\mathcal{E}^{G_0}$ equipped with the natural A-equivariant structure. The pull-back functor

$$\varphi^* : D^b \operatorname{coh}[Y_0/A] \to D^b \operatorname{coh}[Y_0/G]$$

sends an A-equivariant coherent sheaf on Y_0 to the same sheaf considered as a G-equivariant coherent sheaf through the surjective homomorphism $G \twoheadrightarrow A$.

Consider the diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[d, phantom, "\cap"] & \\ & Y_0 \times \mathbb{A}^2 \arrow[dl, "\pi_{Y_0}"'] \arrow[dr, "\pi_{\mathbb{A}^2}"] & \\ Y_0 & & \mathbb{A}^2 \end{tikzcd}$$

where $\mathcal{Z} \subset Y_0 \times \mathbb{A}^2$ is the universal subscheme and $\pi_{\mathbb{A}^2}$ and π_{Y_0} are the natural projections. By taking the quotient of the whole diagram with respect to the action of G, one obtains another diagram

$$\begin{tikzcd} & {[\mathcal{Z}/G]} \arrow[d, phantom, "\cap"] & \\ & {[Y_0 \times \mathbb{A}^2/G]} \arrow[dl, "\pi_{[Y_0/G]}"'] \arrow[dr, "\pi_{[\mathbb{A}^2/G]}"] & \\ {[Y_0/G]} & & {[\mathbb{A}^2/G]}. \end{tikzcd}$$

Then we can define an integral functor

$$\Phi_0 : D^b \operatorname{coh}[Y_0/A] \to D^b \operatorname{coh}[\mathbb{A}^2/G]$$

by

$$\Phi_0(-) = \pi_{[\mathbb{A}^2/G]*}(\mathcal{O}_{[\mathcal{Z}/G]} \otimes \pi^*_{[Y_0/G]}(\varphi^*(-))) ,$$

and another functor

$$\Psi_0 : D^b \operatorname{coh}[\mathbb{A}^2/G] \to D^b \operatorname{coh}[Y_0/A]$$

by

$$\Psi_0(-) = \varphi_*(\pi_{[Y_0/G]*}(\mathcal{O}^{\vee}_{[\mathcal{Z}/G]}[2] \otimes \det \rho_{\mathrm{Nat}} \otimes \pi^*_{[\mathbb{A}^2/G]}(-))),$$

where
$$\mathcal{O}^{\vee}_{[\mathcal{Z}/G]} = \mathbb{R}\mathcal{H}om_{\mathcal{O}_{[Y_0\times\mathbb{A}^2/G]}}(\mathcal{O}_{[\mathcal{Z}/G]}, \mathcal{O}_{[Y_0\times\mathbb{A}^2/G]}).$$
The functor Ψ_0 is both left and right adjoint to Φ_0 since

- the functor $\pi_{[\mathbb{A}^2/G]*}$ is right adjoint to $\pi^*_{[\mathbb{A}^2/G]}$ and left adjoint to
$$\pi^!_{[\mathbb{A}^2/G]}(-) = \pi^*_{[\mathbb{A}^2/G]}(-) \otimes \pi^*_{[Y/G]}(\omega_{[Y_G]})[2] = \pi^*_{[\mathbb{A}^2/G]}(-) \otimes \det \rho_{\mathrm{Nat}}[2]\,,$$
- the functor $\pi_{[Y_0/G]*}$ is right adjoint to $\pi^*_{[Y_0/G]}$ and left adjoint to
$$\pi^!_{[Y_0/G]}(-) = \pi^*_{[Y_0/G]}(-) \otimes \pi^*_{[\mathbb{A}^2/G]}(\omega_{[\mathbb{A}^2/G]})[2] = \pi^*_{[Y_0/G]}(-) \otimes \det \rho_{\mathrm{Nat}}[2]\,,$$
- the functor $-\otimes \mathcal{O}_{[\mathcal{Z}/G]}$ is both left and right adjoint to $-\otimes \mathcal{O}^{\vee}_{[\mathcal{Z}/G]}$, and
- the functor φ_* is both left and right adjoint to φ^*.

By restricting G-actions to G_0-actions and forgetting A-actions, we can also define the functor $\Phi'_0 : D^b \operatorname{coh} Y_0 \to D^b \operatorname{coh}[\mathbb{A}^2/G_0]$ and its adjoint Ψ'_0 in the same way as above, which are equivalences by [KV00, BKR01].

Let α be any object of $D^b \operatorname{coh}[Y_0/A]$ and consider the adjunction morphism $\nu : \alpha \to \Psi_0\Phi_0(\alpha)$. If we send the morphism ν by the pull-back functor
$$\varphi_A^* : D^b \operatorname{coh}[Y_0/A] \to D^b \operatorname{coh} Y_0$$
along the morphism $\varphi_A : Y_0 \to [Y_0/A]$, then the resulting morphism $\varphi_A^*(\nu)$ is an isomorphism in $D^b \operatorname{coh} Y_0$ since Φ'_0 and Ψ'_0 are equivalences. Although the functor φ_A^* is not full, it is faithful and this shows that the morphisms ν is an isomorphism. We can also show that the adjunction morphism $\Phi_0\Psi_0(\beta) \to \beta$ is an isomorphism for any object β of $D^b \operatorname{coh}[\mathbb{A}^2/G]$ in the same way, so that Φ_0 and Ψ_0 are equivalences. □

5. The root stack of a line bundle. For a line bundle $\mathcal{L}$ on a Deligne-Mumford stack $\mathcal{X}$ and a positive integer r, the r-th root of $\mathcal{L}$ is the stack $\pi : \sqrt[r]{\mathcal{L}/\mathcal{X}} \to \mathcal{X}$ over $\mathcal{X}$ such that

- an object over a scheme T is a triple $(\varphi, \mathcal{M}, \phi)$ consisting of a morphism $\varphi : T \to \mathcal{X}$ of stacks, a line bundle $\mathcal{M}$ on T, and an isomorphism $\phi : \mathcal{M}^{\otimes r} \xrightarrow{\sim} \varphi^*\mathcal{L}$ of line bundles on T, and
- a morphism is a commutative diagram
$$\begin{tikzcd} T \arrow[rr, "\varphi''"] \arrow[dr, "\varphi"'] & & T' \arrow[dl, "\varphi'"] \\ & \mathcal{X} & \end{tikzcd}$$
and an isomorphism $\phi'' : \mathcal{M}^{\otimes r} \xrightarrow{\sim} \varphi''^*\mathcal{M}'^{\otimes r}$ making the diagram
$$\begin{tikzcd} \mathcal{M}^{\otimes r} \arrow[rr, "\phi''"] \arrow[dr, "\phi"'] & & \varphi''^*\mathcal{M}'^{\otimes r} \arrow[dl, "\varphi''^*(\phi')"] \\ & \varphi^*\mathcal{L} \cong (\varphi' \circ \varphi'')^*\mathcal{L} & \end{tikzcd}$$

commute.

Let $(\mathcal{M}, \Phi)$ be the universal object on $\sqrt[r]{\mathcal{L}/\mathcal{X}}$, so that $\mathcal{M}$ is a line bundle on $\sqrt[r]{\mathcal{L}/\mathcal{D}}$ and $\Phi : \mathcal{M}^{\otimes r} \to \pi^*\mathcal{L}$ is an isomorphism of line bundles.

The structure morphism $\pi : \sqrt[r]{\mathcal{L}/\mathcal{X}} \to \mathcal{X}$ makes the root stack $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ into an essentially trivial gerb over $\mathcal{X}$ banded by μ_r, where μ_r is the kernel of the r-th power map $\mathbb{G}_m \to \mathbb{G}_m$ between the multiplicative groups. This means that $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ is the $[pt/\mu_r]$-bundle associated with the principal $\mathbb{G}_m$-bundle $L := \mathcal{L} \setminus$ (the zero section).

Now we prove Theorem 1.5:

PROOF OF THEOREM 1.5. For any coherent sheaf $\mathcal{F}$ on $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ and any integer i, one has the adjunction morphism $\pi^*\pi_*(\mathcal{F} \otimes \mathcal{M}^{\otimes i}) \to \mathcal{F} \otimes \mathcal{M}^{\otimes i}$, whose direct sum gives the morphism

$$\bigoplus_{i=0}^{r-1} \pi^*(\pi_*(\mathcal{F} \otimes \mathcal{M}^{\otimes(-i)})) \otimes \mathcal{M}^{\otimes i} \to \mathcal{F}. \tag{5.1}$$

Since this is a morphism of sheaves, one can work locally to show that it is an isomorphism. Take an open set $\mathcal{U} \subset \mathcal{X}$ where the line bundle $\mathcal{L}$ is trivial, so that the root stack is the trivial gerb given by the direct product $\mathcal{U} \times [pt/\mu_r]$ with the classifying stack. Then the sheaf $\mathcal{M}^{\otimes i}|_{\mathcal{U}}$ corresponds to $\mathcal{O}_{\mathcal{U}} \otimes \rho_i$ under the equivalence $\operatorname{coh}(\mathcal{U} \times [pt/\mu_r]) \cong (\operatorname{coh} \mathcal{U}) \otimes (\operatorname{rep} \mu_r)$, where $\operatorname{rep} \mu_r$ is the category of finite-dimensional representations of μ_r and ρ_i is the representation sending $\alpha \in \mu_r$ to $\alpha^i \in \mathbb{G}_m$. This immediately shows that (5.1) is an isomorphism. The same local consideration also shows that $(\pi^* \operatorname{coh} \mathcal{X}) \otimes \mathcal{M}^{\otimes i}$ for $i = 0, \dots, r-1$ are mutually orthogonal, and Theorem 1.5 is proved. □

6. The root stack of a line bundle with a section. Let $(\mathcal{L}, \sigma)$ be a pair of a line bundle $\mathcal{L} \to \mathcal{X}$ and a section $\sigma : \mathcal{X} \to \mathcal{L}$. The stack $\sqrt[r]{(\mathcal{L}, \sigma)/\mathcal{X}}$ of the r-th roots of $(\mathcal{L}, \sigma)$ is the stack such that

- an object over T is a quadruple $(\varphi, \mathcal{M}, \phi, \tau)$ consisting of an object $(\varphi, \mathcal{M}, \phi)$ of $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ over T and a section τ of $\mathcal{M}$ such that $\phi(\tau^{\otimes r}) = \varphi^*\sigma$, and
- a morphism is a morphism (φ'', ϕ'') of $\sqrt[r]{\mathcal{L}/\mathcal{X}}$ such that $\phi''(\tau) = \tau'$.

Assume that $\mathcal{X}$ is a smooth Deligne-Mumford stack and $\overline{\jmath} : \mathcal{D} \to \mathcal{X}$ is a closed embedding of a smooth divisor. The canonical section of the line bundle $\mathcal{O}(\mathcal{D})$ associated with the divisor $\mathcal{D}$ will be denoted by $1 \in \Gamma(\mathcal{O}(\mathcal{D}))$. Let

$$\sqrt[r]{(\mathcal{O}(\mathcal{D}), 1)/\mathcal{X}}\Big|_{\mathcal{D}} \subset \sqrt[r]{(\mathcal{O}(\mathcal{D}), 1)/\mathcal{X}} \tag{6.1}$$

be the substack consisting of objects $(\varphi, \mathcal{M}, \phi)$ such that the morphism $\varphi : T \to \mathcal{X}$ factors through $\overline{\jmath}$. There is a closed embedding

$$\sqrt[r]{\mathcal{O}_{\mathcal{D}}(\mathcal{D})/\mathcal{D}} \hookrightarrow \sqrt[r]{(\mathcal{O}(\mathcal{D}), 1)/\mathcal{X}}\Big|_{\mathcal{D}}$$

sending an r-th root $\mathcal{M}$ of $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ to the same $\mathcal{M}$ together with the zero section. The composition of this morphism with the embedding (6.1) will be denoted by j, which fits into the commutative diagram

$$\begin{array}{ccc} \sqrt[r]{\mathcal{O}_{\mathcal{D}}(\mathcal{D})/\mathcal{D}} & \xrightarrow{j} & \sqrt[r]{(\mathcal{O}(\mathcal{D}),1)/\mathcal{X}} \\ {\scriptstyle \pi_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{X}}} \\ \mathcal{D} & \xrightarrow{\overline{\jmath}} & \mathcal{X}. \end{array}$$

The universal line bundle on $\sqrt[r]{(\mathcal{O}(\mathcal{D}),1)/\mathcal{X}}$ will be denoted by $\mathcal{M}$.

The following proposition gives Theorem 1.6:

PROPOSITION 6.1.

(i) *The functor* $j_*\pi_{\mathcal{D}}^* : D^b(\operatorname{coh}\mathcal{D}) \to D^b(\operatorname{coh}\sqrt[r]{(\mathcal{O}(\mathcal{D}),1)/\mathcal{X}})$ *is fully faithful if* $r > 1$.

(ii) *One has a semiorthogonal decomposition*

$$D^b \operatorname{coh}\sqrt[r]{(\mathcal{O}(\mathcal{D}),1)/\mathcal{X}} = \langle j_*\pi_{\mathcal{D}}^* D^b(\operatorname{coh}\mathcal{D}) \otimes \mathcal{M}^{\otimes r-1}, \dots, \\ j_*\pi_{\mathcal{D}}^* D^b(\operatorname{coh}\mathcal{D}) \otimes \mathcal{M}, \pi_{\mathcal{X}}^* D^b \operatorname{coh}\mathcal{X}\rangle .$$

PROOF. (i) For any objects α and β of $D^b(\operatorname{coh}\mathcal{D})$ and any $q \in \mathbb{Z}$, we show that the natural morphism

$$\operatorname{Hom}^q(\alpha,\beta) \to \operatorname{Hom}^q(j_*\pi_{\mathcal{D}}^*\alpha, j_*\pi_{\mathcal{D}}^*\beta) \cong \operatorname{Hom}^q(j^*j_*\pi_{\mathcal{D}}^*\alpha, \pi_{\mathcal{D}}^*\beta) \tag{6.2}$$

is an isomorphism. We may assume that α and β are sheaves. Then we have

$$H^i(j^*j_*\pi_{\mathcal{D}}^*\alpha) \cong \begin{cases} \pi_{\mathcal{D}}^*\alpha & i=0\,, \\ \pi_{\mathcal{D}}^*\alpha\otimes\mathcal{M}^{-1} & i=-1\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{6.3}$$

If $r > 1$, Theorem 1.5 shows

$$\operatorname{Hom}^q(\pi_{\mathcal{D}}^*\alpha\otimes\mathcal{M}^{-1}, \pi_{\mathcal{D}}^*\beta) \cong 0$$

and

$$\operatorname{Hom}^q(\pi_{\mathcal{D}}^*\alpha, \pi_{\mathcal{D}}^*\beta) \cong \operatorname{Hom}^q(\alpha,\beta)$$

for any q. It follows that the spectral sequence

$$\operatorname{Hom}^p(H^{-q}(j^*j_*\pi_{\mathcal{D}}^*\alpha), \pi_{\mathcal{D}}^*\beta) \Rightarrow \operatorname{Hom}^{p+q}(j^*j_*\pi_{\mathcal{D}}^*\alpha, \pi_{\mathcal{D}}^*\beta)$$

degenerates and (6.2) is an isomorphism.

(ii) The subcategory $\pi_{\mathcal{X}}^*(D^b(\operatorname{coh}\mathcal{X}))$ is admissible since the functor $\pi_{\mathcal{X}}^*$ has both right and left adjoints. The subcategories $j_*\pi_{\mathcal{D}}^*D^b(\operatorname{coh}\mathcal{D})\otimes\mathcal{M}^{\otimes i}$ are also admissible since the functor $j_*\pi_{\mathcal{D}}^*$ has both left and right adjoints and the functor $(-)\otimes\mathcal{M}^{\otimes i}$ is an equivalence.

We can deduce that $j_*\pi_{\mathcal{D}}^*D^b(\operatorname{coh}\mathcal{D})\otimes\mathcal{M}^{\otimes i}$ are right orthogonal to $\pi_{\mathcal{X}}^*(D^b(\operatorname{coh}X))$ for $1 \le i \le r-1$ from

$$\begin{aligned} \operatorname{Hom}(\pi_{\mathcal{X}}^*\alpha, j_*(\pi_{\mathcal{D}}^*\beta\otimes\mathcal{M}^{\otimes i})) &\cong \operatorname{Hom}(j^*\pi_{\mathcal{X}}^*\alpha, \pi_{\mathcal{D}}^*\beta\otimes\mathcal{M}^{\otimes i}) \\ &\cong \operatorname{Hom}(\pi_{\mathcal{D}}^*\overline{\jmath}^*\alpha, \pi_{\mathcal{D}}^*\beta\otimes\mathcal{M}^{\otimes i}) \\ &= 0\,, \end{aligned}$$

where $\overline{\jmath} : \mathcal{D} \to \mathcal{X}$ is the closed immersion. Similarly, (6.3) implies

$$\operatorname{Hom}(j_* \pi_{\mathcal{D}}^* \alpha \otimes \mathcal{M}^{\otimes k}, j_* \pi_{\mathcal{D}}^* \beta \otimes \mathcal{M}^{\otimes l}) = 0$$

for $1 \le k < l \le r - 1$.

It remains to show that any object $\mathcal{E}$ of $D^b \operatorname{coh} \sqrt[r]{(\mathcal{O}(D), 1)/\mathcal{X}}$ is obtained from objects of $j_* \pi_{\mathcal{D}}^* (D^b \operatorname{coh} \mathcal{D}) \otimes \mathcal{M}^{\otimes i}$ for $1 \le i \le r - 1$ and $\pi_{\mathcal{X}}^* D^b \operatorname{coh} \mathcal{X}$ by taking shifts and cones. Since $\pi_{\mathcal{X}}$ is an isomorphism outside $\mathcal{D}$, the mapping cone $\operatorname{Cone}(\pi_{\mathcal{X}}^* \pi_{\mathcal{X}*} \mathcal{E} \to \mathcal{E})$ of the adjunction morphism is supported on $\sqrt[r]{\mathcal{O}_{\mathcal{D}}(\mathcal{D})/\mathcal{D}}$. It follows that $\mathcal{E}$ can be obtained from $\pi_{\mathcal{X}}^* \pi_{\mathcal{X}*} \mathcal{E}$ and an object supported on $\sqrt[r]{\mathcal{O}_{\mathcal{D}}(\mathcal{D})/\mathcal{D}}$ by taking cones.

An object supported on $\sqrt[r]{\mathcal{O}_{\mathcal{D}}(\mathcal{D})/\mathcal{D}}$ is obtained from objects of $j_* D^b \operatorname{coh} \sqrt[r]{\mathcal{O}_{\mathcal{D}}(\mathcal{D})/\mathcal{D}}$ by taking cones, which in turn can be obtained from objects of $j_* \pi_{\mathcal{D}}^* D^b (\operatorname{coh} \mathcal{D}) \otimes \mathcal{M}^{\otimes i}$ for $0 \le i \le r - 1$ by Theorem 1.5. Finally, we have to show that an object of $j_* \pi_{\mathcal{D}}^* D^b (\operatorname{coh} \mathcal{D})$ is obtained from objects of $\pi_{\mathcal{X}}^* D^b \operatorname{coh} \mathcal{X}$ and $j_* \pi_{\mathcal{D}}^* D^b (\operatorname{coh} \mathcal{D}) \otimes \mathcal{M}^{\otimes i}$ for $1 \le i \le r - 1$. If α is a sheaf in $D^b(\operatorname{coh} \mathcal{D})$, then $\pi_{\mathcal{X}}^* \overline{\jmath}_* \alpha$ has a filtration whose factors are $j_* \pi_{\mathcal{D}}^* \alpha \otimes \mathcal{M}^{\otimes i}$ for $0 \le i \le r - 1$. Thus $j_* \pi_{\mathcal{D}}^* \alpha$ is obtained from $\pi_{\mathcal{X}}^* \overline{\jmath}_* \alpha$ and $j_* \pi_{\mathcal{D}}^* \alpha \otimes \mathcal{M}^{\otimes i}$ for $1 \le i \le r - 1$ by taking shifts and cones. This concludes the proof of Proposition 6.1. □

COROLLARY 6.2. *If both $\mathcal{X}$ and $\mathcal{D}$ have full exceptional collections, then so does the root stack* $\sqrt[r]{(\mathcal{O}(\mathcal{D}), 1)/\mathcal{X}}$.

7. Iterations of root constructions. A smooth Deligne-Mumford stack $\mathcal{Y}$ is said to be *canonical* if the locus where the structure morphism $\mathcal{Y} \to Y$ to the coarse moduli space Y is not an isomorphism has codimension greater than one [FMN10, Definition 4.4]. The canonical stack has the universal property [FMN10, Theorem 4.6] that any dominant codimension-preserving morphism $f : \mathcal{X} \to Y$ from a smooth stack $\mathcal{X}$ without generic stabilizers factors through $\mathcal{Y} \to Y$ uniquely up to unique 2-arrow;

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed, "\exists ! g"] \arrow[dr, "f"'] & \mathcal{Y} \arrow[d, "\varepsilon"] \\ & Y . \end{tikzcd}$$

For a variety X with at worst quotient singularities, there is a canonical stack $\mathcal{X}^{\mathrm{can}}$ whose coarse moduli space is isomorphic to X, which is determined uniquely up to isomorphism [FMN10, Remark 4.9]. For effective divisors $\mathcal{D}_1, \dots, \mathcal{D}_s$ on $\mathcal{X}^{\mathrm{can}}$ and positive integers $r_1, \dots, r_s$, the fiber product

$$(7.1) \quad \mathcal{X} = \sqrt[r_1]{(\mathcal{O}(\mathcal{D}_1), 1)/\mathcal{X}^{\mathrm{can}}} \times_{\mathcal{X}^{\mathrm{can}}} \cdots \times_{\mathcal{X}^{\mathrm{can}}} \sqrt[r_s]{(\mathcal{O}(\mathcal{D}_s), 1)/\mathcal{X}^{\mathrm{can}}} \xrightarrow{\varphi} \mathcal{X}^{\mathrm{can}}$$

is obtained by iterating root constructions [Cad07, Remark 2.2.5]. If we write the reduced closed substack $(\varphi^{-1}(\mathcal{D}_i))_{\mathrm{red}}$ as $\widetilde{\mathcal{D}}_i$, then one has $\varphi^* \mathcal{D}_i = r_i \widetilde{\mathcal{D}}_i$. The numbers $(r_1, \dots, r_s)$ are called the *divisor multiplicities* of $\mathcal{X}$ [FMN10, Remark 3.7]. If each $\mathcal{D}_i$ is smooth and $\sum_i \mathcal{D}_i$ is a simple normal crossing divisor, then $\mathcal{X}$ and $\widetilde{\mathcal{D}}_i$ are smooth and $\sum_i \widetilde{\mathcal{D}}_i$ is a simple normal crossing divisor [FMN10, Section 1.3.b].

The following lemma can be proved in just the same way as (2) of [FMN10, Theorem 5.2]. We give a proof for the reader's convinience.

PROPOSITION 7.1 (cf. [FMN10, Theorem 5.2]). *Let X be a variety over $\mathbb{C}$ with at worst quotient singularities, $\mathcal{X}^{can}$ be its canonical stack, and $\sum_i \mathcal{D}_i$ be an effective simple normal crossing divisor on $\mathcal{X}^{can}$ such that each irreducible component $\mathcal{D}_i$ is smooth. Let further $(r_1, \dots, r_s)$ be a sequence of positive integers and $\mathcal{X}$ be the smooth Deligne-Mumford stack obtained by iterated root constructions as in* (7.1)*. Then $\mathcal{X}$ is characterized by the following properties up to isomorphism:*

- *$\mathcal{X}$ is a smooth separated Deligne-Mumford stack without generic stabilizers.*
- *$\mathcal{X}$ has the same coarse moduli space as $\mathcal{X}^{can}$.*
- *The canonical morphism $\varphi : \mathcal{X} \to \mathcal{X}^{can}$ is an isomorphism outside $\bigcup_i \mathcal{D}_i$.*
- *The pull-back of $\mathcal{D}_i$ is r_i times a prime divisor.*

PROOF. Let $\varphi : \mathcal{X} \to \mathcal{X}^{\mathrm{can}}$ be the stack obtained as in (7.1). Then it follows from [BC10, Section 2.1] that $\mathcal{X}$ has the four properties in the statement.

Conversely, suppose $f : \mathcal{X}' \to \mathcal{X}^{\mathrm{can}}$ is an arbitrary smooth Deligne-Mumford stack with the above properties. Then, by the universal property of the root stack, one has a morphism $g : \mathcal{X}' \to \mathcal{X}$ with $\varphi \circ g = f$. Since $\mathcal{X}'$ has the same coarse moduli space as $\mathcal{X}^{\mathrm{can}}$, g is a surjective morphism. Let $S \to \mathcal{X}$ be an étale atlas and let $\mathcal{Y}$ be the fiber product of S and $\mathcal{X}'$ over $\mathcal{X}$ with the induced morphism $\widetilde{g} : \mathcal{Y} \to S$:

(7.2)
$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "\widetilde{g}"'] & \mathcal{X}' \arrow[d, "g"] \arrow[dr, "f"] & \\ S \arrow[r] & \mathcal{X} \arrow[r, "\varphi"'] & \mathcal{X}^{\mathrm{can}} \end{tikzcd}$$

STEP 1. $\widetilde{g}$ is étale.

Let $U \to \mathcal{Y}$ be an étale atlas. The morphism $U \to S$ is flat since U and S are smooth and $U \to S$ has 0-dimensional fibers. It is also étale over $\mathcal{X}^{\mathrm{can}} \setminus \bigcup_i \mathcal{D}_i$ by our assumption. Moreover, the pull-back of a prime divisor of S to U is a reduced divisor and thus $U \to S$ is unramified in codimension one. Therefore $U \to S$ is étale in codimension one. Then it must be étale by the purity of the branch locus. This implies that $\widetilde{g}$ is étale.

STEP 2. $\mathcal{Y}$ is an algebraic space.

The étale morphism $\widetilde{g} : \mathcal{Y} \to S$ factors through the coarse moduli space Y of $\mathcal{Y}$ by our assumption that S is a scheme. This implies that the morphism from $\mathcal{Y}$ to Y is unramified. Take an étale covering $\{Y_\alpha \to Y\}$ of Y such that $\mathcal{Y} \times_Y Y_\alpha$ is isomorphic to the quotient stack $[U_\alpha / \varGamma_\alpha]$, where U_α is a scheme for each α and $\varGamma_\alpha$ is a finite group acting on U_α [AV02, Lemma 2.2.3]. By the unramifiedness of the morphism $\mathcal{Y} \to Y$, the quotient morphism $U_\alpha \to U_\alpha / \varGamma_\alpha$ is also unramified and hence the action of $\varGamma_\alpha$ on U_α is free. (Suppose a subgroup $H \subset \varGamma_\alpha$ fixes a closed point P of U_α. Then the quotient morphism $h : U_\alpha \to U_\alpha / H$

is unramified and therefore H acts trivially on the tangent space of P. This implies that H acts trivially on a neighbourhood of P. Since $\mathcal{Y}$ doesn't have a generic stabilizer, H must be trivial.) This implies that $\mathcal{Y} = Y$ is an algebraic space.

STEP 3. $\widetilde{g}$ is an isomorphism.

Since $\widetilde{g}$ is an étale surjective separeted morphism of algebraic spaces, the diagonal morphism $\Delta_{\widetilde{g}}$ of $\widetilde{g}$ is an open and closed immersion. Moreover, $\widetilde{g}$ is an isomorphism over an open dense subset of S and therefore $\Delta_{\widetilde{g}}$ is actually an isomorphism. This means that $\widetilde{g}$ becomes an isomorphism if we take a base change by $\widetilde{g}$ itself. Since $\widetilde{g}$ is étale surjective, $\widetilde{g}$ is actually an isomoprhism. This proves that g is an isomorphism. □

The following is the main result in this section:

PROPOSITION 7.2. *Let $\mathcal{X}$ be a two-dimensional smooth separated Deligne-Mumford stack without generic stabilizers. Assume that*

- *the canonical morphism $\varphi : \mathcal{X} \to \mathcal{X}^{can}$ to the canonical stack $\mathcal{X}^{can}$ of the coarse moduli space X is an isomorphism outside a simple normal crossing divisor $\sum_i \mathcal{D}_i$ on $\mathcal{X}^{can}$,*
- *the pull-back $\varphi^* \mathcal{D}_i$ is a multiple of a prime divisor, and*
- *each irreducible component $\mathcal{D}_i$ is a smooth rational stack.*

Then there exists an exceptional collection $(E_1, \dots, E_\ell)$ and a semiorthogonal decomposition

$$D^b \operatorname{coh} \mathcal{X} = \langle E_1, \dots, E_\ell, \varphi^* D^b \operatorname{coh} \mathcal{X}^{can} \rangle . \tag{7.3}$$

PROOF. Put

$$\mathcal{X}_1 = \sqrt[r_2]{(\mathcal{O}(\mathcal{D}_2), 1)/\mathcal{X}^{\mathrm{can}}} \times_{\mathcal{X}^{\mathrm{can}}} \cdots \times_{\mathcal{X}^{\mathrm{can}}} \sqrt[r_s]{(\mathcal{O}(\mathcal{D}_s), 1)/\mathcal{X}^{\mathrm{can}}}$$

and let $\mathcal{D} \subset \mathcal{X}_1$ be the prime divisor corresponding to D_1. Then $\mathcal{X}$ is isomorphic to $\sqrt[r_1]{(\mathcal{O}(\mathcal{D}), 1)/\mathcal{X}_1}$ and one has a semiorthogonal decomposition

$$D^b \operatorname{coh} \mathcal{X} = \langle j_* \pi_{\mathcal{D}}^* D^b(\operatorname{coh} \mathcal{D}) \otimes \mathcal{M}^{\otimes r_1 - 1}, \dots, j_* \pi_{\mathcal{D}}^* D^b(\operatorname{coh} \mathcal{D}) \otimes \mathcal{M}, \varphi^* D^b \operatorname{coh} \mathcal{X}_1 \rangle$$

by Proposition 6.1. Since D_1 is a smooth rational curve and $\sum_i \mathcal{D}_i$ is a simple normal crossing divisor, the divisor $\mathcal{D}$ is smooth and its coarse moduli space is a smooth rational curve. It follows that $\mathcal{D}$ is isomorphic to a weighted projective line in the sense of Geigle and Lenzing [GL87], so that the derived category $D^b \operatorname{coh} \mathcal{D}$ and hence the right orthogonal to $\varphi^* D^b \operatorname{coh} \mathcal{X}_1$ in $D^b \operatorname{coh} \mathcal{X}$ has a full exceptional collection. Now the assertion follows from induction on s. □

We end this section with the following lemma:

LEMMA 7.3. *The stack $[Y_0/A]$ satisfies the assumption of Proposition 7.2.*

PROOF. Y_0 is isomorphic to the minimal resolution of $\mathbb{A}^2/G_0$, and let $E = \bigcup_i D_i$ be the exceptional divisor. The canonical morphism is clearly an isomorphism outside E, which is a normal crossing divisor. Since A is a cyclic group whose action on Y_0 has no generic stabilizers, and E is a simple normal crossing divisor, the locus of $[Y_0/A]$ where the canonical morphism $[Y_0/A] \to [Y_0/A]^{\mathrm{can}}$ is not an isomorphism consists of disjoint union of irreducible exceptional divisors. Each of these irreducible divisors are smooth rational curves, and Lemma 7.3 is proved. □

8. Cyclic case of Theorem 1.4. Let X be a surface with at worst quotient singularities and consider the diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "q"] \arrow[d, "p"'] & \mathcal{X} \arrow[d, "\pi"] \\ Y \arrow[r, "\tau"] & X \end{tikzcd}$$

where $\mathcal{X}$ is the canonical stack associated with X, $\tau : Y \to X$ is the minimal resolution, and $\mathcal{Z}$ is the reduced part of the fiber product $Y \times_X \mathcal{X}$. We consider the integral functor

$$\Phi := q_* \circ p^* : D^b(\operatorname{coh} Y) \to D^b(\operatorname{coh} \mathcal{X}),$$

whose right adjoint will be denoted by Ψ.

PROPOSITION 8.1. *Assume X has only cyclic quotient singularities. Then Φ is fully faithful and there is a semiorthogonal decomposition*

$$D^b \operatorname{coh} \mathcal{X} = \langle E_1, \dots, E_\ell, \Phi(D^b \operatorname{coh} Y) \rangle$$

where $E_1, \dots, E_\ell$ is an exceptional collection.

PROOF. If X is the global quotient $\mathbb{A}^2/G$ for a finite small subgroup G of $GL_2(\mathbb{C})$, the proof of [Ish02, Theorem 3.1] shows that $\mathcal{Z}$ is the quotient stack of the universal subscheme in $Y \times \mathbb{A}^2$ by the action of G under the identification of Y with G-Hilb($\mathbb{A}^2$). Hence the assertion in this case follows from Theorem 3.1.

In the general case, the composition $\Psi \circ \Phi$ is an integral functor with respect to some kernel object $\mathcal{P}$ on $Y \times Y$. By the local case above, $\mathcal{P}$ is etale locally the structure sheaf of the diagonal. Hence the kernel of $\Psi \circ \Phi$ is a line bundle on the diagonal, which implies that Φ is fully faithful. Since the singularities of X are isolated, the semiorthogonal decomposition comes from local contributions around each singular points, where the assertion holds by the global quotient case above. □

This yields Step 4 in Introduction.

9. Proofs of Theorem 1.3 and Theorem 1.4. To prove Theorem 1.3, it remains to show the isomorphism of functors between

$$\Phi' : D^b \operatorname{coh} Y \to D^b \operatorname{coh}[\mathbb{A}^2/G]$$

and the composition

$$D^b \operatorname{coh} Y \xrightarrow{\Phi_3} D^b \operatorname{coh} Y_2 \xrightarrow{\Phi_2} D^b \operatorname{coh} \mathcal{Y}_1 \xrightarrow{\Phi_1} D^b \operatorname{coh}[Y_0/A] \xrightarrow{\Phi_0} D^b \operatorname{coh}[\mathbb{A}^2/G].$$

It suffices to show that the functor $\Phi_0 \circ \Phi_1 \circ \Phi_2$ is isomorphic to the functor defined by the universal family parameterized by the moduli space $Y_2 = A$-Hilb(G_0-Hilb($\mathbb{A}^2$)). Since Φ_2 is the integral functor defined by the kernel object $\mathcal{O}_{(Y_2 \times_{Y_1} \mathcal{Y}_1)_{\mathrm{red}}}$ and Φ_1 is the pull-back functor, $\Phi_1 \circ \Phi_2$ is an integral functor whose kernel object is the pull-back of $\mathcal{O}_{(Y_2 \times_{Y_1} \mathcal{Y}_1)_{\mathrm{red}}}$ by the flat morphism $Y_2 \times [Y_0/A] \to Y_2 \times \mathcal{Y}_1$. This object is isomorphic to $\mathcal{O}_{(Y_2 \times [Y_0/A])_{\mathrm{red}}}$ and therefore $\Phi_1 \circ \Phi_2$ is isomorphic to the functor defined by the universal family of $Y_2 = A\text{-Hilb}(Y_0)$. Then by [IINdC13, Lemma 2.2], $\Phi_0 \circ \Phi_1 \circ \Phi_2$ is isomorphic to the functor defined by the universal family parameterized by the moduli space $Y_2 = A\text{-Hilb}(G_0\text{-Hilb}(\mathbb{A}^2))$.

To prove Theorem 1.4, we want to replace Theorem 3.1 with Theorem 1.3 in the proof of Proposition 8.1. In order to do that, consider the resolution $\tau_2 : Y_2 \to X$ obtained by successively blowing up Y so that it is isomorphic to $A\text{-Hilb}(G_0\text{-Hilb}(\mathbb{A}^2))$ over an etale neighbourhood of a singular point of X (whose corresponding point in $\mathcal{X}$ has stabilizer group G). Let $\widetilde{\mathcal{Z}}$ be the reduced part of the fiber product $Y_2 \times_X \mathcal{X}$. By the lemma below, for each singular point P of X whose neighbourhood is the quotient by a group $G_P \subset GL(2, \mathbb{C})$, there is a sheaf $\mathcal{F}_P$ on $Y_2 \times \mathcal{X}$ supported on $\tau^{-1}(P) \times BG_P$ with an extension

$$0 \to \mathcal{O}_{\widetilde{\mathcal{Z}}} \to \mathcal{E} \to \bigoplus_P \mathcal{F}_P \to 0$$

such that $\mathcal{E}$ is isomorphic to the universal family parameterized by $A\text{-Hilb}(G_0\text{-Hilb}(\mathbb{A}^2))$ for $G = G_P$ in an etale neighbourhood of each P. If we define Φ as the integral functor whose kernel object is $\mathcal{E}$, then we can argue as in the proof of Proposition 8.1 to obtain Theorem 1.4.

LEMMA 9.1. *Consider the local situation $X = \mathbb{A}^2/G$ with $Y = G\text{-Hilb}(\mathbb{A}^2)$ and $Y_2 = A\text{-Hilb}(G_0\text{-Hilb}(\mathbb{A}^2))$. Let $\widetilde{\mathcal{Z}}$ be the reduced part of the figer product $Y_2 \times_X \mathbb{A}^2$ and let $\mathcal{E}$ be the universal family parameterized by Y_2. Then there is an exact sequence*

$$0 \to \mathcal{O}_{\widetilde{\mathcal{Z}}} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a G-equivariant coherent sheaf on $Y_2 \times \mathbb{A}^2$ supported on $\tau_2^{-1}(0) \times \{0\}$.

PROOF. The universal sheaf $\mathcal{E}$ is the structure sheaf of the reduced part of the fiber product $Y_2 \times_{Y_1} Y_0 \times_{\mathbb{A}^2/G_0} \mathbb{A}^2$ and there is a morphism $Y_2 \times_{Y_1} Y_0 \times_{\mathbb{A}^2/G_0} \mathbb{A}^2 \to \mathcal{Z}$ which implies a map $\mathcal{O}_{\widetilde{\mathcal{Z}}} \to \mathcal{E}$. It is an isomorphism over the smooth locus of X and since $\mathcal{O}_{\widetilde{\mathcal{Z}}}$ is torsion-free as a coherent sheaf of $\mathcal{O}_{Y_2}$-modules, this map is injective. □

10. Invertible polynomials. Let n be a positive integer. An integer $n \times n$-matrix $A = (a_{ij})_{i,j=1}^n$ with non-zero determinant gives a polynomial $W \in \mathbb{C}[x_1, \dots, x_n]$ by

$$W = \sum_{i=1}^n x_1^{a_{i1}} \cdots x_n^{a_{in}} .$$

Non-zero coefficients of W can be absorbed by rescaling x_i. A polynomial obtained in this way is called an *invertible polynomial* if it has an isolated critical point at the origin. Invertible polynomials play essential role in transposition mirror symmetry of Berglund and Hübsch [BH93], which has attracted much attention recently (cf. e.g. [Bor13, CR11, Kra, Tak10] and references therein). The quotient ring $R = \mathbb{C}[x_1, \dots, x_n]/(W)$ is naturally graded by the abelian group L generated by $n+1$ elements $\vec{x}_i$ and $\vec{c}$ with relations

$$a_{i1}\vec{x}_1 + \cdots + a_{in}\vec{x}_n = \vec{c}\,, \quad i = 1, \ldots, n\,.$$

The abelian group L is the group of characters of K defined by

$$K = \{(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}} = \cdots = \alpha_1^{a_{n1}} \cdots \alpha_n^{a_{nn}}\}\,.$$

The group $G_{\max}$ of *maximal diagonal symmetries* is defined as the kernel of the map

$$\begin{array}{ccc} K & \to & \mathbb{C}^\times \\ \Psi & & \Psi \\ (\alpha_1, \ldots, \alpha_n) & \mapsto & \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}}, \end{array}$$

so that there is an exact sequence

$$1 \to G_{\max} \to K \to \mathbb{C}^\times \to 1$$

of abelian groups. Let

$$\mathcal{X} = [(W^{-1}(0) \setminus \{0\})/K]$$

be the quotient stack of $W^{-1}(0) \setminus \{0\}$ by the natural action of K. It is a smooth Deligne-Mumford stack since W has an isolated critical point at the origin and the action of K at any point in $W^{-1}(0) \setminus \{0\}$ has a finite isotropy group.

LEMMA 10.1. *The coarse moduli space of $\mathcal{X}$ is a rational variety. Moreover, each codimension one irreducible component of the locus where $\mathcal{X}$ has non-trivial stabilizers is also rational and these components form a simple normal crossing divisor.*

PROOF. Since the K-action on $(\mathbb{C}^\times)^n$ is free, the open dense substack

$$\mathcal{U} = [(W^{-1}(0) \cap (\mathbb{C}^\times)^n)/K]$$

of $\mathcal{X}$ is a scheme, which is an affine linear subspace of

$$[(\mathbb{C}^\times)^n/K] \cong (\mathbb{C}^\times)^{n-1}$$

considered as an open subscheme of $\mathbb{C}^{n-1}$. This shows that $\mathcal{X}$ is rational. A divisor with a non-trivial generic stabilizer is the closure of either

$$W^{-1}(0) \cap \{x_i = 0\} \cap \{x_k \neq 0 \text{ for } k \neq i\}$$

for some i or

$$\{x_i = x_j = 0\} \cap \{x_k \neq 0 \text{ for } k \neq i, j\}$$

for some $i \neq j$. (If $\{x_i = x_j = 0\}$ is not contained in $W^{-1}(0)$, then $W^{-1}(0) \cap \{x_i = x_j = 0\}$ has codimension greater than one.) The quotient of the former also contains an affine subspace of a tours, and the quotient of the latter is a toric stack. Hence they are rational.

Since the stabilizer group of any point on $\mathcal{X}$ are abelian and therefore locally diagonalizable, the union of such divisors has normal crossings. Moreover, at each point on the union, different local components have different stabilizer subgroups in K. Hence the union has simple normal crossings. $\square$

Now assume $n = 4$ so that $\dim \mathcal{X} = 2$ and let Y be the minimal resolution of the coarse moduli space of $\mathcal{X}$. Since Y is a rational surface, one has a full exceptional collection on Y by Orlov [Orl92]. Let $\mathcal{X}^{\mathrm{can}}$ be the canonical stack associated with the coarse moduli space of $\mathcal{X}$. Then Theorem 1.4 gives a full exceptional collection on $\mathcal{X}^{\mathrm{can}}$. Since $\mathcal{X}$ can be obtained by successive root constructions from $\mathcal{X}^{\mathrm{can}}$, Proposition 7.2 and Lemma 10.1 give the following:

THEOREM 10.2. *The two-dimensional Deligne-Mumford stack associated with an invertible polynomial in four variables has a full exceptional collection.*

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DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
HIROSHIMA UNIVERSITY
1–7–1 KAGAMIYAMA
739–8521 HIGASHI-HIROSHIMA
JAPAN

E-mail address: akira141@hiroshima-u.ac.jp

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
MACHIKANEYAMA 1–1, TOYONAKA
560–0043 OSAKA
JAPAN

E-mail address: kazushi@math.sci.osaka-u.ac.jp