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STRONG CONVERGENCE THEOREM OF CESÀRO MEANS WITH RESPECT TO THE WALSH SYSTEM

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Abstract. We prove that Cesàro means of one-dimensional Walsh-Fourier series are uniformly bounded operators in the martingale Hardy space H_p for $0 < p < 1/(1+\alpha)$.

1. Introduction. The definitions and notations used in this introduction can be found in the next section. It is well-known (see, e.g., [11, p.125]) that Walsh-Paley system is not a Schauder basis in the space $L_1(G)$. Moreover, there is a function F in the dyadic Hardy space $H_1(G)$, such that the partial sums of the Walsh-Fourier series of F are not bounded in the L_1-norm. However, in Simon [19] the following estimation was obtained: for all $F \in H_1(G)$

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|S_kF\|_1}{k} \le c\,\|F\|_{H_1}, \quad (n = 2, 3, \dots),$$

where S_kF denotes the k-th partial sum of the Walsh-Fourier series of F (For the trigonometric analogue see in Smith [21], for the Vilenkin system in Gát [6], for a more general, so-called Vilenkin-like system in Blahota [1].). Simon [16] (see also [27] and [34]) proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n}\sum_{k=1}^{n}\frac{\|S_kF\|_p^p}{k^{2-p}} \le c_p\,\|F\|_{H_p}^p, \quad (0 < p \le 1,\ n = 2, 3, \dots), \tag{1}$$

for all $F \in H_p$, where $[p]$ denotes integer part of p.

In [25] it was proven that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0 < p < 1)$ in (1) is given exactly.

Weisz [35] considered the norm convergence of Fejér means of Walsh-Fourier series and proved that

$$\|\sigma_nF\|_{H_p} \le c_p\,\|F\|_{H_p}, \quad F \in H_p, \quad (1/2 < p < \infty,\ n = 1, 2, 3, \dots), \tag{2}$$

where the constant $c_p > 0$ depends only on p.

Inequality (2) immediately implies that

$$\frac{1}{n^{2p-1}}\sum_{k=1}^{n}\frac{\|\sigma_kF\|_{H_p}^p}{k^{2-2p}} \le c_p\,\|F\|_{H_p}^p, \quad (1/2 < p < \infty).$$

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If (2) also hold, for $0 < p \leq 1/2$, then we would have

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k F\|_{H_p}^p}{k^{2-2p}} \leq c_p \|F\|_{H_p}^p , \quad (0 < p \leq 1/2, \ n = 2, 3, \ldots) , \tag{3}$$

but in [22] it was proven that the assumption $p > 1/2$ is essential. In particular, there was proven that there exists a martingale $F \in H_p$ $(0 < p \leq 1/2)$, such that $\sup_n \|\sigma_n F\|_p = +\infty$.

However, in [26] (see also [3]) it was proven that (3) holds, though (2) is not true for $0 < p \leq 1/2$.

The weak-type (1,1) inequality for the maximal operator of Fejér means σ^* can be found in Schipp [14] (see also [13]). Fujji [5] and Simon [18] verified that σ^* is bounded from H_1 to L_1. Weisz [30] generalized this result and proved the boundedness of σ^* from the space H_p to the space L_p for $p > 1/2$. Simon [17] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ is due to Goginava [8] (see also [4]). Weisz [31] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$. In [23, 24] it was proven that the maximal operators $\widetilde{\sigma}_p^*$ defined by

$$\widetilde{\sigma}_p^* F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n F|}{n^{1/p-2} \log^{2[1/2+p]} n} , \quad (0 < p \leq 1/2, \ n = 2, 3, \ldots) \tag{4}$$

is bounded from the Hardy space H_p to the space L_p, where $F \in H_p$ and $\left[1/2 + p\right]$ denotes integer part of $1/2+p$. Moreover, there was also shown that sequence $\{n^{1/p-2} \log^{2[1/2+p]} n \ : \ n = 2, 3, \ldots\}$ in (4) can not be improved.

Weisz [33] proved that the maximal operator $\sigma^{\alpha,*}$ $(0 < \alpha < 1)$ of the Cesàro means of Walsh system is bounded from the martingale space H_p to the space L_p for $p > 1/(1+\alpha)$. Goginava [9] gave a counterexample, which shows that the boundedness does not hold for $0 < p \leq 1/(1+\alpha)$. Recently, Weisz and Simon [20] show that the maximal operator $\sigma^{\alpha,*}$ is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha),\infty}$. An analogical result for Walsh-Kaczmarz system was proven in [7].

In [10] Goginava investigated the behaviour of Cesàro means of Walsh-Fourier series in detail. For some approximation properties of the two dimensional case see paper of Nagy [12].

The main aim of this paper is to generalize estimate (3) for Cesàro means, when $0 < p < 1/(1+\alpha)$. We also consider the weighted maximal operator of (C,α) means and proved some new $\left(H_p, L_p\right)$-type inequalities for it.

We note that the case $p = 1/(1+\alpha)$ was considered in [2].

2. Definitions and Notations. Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given so that the measure of a singleton is $1/2$.

Define the group G as the complete direct product of the group $\mathbb{Z}_2$ with the product of the discrete topologies of $\mathbb{Z}_2$'s. The elements of G are represented by sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \qquad (x_k = 0, 1) \ .$$

It is easy to give a base for the neighborhood of G

$$\begin{aligned} I_0(x) &:= G, \\ I_n(x) &:= \{y \in G \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \ (x \in G, \ n \in \mathbb{N}) \,. \end{aligned}$$

Denote $I_n := I_n(0)$ and $\overline{I_n} := G \setminus I_n$. Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G \qquad (n \in \mathbb{N}) \ .$$

Denote

$$I_M^{k,l} := \begin{cases} I_M(0, \ldots, 0, x_k = 1, 0, \ldots, 0, x_l = 1, x_{l+1}, \ldots, x_{M-1}), & k < l < M, \\ I_M(0, \ldots, 0, x_k = 1, 0, \ldots, 0), & l = M \,. \end{cases}$$

It is evident

$$\overline{I_M} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_M^{k,l} \right) \bigcup \left(\bigcup_{k=0}^{M-1} I_M^{k,M} \right) . \tag{5}$$

If $n \in \mathbb{N}$, then every n can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j 2^j$, where $n_j \in Z_2$ $(j \in \mathbb{N})$ and only finite number of n_j's differ from zero, that is, n is expressed in the number system of base 2. Let $|n| := \max \{j \in \mathbb{N}, n_j \neq 0\}$, that is $2^{|n|} \leq n \leq 2^{|n|+1}$.

The norm (or quasi-norm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \left(\int_G |f|^p \, d\mu \right)^{1/p}, \ (0 < p < \infty) \ .$$

The space $L_{p,\infty}(G)$ consists of all measurable functions f, for which

$$\|f\|_{L_{p,\infty}(G)} := \sup_{\lambda > 0} \lambda \mu (f > \lambda)^{1/p} < \infty \,.$$

Next, we introduce on G an orthonormal system which is called the *Walsh system.* At first, define the functions $r_k(x) : G \to \mathbb{C}$, the so-called Rademacher functions as

$$r_k(x) := (-1)^{x_k} \qquad (x \in G, \ k \in \mathbb{N}) \ .$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x) \, (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \qquad (n \in \mathbb{N}) \ .$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see, e.g., [28]).

If $f \in L_1(G)$, then we can establish Fourier coefficients, partial sums of the Fourier series, Fejér means, Dirichlet and Fejér kernels in the usual manner:

$$\begin{aligned}
\widehat{f}(n) &: = \int_G f w_n d\mu, && (n \in \mathbb{N}), \\
S_n f &: = \sum_{k=0}^{n-1} \widehat{f}(k) w_k, && (n \in \mathbb{N}_+), \\
\sigma_n f &: = \frac{1}{n} \sum_{k=1}^{n} S_k f, && (n \in \mathbb{N}_+), \\
D_n &: = \sum_{k=0}^{n-1} w_k, && (n \in \mathbb{N}_+), \\
K_n &: = \frac{1}{n} \sum_{k=1}^{n} D_k, && (n \in \mathbb{N}_+),
\end{aligned}$$

respectively. Recall that (see e.g., [15])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases} \tag{6}$$

The Cesàro means ((C, α)-means) are defined as

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 1, \qquad A_n^\alpha := \frac{(\alpha+1)\cdots(\alpha+n)}{n!} \qquad \alpha \neq -1, -2, \dots. \tag{7}$$

It is well known that

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \; A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \; A_n^\alpha \backsim n^\alpha$$

and

$$\sup_n \int_G \left| K_n^\alpha \right| d\mu \leq c < \infty, \tag{8}$$

where K_n^α is n-th Cesàro kernel.

The σ-algebra generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by F_n $(n \in \mathbb{N})$. Denote by $F = (F_n, n \in \mathbb{N})$ the martingale with respect to F_n $(n \in \mathbb{N})$ (for details see, e.g., [29]).

The maximal function of a martingale F is defined by

$$F^* := \sup_{n \in \mathbb{N}} |F_n|.$$

In the case $f \in L_1(G)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n\in\mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right| .$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G)$ consist of all martingales such that

$$\|F\|_{H_p} := \|F^*\|_p < \infty .$$

A bounded measurable function a is a p-atom, if there exist a dyadic interval I such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I .$$

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and for every $k \in \mathbb{N}$ the limit

$$\widehat{F}(k) := \lim_{n\to\infty} \int_G F_n w_k d\mu \tag{9}$$

exists and it is called the k-th Walsh-Fourier coefficients of F.

Denote by $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)\,(x \in G,\ n \in \mathbb{N})$. If $F := (S_{2^n} f : n \in \mathbb{N})$ is the regular martingale generated by $f \in L_1(G)$, then

$$\widehat{F}(k) = \int_G f w_k d\mu =: \widehat{f}(k), \qquad k \in \mathbb{N}.$$

For $0 < \alpha \leq 1$, let consider maximal operators

$$\sigma^{\alpha,*} F := \sup_{n\in\mathbb{N}} \left|\sigma_n^\alpha F\right|, \ \ \overset{\sim}{\sigma}{}_p^{\alpha,*} F := \sup_{n\in\mathbb{N}} \frac{\left|\sigma_n^\alpha F\right|}{(n+1)^{1/p-1-\alpha}}, \ 0 < p < 1/(1+\alpha) .$$

For the martingale

$$F = \sum_{n=0}^{\infty} (F_n - F_{n-1})$$

the conjugate transforms are defined as

$$\widetilde{F^{(t)}} := \sum_{n=0}^{\infty} r_n(t)(F_n - F_{n-1}),$$

where $t \in G$ is fixed. Note that $\widetilde{F^{(0)}} = F$.

As it is well-known (see, e.g., [29])

$$\left\|\widetilde{F^{(t)}}\right\|_{H_p} = \|F\|_{H_p}, \quad \|F\|_{H_p}^p \sim \int_G \left\|\widetilde{F^{(t)}}\right\|_p^p dt, \quad \widetilde{\sigma_m^\alpha F^{(t)}} = \sigma_m^\alpha \widetilde{F^{(t)}} . \tag{10}$$

3. Formulation of main results.

THEOREM 1. *a) Let* $0 < \alpha < 1$ *and* $0 < p < 1/(1+\alpha)$. *Then there exists absolute constant* $c_{\alpha,p}$, *depending on* α *and* p, *such that for all* $F \in H_p(G)$

$$\left\| \overset{\sim}{\sigma}_p^{\alpha,*} F \right\|_p \leq c_{\alpha,p} \left\| F \right\|_{H_p} .$$

b) Let $0 < \alpha < 1$, $0 < p < 1/(1+\alpha)$ *and* $\varphi : \mathbb{N}_+ \to [1, \infty)$ *be a nondecreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{n^{1/p-1-\alpha}}{\varphi(n)} = \infty . \tag{11}$$

Then the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n^\alpha f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_p(G)$ *to the space* $L_p(G)$.

THEOREM 2. *Let* $0 < \alpha < 1$ *and* $0 < p < 1/(1+\alpha)$. *Then there exists absolute constant* $c_{\alpha,p}$, *depending on* α *and* p, *such that for all* $F \in H_p$

$$\sum_{m=1}^{\infty} \frac{\left\| \sigma_m^\alpha F \right\|_{H_p}^p}{m^{2-(1+\alpha)p}} \leq c_{\alpha,p} \left\| F \right\|_{H_p}^p .$$

4. Auxiliary Propositions. The dyadic Hardy martingale spaces $H_p(G)$ have an atomic characterization, when $0 < p \leq 1$:

LEMMA 1 (Weisz [32]). *A martingale* $F = (F_n,\ n \in \mathbb{N})$ *is in* H_p $(0 < p \leq 1)$ *if and only if there exists a sequence* $(a_k,\ k \in \mathbb{N})$ *of p-atoms and a sequence* $(\mu_k,\ k \in \mathbb{N})$ *of a real numbers*, *such that for every* $n \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \tag{12}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty .$$

Moreover,

$$\|F\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} ,$$

where the infimum is taken over all decompositions of F *of the form (12).*

By using Lemma 1 we can easily proved the following:

LEMMA 2 (Weisz [29]). *Suppose that an operator T is σ-linear and for some $0 < p \leq 1$*

$$\int_{\bar{I}} |Ta|^p \, d\mu \leq c_p < \infty,$$

for every p-atom a, where I denote the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p} \, .$$

To prove our main results we also need the following estimations:

LEMMA 3 ([2]). *Let $0 < \alpha < 1$ and $n > 2^M$. Then*

$$\int_{I_M} \left|K_n^\alpha (x+t)\right| d\mu (t) \leq \frac{c_\alpha 2^{\alpha l + k}}{n^\alpha 2^M},$$

for $x \in I_{l+1}(e_k + e_l)$, $(k = 0, \ldots, M-2, l = k+1, \ldots, M-1)$ and

$$\int_{I_M} \left|K_n^\alpha (x+t)\right| d\mu (t) \leq \frac{c_\alpha 2^k}{2^M},$$

for $x \in I_M(e_k)$, $(k = 0, \ldots, M-1)$.

5. Proof of Theorems.

PROOF OF THEOREM 1. Since σ_n is bounded from L_∞ to L_∞ (the boundedness follows from (8)) according to Lemma 2 the proof of Theorem 1 will be complete if we show

$$\sup \int_{\overline{I_M}} \left|\overset{\sim}{\sigma}{}_p^{\alpha,*} a\right|^p d\mu < \infty,$$

where the supremum is taken over all p-atoms a. We may assume that a is an arbitrary p-atom, with support I, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_n^\alpha(a) = 0$, when $n \leq 2^M$. Therefore, we can suppose that $n > 2^M$.

Let $x \in I_M$. Since $\|a\|_\infty \leq c2^{M/p}$ we obtain

$$\begin{aligned} \left|\sigma_n^\alpha a(x)\right| &\leq \int_{I_M} |a(t)| \left|K_n^\alpha (x+t)\right| d\mu(t) \\ &\leq \|a(x)\|_\infty \int_{I_M} \left|K_n^\alpha (x+t)\right| d\mu(t) \\ &\leq c_\alpha 2^{M/p} \int_{I_M} \left|K_n^\alpha (x+t)\right| d\mu(t) \, . \end{aligned}$$

Let $x \in I_M^{k,l}$, $0 \leq k < l < M$. Then from Lemma 3 we get

$$\left|\sigma_n^\alpha a(x)\right| \leq \frac{c_{\alpha,p} 2^{M(1/p-1)} 2^{\alpha l + k}}{n^\alpha} \, . \tag{13}$$

Let $x \in I_M^{k,M},\ 0 \leq k < M$. Then from Lemma 3 we have

$$\left|\sigma_n^{\alpha} a(x)\right| \leq c_{\alpha,p} 2^{M(1/p-1)+k}. \tag{14}$$

By combining (5), (13) and (14) we obtain

$$\begin{aligned}
&\int_{\overline{I_M}} \sup_{n\in\mathbb{N}} \left|\frac{\sigma_n^{\alpha} a(x)}{n^{1/p-1-\alpha}}\right|^p d\mu(x) \\
=& \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \sum_{x_j=0,\, j\in\{l+1,\dots,M-1\}}^{1} \int_{I_M^{k,l}} \sup_{n>2^M} \left|\frac{\sigma_n^{\alpha} a(x)}{n^{1/p-1-\alpha}}\right|^p d\mu(x) \\
&+\sum_{k=0}^{M-1} \int_{I_M^{k,M}} \sup_{n>2^M} \left|\frac{\sigma_n^{\alpha} a(x)}{n^{1/p-1-\alpha}}\right|^p d\mu(x) \\
\leq& \frac{1}{2^{M(1-(1+\alpha)p)}} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \sum_{x_j=0,\, j\in\{l+1,\dots,N-1\}}^{1} \int_{I_M^{k,l}} \sup_{n>2^M} \left|\sigma_n^{\alpha} a(x)\right|^p d\mu(x) \\
&+\frac{1}{2^{M(1-(1+\alpha)p)}} \sum_{k=0}^{M-1} \int_{I_M^{k,M}} \sup_{n>2^M} \left|\sigma_n^{\alpha} a(x)\right|^p d\mu(x) \\
\leq& \frac{c_{\alpha,p}}{2^{M(1-(1+\alpha)p)}} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{2^l} \frac{2^{M(1-p)} 2^{(\alpha l+k)p}}{2^{M\alpha p}} \\
&+\frac{c_{\alpha,p}}{2^{M(1-(1+\alpha)p)}} \frac{1}{2^M} \sum_{k=0}^{M-1} 2^{M(1-p)+pk} \\
\leq& c_{\alpha,p} \sum_{k=0}^{M-2} 2^{kp} \sum_{l=k+1}^{M-1} \frac{1}{2^{l(1-\alpha p)}} \\
&+\frac{c_{\alpha,p}}{2^{M(1-(1+\alpha)p)}} \sum_{k=0}^{M-1} \frac{2^{pk}}{2^{pM}} \leq c_{\alpha,p} < \infty .
\end{aligned}$$

It is easy to show that under condition (11), there exists a sequence of positive integers $\{n_k,\ k \in \mathbb{N}_+\}$, such that

$$\lim_{k\to\infty} \frac{\left(2^{2n_k}+2\right)^{1/p-1-\alpha}}{\varphi\left(2^{2n_k}+2\right)} = \infty.$$

Let

$$f_{n_k} = D_{2^{2n_k+1}} - D_{2^{2n_k}}.$$

It is evident

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = 2^{2n_k}, \dots, 2^{2n_k+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write

$$S_i f_{n_k} = \begin{cases} D_i - D_{2^{2n_k}}, & \text{if } i = 2^{2n_k} + 1, \ldots, 2^{2n_k+1} - 1, \\ f_{n_k}, & \text{if } i \geq 2^{2n_k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

From (6) we get

$$\left\| f_{n_k} \right\|_{H_p} = \left\| f_{n_k}^* \right\|_p = \left\| D_{2^{2n_k+1}} - D_{2^{2n_k}} \right\|_p \leq c 2^{2n_k(1-1/p)}. \tag{16}$$

Since $A_0^{\alpha-1} = 1$, by (15) we can write

$$\begin{aligned}
\frac{\left| \sigma^{\alpha}_{2^{2n_k}+1} f_{n_k} \right|}{\varphi\left(2^{2n_k}+1\right)} &= \frac{1}{\varphi\left(2^{2n_k}+1\right) A^{\alpha}_{2^{2n_k}+1}} \left| \sum_{j=1}^{2^{2n_k}+1} A^{\alpha-1}_{2^{2n_k}+1-j} S_j f_{n_k} \right| \\
&= \frac{1}{\varphi\left(2^{2n_k}+1\right) A^{\alpha}_{2^{2n_k}+1}} \left| \sum_{j=2^{2n_k}+1}^{2^{2n_k}+1} A^{\alpha-1}_{2^{2n_k}+1-j} S_j f_{n_k} \right| \\
& \frac{1}{\varphi\left(2^{2n_k}+1\right) A^{\alpha}_{2^{2n_k}+1}} \left| A_0^{\alpha-1} \left(D_{2^{2n_k}+1} - D_{2^{2n_k}} \right) \right| \\
&= \frac{1}{\varphi\left(2^{2n_k}+1\right) A^{\alpha}_{2^{2n_k}+1}} \left| A_0^{\alpha-1} w_{2^{2n_k}} \right| \\
&\geq \frac{c}{\varphi\left(2^{2n_k}+1\right) \left(2^{2n_k}+1\right)^{\alpha}}.
\end{aligned}$$

From (16) we have

$$\begin{aligned}
&\frac{c/\left(\varphi\left(2^{2n_k}+1\right)\left(2^{2n_k}+1\right)^{\alpha}\right) \mu\left\{x : \left| \widetilde{\sigma}^{\alpha,*} f \right| \geq c/\left(\varphi\left(2^{2n_k}+1\right)\left(2^{2n_k}+1\right)^{\alpha}\right)\right\}^{1/p}}{\left\| f_{n_k} \right\|_{H_p}} \\
&\geq \frac{c}{\varphi\left(2^{2n_k}+1\right)\left(2^{2n_k}+1\right)^{\alpha}} \frac{1}{2^{2n_k(1-1/p)}} \geq \frac{c\left(2^{2n_k}+1\right)^{1/p-1-\alpha}}{\varphi\left(2^{2n_k}+1\right)} \to \infty, \text{ as } k \to \infty.
\end{aligned}$$

Theorem 1 is proven. □

PROOF OF THEOREM 2. Suppose that

$$\sum_{m=1}^{\infty} \frac{\left\| \sigma_m^{\alpha} F \right\|_p^p}{m^{2-(1+\alpha)p}} \leq \| F \|_{H_p}^p.$$

Then by using (10) we have

$$\sum_{m=1}^{\infty}\frac{\left\|\sigma_m^{\alpha}F\right\|_{H_p}^{p}}{m^{2-(1+\alpha)p}}=\sum_{m=1}^{\infty}\frac{\int_G\left\|\widetilde{\sigma_m^{\alpha}F^{(t)}}\right\|_p^p dt}{m^{2-(1+\alpha)p}}\leq\int_G\sum_{m=1}^{n}\frac{\left\|\sigma_m^{\alpha}\widetilde{F^{(t)}}\right\|_p^p}{m^{2-(1+\alpha)p}}dt \tag{17}$$
$$\leq\int_G\left\|\widetilde{F^{(t)}}\right\|_{H_p}^p dt\sim\int_G\|F\|_{H_p}^p dt=\|F\|_{H_p}^p .$$

According to Theorem 1 and (17) the proof of Theorem 2 will be complete, if we show

$$\sum_{m=1}^{\infty}\frac{\left\|\sigma_m^{\alpha}a\right\|_p^p}{m^{2-(1+\alpha)p}}\leq c_{\alpha}<\infty,$$

for every p-atom a. Analogously to first part of Theorem 1 we can assume that $n>2^M$ and a be an arbitrary p-atom, with support I, $\mu(I)=2^{-M}$ and $I=I_M$.

Let $x\in I_M$. Since σ_n is bounded from L_∞ to L_∞ (the boundedness follows from (8)) and $\|a\|_\infty\leq c2^{M/p}$ we obtain

$$\begin{aligned}\int_{I_M}\left|\sigma_m^{\alpha}a\right|^p d\mu &\leq\int_{I_M}\left\|K_m^{\alpha}\right\|_1^p\|a\|_\infty^p d\mu\\ &\leq c_{\alpha,p}\int_{I_M}\|a\|_\infty^p d\mu\leq c_{\alpha,p}<\infty .\end{aligned}$$

Hence

$$\begin{aligned}\sum_{m=2^M+1}^{\infty}\frac{\int_{I_M}\left|\sigma_m^{\alpha}a\right|^p d\mu}{m^{2-(1+\alpha)p}} &\leq c_{\alpha,p}\sum_{m=2^M+1}^{\infty}\frac{1}{m^{2-(1+\alpha)p}}\\ &\leq\frac{c_{\alpha,p}}{2^{M(1-(1+\alpha)p)}}\leq c_{\alpha,p}<\infty.\end{aligned}$$

By combining (5), (13) and (14) analogously to first part of Theorem 1 we can write

$$\begin{aligned}&\sum_{m=2^M+1}^{\infty}\frac{\int_{\overline{I_M}}\left|\sigma_m^{\alpha}a\right|^p d\mu}{m^{2-(1+\alpha)p}}\\ &=\sum_{m=2^M+1}^{\infty}\left(\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\sum_{x_j=0,j\in\{l+1,\dots,M-1\}}^{1}\frac{\int_{I_M^{k,l}}\left|\sigma_m^{\alpha}a\right|^p d\mu}{m^{2-(1+\alpha)p}}+\sum_{k=0}^{M-1}\frac{\int_{I_M^{k,M}}\left|\sigma_m^{\alpha}a\right|^p d\mu}{m^{2-(1+\alpha)p}}\right)\\ &\leq\sum_{m=2^M+1}^{\infty}\left(\frac{c_{\alpha,p}2^{M(1-p)}}{m^{2-p}}\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{p(\alpha l+k)}}{2^l}+\frac{c_{\alpha,p}2^{M(1-p)}}{m^{2-(1+\alpha)p}}\sum_{k=0}^{M-1}\frac{2^{pk}}{2^M}\right)\\ &<c_{\alpha,p}2^{M(1-p)}\sum_{m=2^M+1}^{\infty}\frac{1}{m^{2-p}}+c_{\alpha,p}\sum_{m=2^M+1}^{\infty}\frac{1}{m^{2-(1+\alpha)p}}\leq c_{\alpha,p}<\infty,\end{aligned}$$

which completes the proof of Theorem 2. □

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