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# SCHRÖDINGER UNCERTAINTY RELATION AND CONVEXITY FOR THE MONOTONE PAIR SKEW INFORMATION

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**Abstract.** Furuichi and Yanagi showed a Schrödinger uncertainty relation for the Wigner-Yanase-Dyson skew information, which is a special monotone pair skew information. In this paper, we give a Schrödinger uncertainty relation based on a monotone pair skew information, and extend the result of Furuichi and Yanagi. Moreover, we show that some monotone pair skew information becomes a metric adjusted skew information and therefore the convexity of it follows from known results.

## 1. Introduction.

Wigner-Yanase skew information

$$I_\rho(A) := \frac{1}{2}\mathrm{Tr}\big((i[\rho^{1/2}, A])^2\big) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^{1/2}A\rho^{1/2}A) \tag{1.1}$$

was defined in [14]. This quantity can be considered as a kind of the degree for non-commuatativity between a quantum state (density matrix) $\rho$ and an observable (self-adjoint operator) $A$. Here the commutator is defined by $[A, B] = AB - BA$. This quantity was generalized by Dyson as

$$I_{\rho,\alpha}(A) := \frac{1}{2}\mathrm{Tr}\big((i[\rho^\alpha, A])(i[\rho^{1-\alpha}, A])\big) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^\alpha A\rho^{1-\alpha}A)\,, \quad \alpha \in [0, 1]\,, \tag{1.2}$$

and it is known as the Wigner-Yanase-Dyson skew information. The relations between skew information and uncertainty relation have been studied by many authors. See for example the references [2, 4, 12, 13, 15, 16].

Heisenberg's uncertainty relation for a density matrix $\rho$ and any pair of observables $A$, $B$ is the following inequality:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2\,. \tag{1.3}$$

Here the variance $V_\rho(A)$ for $\rho$ and an observable $A$ is defined by $V_\rho(A) = \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2$. The uncertainty relation is one of the most significant consequences of non-commutativity in quantum mechanics and is a key point in which quantum probability differs from classical probability. The further strong relation was given by Schrödinger:

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\{\mathrm{Cov}_\rho(A, B)\}|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2\,. \tag{1.4}$$

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Recall that the covariance $\mathrm{Cov}_\rho(A, B)$ for $\rho$ and two observables $A$ and $B$ is given by $\mathrm{Cov}_\rho(A, B) = \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)$. As a generalized version, Furuichi gave the Schrödinger uncertainty relation for mixed states in [2]:

$$U_\rho(A)U_\rho(B) - |\mathrm{Re}\{\mathrm{Corr}_\rho(A, B)\}|^2 \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2 , \tag{1.5}$$

where $U_\rho(A) := \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho(A))^2}$. The correlation measure is defined by

$$\mathrm{Corr}_\rho(A, B) = \mathrm{Tr}(\rho A^* B) - \mathrm{Tr}(\rho^{1/2} A^* \rho^{1/2} B) \tag{1.6}$$

for any operators $A, B$. Since $|\mathrm{Im}\{\mathrm{Corr}_\rho(A, B)\}|^2 = \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2$ for the observables $A$ and $B$, the inequality (1.5) is equivalent to the inequality

$$U_\rho(A)U_\rho(B) \geq |\mathrm{Corr}_\rho(A, B)|^2 . \tag{1.7}$$

In [6], Furuichi and Yanagi recently gave a generalization of the Schrödinger-type uncertainty relation: for $\alpha \in [1/2, 1]$, any density matrix $\rho$ and observables $A,\ B$,

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\mathrm{Corr}_{\rho,\alpha}(A, B)|^2 , \tag{1.8}$$

where $U_{\rho,\alpha}(A) := \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_{\rho,\alpha}(A))^2}$ and the generalized correlation measure is given by

$$\mathrm{Corr}_{\rho,\alpha}(A, B) = \mathrm{Tr}(\rho A^* B) - \mathrm{Tr}(\rho^\alpha A^* \rho^{1-\alpha} B) . \tag{1.9}$$

The inequalities are refinements of Heisenberg's uncertainty relation (1.3) because $0 \leq I_{\rho,\alpha}(A) \leq U_{\rho,\alpha}(A) \leq V_\rho(A)$.

In [3], for a monotone pair $(f, g)$ of operator monotone functions, Furuichi introduced the $(f, g)$-skew information by

$$\begin{aligned} I_{\rho,(f,g)}(A) &:= \frac{1}{2}\mathrm{Tr}\big((i[f(\rho), A])(i[g(\rho), A])\big) \\ &= \mathrm{Tr}\big(f(\rho)g(\rho)A^2\big) - \mathrm{Tr}\big(f(\rho)Ag(\rho)A\big) . \end{aligned} \tag{1.10}$$

For $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha} (0 < \alpha < 1)$, $I_{\rho,(f,g)}(A)$ reduces to $I_{\rho,\alpha}(A)$. In [10], for all monotone pair $(f, g)$ of operator monotone functions, which are compatible in logarithmic increase (CLI monotone pair, in short), the present authors extended Yanagi's uncertainty relation in [15]. The purpose of this paper is to obtain a Schrödinger-type uncertainty relation based on the CLI monotone pair skew information satisfying the condition (2.11). We will see that in special cases this result reduces to (1.7) and (1.8).

Next we discuss the convexity for the monotone pair skew information. The convexity of the skew information means that the information content should decrease when two states are mixed. The convexity of the Wigner-Yanase skew information $I_\rho$ was proved by Wigner and Yananse [14], and the convexity of the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}$, $\alpha \in (0, 1)$, was remarkably established by Lieb [11]. We study the convexity of the monotone pair skew information in order to see that it is really a physically meaningful information measure. By using the Morozova-Chentsov function, Hansen gave the notion of metric adjusted skew information on the state space of a quantum system and showed the convexity for the metric adjusted skew information [9]. The Wigner-Yanase-Dyson skew information turns out to be a metric adjusted skew information as a special case. See [1, 6, 8, 16]. We cannot show the convexity of the monotone pair skew information in general. However, investigating the relations between the monotone pair skew information and the metric adjusted skew information, we can show the convexity of the monotone pair skew information in some cases.

This paper is organized as follows: In Section 2, we briefly review some uncertainty relations and state the main result (Theorem 2.3). Then we give some examples showing that it extends the previous results. In Section 3, we show that the monotone pair skew information becomes the metric adjusted skew information in some cases. Section 4 is devoted to the proof of Theorem 2.3.

**2. Trace inequalities and main result.** In this section we briefly review the uncertainty relation for skew informations and state the main result.

Let $M_n$ (resp. $M_{n,sa}$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices). Let $D_n$ be the set of strictly positive elements of $M_n$ and $D_n^1 \subset D_n$ the set of strictly positive density matrices, that is,

$$D_n^1 = \{\rho \in M_n\ ;\ \mathrm{Tr}(\rho) = 1,\ \rho > 0\}\,.$$

Let $\rho \in D_n^1$ be a fixed density matrix. For any $A \in M_{n,sa}$, define $A_0 := A - \mathrm{Tr}(\rho A)I$, where $I \in M_n$ is the identity matrix. The variance $V_\rho(A)$ for $\rho$ and $A$ is defined by $V_\rho(A) = \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2 = \mathrm{Tr}(\rho A_0^2)$.

For $0 \le \alpha \le 1$, we introduce the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(A)$ and some other quantities:

$$\begin{aligned}
(2.1)\qquad I_{\rho,\alpha}(A) &:= \frac{1}{2}\mathrm{Tr}\big((i[\rho^\alpha, A_0])(i[\rho^{1-\alpha}, A_0])\big) \\
&= \mathrm{Tr}(\rho A_0^2) - \mathrm{Tr}(\rho^\alpha A_0 \rho^{1-\alpha} A_0)\,, \\
(2.2)\qquad J_{\rho,\alpha}(A) &:= \frac{1}{2}\mathrm{Tr}\big(\{\rho^\alpha, A_0\}\{\rho^{1-\alpha}, A_0\}\big) \\
&= \mathrm{Tr}(\rho A_0^2) + \mathrm{Tr}(\rho^\alpha A_0 \rho^{1-\alpha} A_0)\,, \\
U_{\rho,\alpha}(A) &:= \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_{\rho,\alpha}(A))^2} \\
(2.3)\qquad &= \sqrt{I_{\rho,\alpha}(A) J_{\rho,\alpha}(A)}\,.
\end{aligned}$$

Here, $\{A, B\} = AB + BA$ is the anti-commutator. In [15], using the quantity $U_{\rho,\alpha}$, Yanagi gave the generalized uncertainty relation of Heisenberg-type:

$$(2.4)\qquad U_{\rho,\alpha}(A) U_{\rho,\alpha}(B) \ge \alpha(1-\alpha)|\mathrm{Tr}(\rho[A, B])|^2\,.$$

One notes that in the special case of $\alpha = 1/2$, it was shown by Luo [12]. The inequality is a refinement of Heisenberg's uncertainty relation (1.3) in the sense of $0 \le I_{\rho,\alpha}(A) \le U_{\rho,\alpha}(A) \le V_\rho(A)$.

On the other hand, Furuichi and Yanagi gave a Schrödinger-type uncertainty relation by using the correlation measure and $U_{\rho,\alpha}$: for $\alpha \in [1/2, 1]$, $\rho \in D_n^1$ and $A, B \in M_{n,sa}$,

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq 4\alpha(1-\alpha)|\mathrm{Corr}_{\rho,\alpha}(A, B)|^2 . \tag{2.5}$$

In the case of $\alpha = 1/2$, the inequality (2.5) reduces to the result (1.7) of Furuichi [2], and thus (2.5) is a refinement of Schrödinger uncertainty relation.

To give the further generalized trace inequality, Furuichi [3] introduced the $(f, g)$-skew information associated to operator monotone functions $f$ and $g$. In [10], confining to the monotone pair $(f, g)$, which are compatible in logarithmic increase, we have generalized the uncertainty relation (see (2.10) below). The purpose here is to get a Schrödinger-type uncertainty relation for the monotone pair skew information.

Let us recall the CLI monotone pair $(f, g)$-skew information from [10].

DEFINITION 2.1. Let $f(x)$ and $g(x)$ be nonnegative operator monotone functions defined on the interval $[0, 1]$. We call the pair $(f, g)$ a compatible in log-increase, monotone pair (CLI monotone pair, in short) if

(a) $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [0, 1]$,

(b) $f(x)$ and $g(x)$ are differentiable on $(0, 1)$ and

$$0 < \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} < \infty ,$$

where $F(x) = \log f(x)$ and $G(x) = \log g(x)$.

Notice that two increasing or decreasing functions $f(x)$ and $g(x)$ on $(0, 1)$ satisfy the condition (a) in Definition 2.1. It can be easily shown that the pair $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha}$, $0 < \alpha < 1$, is a CLI monotone pair.

For each CLI monotone pair $(f, g)$ we define the $(f, g)$-skew information $I_{\rho,(f,g)}$ and other quantities.

DEFINITION 2.2. Let $\rho \in D_n^1$ and $(f, g)$ be a CLI monotone pair. For $A, B \in M_{n,sa}$, we define

$$\begin{aligned} I_{\rho,(f,g)}(A) &:= \frac{1}{2}\mathrm{Tr}\big(i[f(\rho), A_0]i[g(\rho), A_0]\big) \\ &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2\big) - \mathrm{Tr}\big(f(\rho)A_0 g(\rho)A_0\big) , \end{aligned} \tag{2.6}$$

$$\begin{aligned} J_{\rho,(f,g)}(A) &:= \frac{1}{2}\mathrm{Tr}\big(\{f(\rho), A_0\}\{g(\rho), A_0\}\big) \\ &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2\big) + \mathrm{Tr}\big(f(\rho)A_0 g(\rho)A_0\big) , \end{aligned} \tag{2.7}$$

$$U_{\rho,(f,g)}(A) := \sqrt{I_{\rho,(f,g)}(A)J_{\rho,(f,g)}(A)} . \tag{2.8}$$

By the condition (a) in Definition 2.1, $I_{\rho,(f,g)}(A) \geq 0$ for $A \in M_{n,sa}$ (for the proof, see (4.3) below). It holds that for $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha}$, $0 < \alpha < 1$, $I_{\rho,(f,g)} =$

$I_{\rho,\alpha}$, $J_{\rho,(f,g)} = J_{\rho,\alpha}$ and $U_{\rho,(f,g)} = U_{\rho,\alpha}$. For each CLI monotone pair $(f, g)$ we let

$$\beta_{(f,g)} := \min\left\{\frac{m}{(1+m)^2}, \frac{M}{(1+M)^2}\right\}, \tag{2.9}$$

where $m = \inf_{0<x<1} G'(x)/F'(x)$ and $M = \sup_{0<x<1} G'(x)/F'(x)$. In [10], we have generalized the uncertainty relation (2.4) as follows:

$$U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) \geq \beta_{(f,g)}|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)|^2, \quad A, B \in M_{n,sa}. \tag{2.10}$$

In order to get a Schrödinger-type uncertainty relation, we also introduce the $(f, g)$-correlation measure $\mathrm{Corr}_{\rho,(f,g)}(A, B)$ by

$$\mathrm{Corr}_{\rho,(f,g)}(A, B) := \mathrm{Tr}(f(\rho)g(\rho)A^*B) - \mathrm{Tr}(f(\rho)A^*g(\rho)B)$$

for any $A, B \in M_n$. Note that $I_{\rho,(f,g)}(A) = \mathrm{Corr}_{\rho,(f,g)}(A, A)$ and $\mathrm{Corr}_{\rho,(f,g)}(A, B) = \mathrm{Corr}_{\rho,(f,g)}(A_0, B_0)$ for $A, B \in M_{n,sa}$.

Now we state our main result. This is a generalization of the inequality (2.5) (see Remark 2.4 below).

THEOREM 2.3. *Let $(f, g)$ be a CLI monotone pair satisfying*

$$g(x),\ f(x)/g(x) \text{ are increasing on } (0,1). \tag{2.11}$$

*For any $\rho \in D_n^1$, the following inequality*

$$U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) \geq 4\beta_{(f,g)}|\mathrm{Corr}_{\rho,(f,g)}(A, B)|^2, \quad A, B \in M_{n,sa} \tag{2.12}$$

*holds.*

The proof of this theorem is given in Section 4. Before going on, we consider some examples.

REMARK 2.4. (a) Let $f(x) = x^\alpha$ and $g(x) = x^\gamma$, $0 < \gamma \leq \alpha < 1$ on $[0, 1]$. The pair $(f, g)$ is a CLI monotone pair satisfying (2.11) and $\beta_{(f,g)} = \alpha\gamma/(\alpha+\gamma)^2$. In particular, if $\gamma = 1-\alpha$ $(1/2 \leq \alpha < 1)$, then $\beta_{(f,g)} = \alpha(1-\alpha)$ and the relation (2.12) reduces to (2.5) (Furuichi and Yanagi's result of [6]).

(b) Let $f(x) = g(x) = (x^\alpha + x^{1-\alpha})/2$, $\alpha \in (0, 1)$ on $[0, 1]$. The pair $(f, g)$ is a CLI monotone pair satisfying (2.11) and $\beta_{(f,g)} = 1/4$.

(c) Let $f(x) = \left(\frac{x}{1+x}\right)^\alpha$ and $g(x) = \left(\frac{x}{1+x}\right)^\gamma$, $0 < \gamma \leq \alpha < 1$ on $[0, 1]$. Then $f$ and $g$ are operator monotone functions (see [5]). The pair $(f, g)$ is a CLI monotone pair satisfying (2.11) and $\beta_{(f,g)} = \alpha\gamma/(\alpha+\gamma)^2$.

In the following remark, we show that the two uncertainty relations (2.10) and (2.12) are not compatible.

REMARK 2.5. Notice that there is no general ordering between $|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)|^2$ in (2.10) and $4|\mathrm{Corr}_{\rho,(f,g)}(A, B)|^2$ in (2.12). We take $f(x) = g(x) = \sqrt{x}$,

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This example was introduced in [2, Remark 2]. Then we have $|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)|^2 = 0$ and $4|\mathrm{Corr}_{\rho,(f,g)}(A,B)|^2 = (2-\sqrt{3})^2$. On the other hand, we take $f(x) = x^{2/3}$, $g(x) = x^{1/3}$,

$$\rho = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can see this example in [3, Counter example 4.1]. Then we have $|\mathrm{Tr}\big(f(\rho)g(\rho)[A,B]\big)|^2 = 1$ and $4|\mathrm{Corr}_{\rho,(f,g)}(A,B)|^2 = (2-3^{2/3}+3^{1/3})^2/4 \approx 0.46387$. Therefore we can not conclude that one of the inequalities in (2.10) and (2.12) is stronger than the other.

**3. Monotone pair skew informations and metric adjusted skew informations.** In this section, we investigate the relation between a monotone pair skew information and a metric adjusted skew information. Our aim is to show that some of monotone pair skew informations can be shown as metric adjusted skew informations. Since a metric adjusted skew information is convex with respect to the density matrix, we then show the convexity of the monotone pair skew information $I_{\rho,(f,g)}(A)$ for $\rho$ in some cases.

We introduce the notion of quantum Fisher information (see [8] and references therein). We denote by $\mathcal{F}_{op}$ the set of functions $h : (0,\infty) \to (0,\infty)$ such that

(i) $h$ is operator monotone, i.e., $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$ on $M_n$,
(ii) $h(x) = xh(x^{-1})$ for all $x > 0$,
(iii) $h(1) = 1$.

We have the following examples as elements of $\mathcal{F}_{op}$ (see [7, 8]):

$$h_{RLD}(x) = \frac{2x}{x+1}, \quad h_{SLD}(x) = \frac{1+x}{2}, \quad h_{BKM}(x) = \frac{x-1}{\log x},$$

$$h_{WY}(x) = \Big(\frac{1+\sqrt{x}}{2}\Big)^2, \quad h_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)}, \quad \alpha \in (0,1).$$

For $h \in \mathcal{F}_{op}$, we define $h(0) = \lim_{x\to 0} h(x)$. We say that $h$ is regular if $h(0) > 0$ and non-regular if $h(0) = 0$. Among the above examples, $h_{WYD}$, $h_{WY}$, and $h_{SLD}$ are regular and $h_{RLD}$ and $h_{BKM}$ are non-regular.

The Morozova-Chentsov function $c$ associated with $h \in \mathcal{F}_{op}$ is given by

$$c(x,y) = \frac{1}{yh(xy^{-1})} \quad x, y > 0. \tag{3.1}$$

In [9], for a Morozova-Chentsov function $c$ associated with a regular monotone function $h$, the corresponding metric adjusted skew information is defined by

$$I_\rho^c(A) := \frac{h(0)}{2}\mathrm{Tr}\big(i[\rho,A]c(L_\rho,R_\rho)i[\rho,A]\big), \quad A \in M_{n,sa}, \tag{3.2}$$

where $L_\rho$ and $R_\rho$ are respectively the left and right multiplication operators by $\rho$: $L_\rho(A) = \rho A$ and $R_\rho(A) = A\rho$, and they are positive definite and mutually commuting. The Wigner-Yanase skew information $I_\rho$ is the metric adjusted skew information associated with $h_{WY}$ and the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}$ is the metric adjusted skew information associated with $h_{WYD}$. The following result was shown by Hansen.

THEOREM 3.1. ([9, Theorem 3.7]) *Let $c$ be a Morozova-Chentsov function associated with a regular monotone function $h$. For each $A \in M_{n,sa}$ the metric adjusted skew information $I_\rho^c(A)$ defined in* (3.2) *is a convex function of $\rho$.*

Cai and Hansen [1] introduced a metric adjusted skew information associated with a non-regular monotone metric by setting

$$I_\rho^c(A) := \mathrm{Tr}\big(i[\rho, A]c(L_\rho, R_\rho)i[\rho, A]\big), \quad A \in M_{n,sa}. \tag{3.3}$$

This type of metric adjusted skew information is unbounded and can no longer be extended from the state manifold to the state space. The following was proven by Cai and Hansen.

THEOREM 3.2. ([1, Theorem 5.1]) *Let $h$ be a non-regular function in $\mathcal{F}_{op}$ with corresponding Morozova-Chentsov function $c$. For each $A \in M_{n,sa}$ the metric adjusted skew information $I_\rho^c(A)$ defined in* (3.3) *is a convex function of $\rho$.*

We would like to find the relation between the $(f, g)$-skew information and the metric adjusted skew information. Consider the inner product $\langle\cdot,\cdot\rangle$ on $M_n \times M_n$ given by $\langle A, B\rangle = \mathrm{Tr}(A^*B)$, $A, B \in M_n$. For a CLI monotone pair $(f, g)$, the $(f, g)$-skew information in (4.3) can be expressed as

$$\begin{aligned}
I_{\rho,(f,g)}(A) &= \frac{1}{2}\mathrm{Tr}\big(i[f(\rho), A]i[g(\rho), A]\big)\\
&= \frac{1}{2}\langle i[f(\rho), A], i[g(\rho), A]\rangle\\
&= \frac{1}{2}\langle i(f(L_\rho) - f(R_\rho))(A), i(g(L_\rho) - g(R_\rho))(A)\rangle\\
&= \frac{1}{2}\langle A, \big(i(f(L_\rho) - f(R_\rho))\big)^*\big(i(g(L_\rho) - g(R_\rho))\big)(A)\rangle\\
&= \frac{1}{2}\langle i[\rho, A], \big((i(L_\rho - R_\rho))^{-1}\big)^*\big(i(f(L_\rho) - f(R_\rho))\big)^*\\
&\qquad \big(i(g(L_\rho - g(R_\rho))\big)\big(i(L_\rho - R_\rho)\big)^{-1}(i[\rho, A])\rangle\\
&= \frac{1}{2}\mathrm{Tr}\big(i[\rho, A]c(L_\rho, R_\rho)(i[\rho, A])\big),
\end{aligned} \tag{3.4}$$

where

$$c(x, y) = \frac{(f(x) - f(y))(g(x) - g(y))}{(x - y)^2}. \tag{3.5}$$

Now we want to find, if possible, a function $h \in \mathcal{F}_{op}$ satisfying

$$h(xy^{-1}) = \frac{1}{yc(x, y)} = \frac{y\big(x/y - 1\big)^2}{f(y)g(y)\big(f(x)/f(y) - 1\big)\big(g(x)/g(y) - 1\big)}. \tag{3.6}$$

In the following we consider some examples where the monotone pair $(f, g)$ is related to a function $h \in \mathcal{F}_{op}$ by the relation (3.6). In these cases therefore, by Theorem 3.1 and Theorem 3.2, the corresponding $(f, g)$-skew information $I_{\rho,(f,g)}$ is convex in state variables $\rho$.

EXAMPLE 3.3. If $f$ and $g$ satisfy the conditions

$$f(y)g(y)=y\,,\quad \frac{f(x)}{f(y)}=l_1(xy^{-1})\,,\quad \frac{g(x)}{g(y)}=l_2(xy^{-1})$$

for some functions $l_1, l_2$, then it is not hard to see that $f(x) = ax^\alpha$ and $g(x) = a^{-1}x^{1-\alpha}$ for some $a > 0$ and $0 < \alpha < 1$. In this case, the $(f, g)$-skew information is the Wigner-Yanase-Dyson skew information and equals the metric adjusted skew information related to the regular monotone function $h_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}$. Indeed, by (3.2) and (3.4) we have

$$\begin{aligned} I_{\rho,(f,g)}(A) &= \frac{1}{2}\mathrm{Tr}\big(i[f(\rho), A]i[g(\rho), A]\big) \\ &= \frac{h_{WYD}(0)}{2}\mathrm{Tr}\big(i[\rho, A]c(L_\rho, R_\rho)(i[\rho, A])\big)\,, \end{aligned}$$

where $c(x, y) = \frac{(x^\alpha-y^\alpha)(x^{1-\alpha}-y^{1-\alpha})}{\alpha(1-\alpha)(x-y)^2}$.

EXAMPLE 3.4. For $1 \le a$ and $e^{-a} < k < 1$, let $kI \le \rho \in D_n^1$ and $f(x) = a + \log x$ and $g(x) = 2x$ on $[k, 1]$. The functions $f$ and $g$ satisfy conditions in Definition 2.1 and (2.11). The $(f, g)$-skew information equals the metric adjusted skew information $I_\rho^c$ for the Morozova-Chentsov function $c$ related to the non-regular monotone function $h_{BKM}(x) = (x-1)/\log x$. Indeed, by (3.3) and (3.4) we have

$$\begin{aligned} I_{\rho,(f,g)}(A) &= \frac{1}{2}\mathrm{Tr}\big(i[f(\rho), A]i[g(\rho), A]\big) \\ &= \mathrm{Tr}\big(i[\rho, A]c(L_\rho, R_\rho)(i[\rho, A])\big)\,, \end{aligned}$$

where $c(x, y) = (\log x - \log y)/(x - y)$.

EXAMPLE 3.5. For $0 < 3/2a < k < 1$, let $kI \le \rho \in D_n^1$ and $f(x) = x^2$ and $g(x) = a - 1/x$ on $[k, 1]$. The functions $f$ and $g$ satisfy conditions in Definition 2.1 and (2.11). The $(f, g)$-skew information equals the metric adjusted skew information $I_\rho^c$ for the Morozova-Chentsov function $c$ related to the non-regular monotone function $h_{RLD}(x) = 2x/(x + 1)$. Indeed, by (3.3) and (3.4) we have

$$\begin{aligned} I_{\rho,(f,g)}(A) &= \frac{1}{2}\mathrm{Tr}\big(i[f(\rho), A]i[g(\rho), A]\big) \\ &= \mathrm{Tr}\big(i[\rho, A]c(L_\rho, R_\rho)(i[\rho, A])\big)\,, \end{aligned}$$

where $c(x, y) = (x + y)/2xy$.

**4. Proof of Theorem 2.3.** In this section we give a proof of Theorem 2.3. We adopt a similar method used in [10].

Let $\rho = \sum_l \lambda_l|\phi_l\rangle\langle\phi_l| \in D_n^1$, $0 < \lambda_1 \le \cdots \le \lambda_n \le 1$, where $\{\phi_l\}_{l=1}^n$ is an orthonormal set in $\boldsymbol{C}^n$ and $\mathrm{Tr}(\rho) = \sum_l \lambda_l = 1$. Let $(f, g)$ be a CLI monotone pair satisfying (2.11). By a simple calculation, we have

$$
\begin{aligned}
\mathrm{Tr}(f(\rho)g(\rho)A_0B_0) &= \sum_l f(\lambda_l)g(\lambda_l)\langle A_0\phi_l, B_0\phi_l\rangle \\
&= \sum_{l,m} f(\lambda_l)g(\lambda_l)\langle \phi_m, B_0\phi_l\rangle\langle A_0\phi_l, \phi_m\rangle \\
&= \sum_{l<m} \big(f(\lambda_l)g(\lambda_l)a_{lm}b_{ml} + f(\lambda_m)g(\lambda_m)a_{ml}b_{lm}\big) \\
&\quad + \sum_l f(\lambda_l)g(\lambda_l)a_{ll}b_{ll}
\end{aligned} \tag{4.1}
$$

and

$$
\begin{aligned}
\mathrm{Tr}(f(\rho)A_0g(\rho)B_0) &= \sum_{l,m} f(\lambda_l)g(\lambda_m)\langle \phi_m, B_0\phi_l\rangle\langle A_0\phi_l, \phi_m\rangle \\
&= \sum_{l<m} \big(f(\lambda_l)g(\lambda_m)a_{lm}b_{ml} + f(\lambda_m)g(\lambda_l)a_{ml}b_{lm}\big) \\
&\quad + \sum_l f(\lambda_l)g(\lambda_l)a_{ll}b_{ll}
\end{aligned} \tag{4.2}
$$

for any $A, B \in M_{n,sa}$, where $a_{ml} = \langle \phi_m, A_0\phi_l\rangle$ and $b_{ml} = \langle \phi_m, B_0\phi_l\rangle$. From (4.1) and (4.2) we get

$$
\begin{aligned}
I_{\rho,(f,g)}(A) &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2\big) - \mathrm{Tr}\big(f(\rho)A_0g(\rho)A_0\big) \\
&= \sum_{l<m} \big(f(\lambda_m) - f(\lambda_l)\big)\big(g(\lambda_m) - g(\lambda_l)\big)|a_{ml}|^2 ,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
J_{\rho,(f,g)}(A) &= \mathrm{Tr}\big(f(\rho)g(\rho)A_0^2\big) + \mathrm{Tr}\big(f(\rho)A_0g(\rho)A_0\big) \\
&\geq \sum_{l<m} \big(f(\lambda_m) + f(\lambda_l)\big)\big(g(\lambda_m) + g(\lambda_l)\big)|a_{ml}|^2 ,
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
\mathrm{Corr}_{\rho,(f,g)}(A, B) &= \mathrm{Corr}_{\rho,(f,g)}(A_0, B_0) \\
&= \mathrm{Tr}\big(f(\rho)g(\rho)A_0B_0\big) - \mathrm{Tr}\big(f(\rho)A_0g(\rho)B_0\big) \\
&= \sum_{l\neq m} f(\lambda_m)\big(g(\lambda_m) - g(\lambda_l)\big)a_{ml}b_{lm} .
\end{aligned} \tag{4.5}
$$

To prove Theorem 2.3, we use the following lower bound of a function coming from a CLI monotone pair $(f, g)$:

$$
\min_{x,y\in[0,1]} L(x, y) \geq 4\beta_{(f,g)} , \tag{4.6}
$$

where $L(x, y) = \frac{(f(x)^2-f(y)^2)(g(x)^2-g(y)^2)}{(f(x)g(x)-f(y)g(y))^2}$ and $\beta_{(f,g)}$ is defined in (2.9). See the proof of Proposition 3.1 in [10].

PROOF OF THEOREM 2.3. By (2.11) and (4.5), we have

$$|\mathrm{Corr}_{\rho,(f,g)}(A,B)| \le \sum_{l<m} \big(f(\lambda_m)+f(\lambda_l)\big)\big(g(\lambda_m)-g(\lambda_l)\big)|a_{lm}||b_{lm}|$$
$$\le \sum_{l<m} \big(f(\lambda_m)g(\lambda_m)-f(\lambda_l)g(\lambda_l)\big)|a_{lm}||b_{lm}|$$

for any $A, B \in M_{n,sa}$. It follows from (4.6) that

$$4\beta_{(f,g)}|\mathrm{Corr}_{\rho,(f,g)}(A,B)|^2 \le 4\beta_{(f,g)}\bigg(\sum_{l<m} |f(\lambda_m)g(\lambda_m)-f(\lambda_l)g(\lambda_l)\,||a_{lm}b_{lm}|\bigg)^2$$
$$\le \bigg(\sum_{l<m} \sqrt{\big(f(\lambda_m)^2-f(\lambda_l)^2\big)\big(g(\lambda_m)^2-g(\lambda_l)^2\big)}|a_{lm}b_{lm}|\bigg)^2 .$$

By Schwarz inequality, we have

$$\begin{aligned} 4\beta_{(f,g)}|\mathrm{Corr}_{\rho,(f,g)}(A,B)|^2 &\le \sum_{l<m} \big(f(\lambda_m)-f(\lambda_l)\big)\big(g(\lambda_m)-g(\lambda_l)\big)|a_{ml}|^2 \\ &\quad\times \sum_{l<m} \big(f(\lambda_m)+f(\lambda_l)\big)\big(g(\lambda_m)+g(\lambda_l)\big)|b_{lm}|^2 \\ &\le I_{\rho,(f,g)}(A)J_{\rho,(f,g)}(B) \end{aligned} \tag{4.7}$$

and similarly

$$4\beta_{(f,g)}|\mathrm{Corr}_{\rho,(f,g)}(A,B)|^2 \le I_{\rho,(f,g)}(B)J_{\rho,(f,g)}(A) .$$

Hence by multiplying the above two inequalities, we have

$$4\beta_{(f,g)}|\mathrm{Corr}_{\rho,(f,g)}(A,B)|^2 \le U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) . \qquad \square$$

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