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THE FUNCTOR OF TORIC VARIETIES ASSOCIATED WITH WEYL CHAMBERS AND LOSEV-MANIN MODULI SPACES

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Abstract. A root system R of rank n defines an n-dimensional smooth projective toric variety $X(R)$ associated with its fan of Weyl chambers. We give a simple description of the functor of $X(R)$ in terms of the root system R and apply this result in the case of root systems of type A to give a new proof of the fact that the toric variety $X(A_n)$ is the fine moduli space $\overline{L}_{n+1}$ of stable $(n+1)$-pointed chains of projective lines investigated by Losev and Manin.

Introduction. Let $R \subset E$ be a root system of rank n in an n-dimensional Euclidean space E and let $M(R) \subset E$ be its root lattice. The toric variety $X(R)$ corresponding to the root system R is the smooth projective toric variety associated with the fan of Weyl chambers $\Sigma(R)$ in the dual space E^* with respect to the lattice $N(R) \subset E^*$ dual to the root lattice $M(R)$. It was shown by Klyachko in [17] (see also [18]) that if G is a semisimple algebraic group corresponding to R and B is a Borel subgroup in G then the toric variety $X(R)$ can be characterised as the closure of a general orbit of a maximal torus $T \subset G$ acting on the flag variety G/B. The natural representation of the Weyl group $W(R)$ on the cohomology of $X(R)$ has been studied by Procesi [21], Dolgachev-Lunts [11], and Stembridge [23].

The present paper is inspired by a paper of Losev and Manin [20], in which fine moduli spaces $\overline{L}_n$ of stable n-pointed chains of projective lines were constructed and it was observed that the Losev-Manin moduli space $\overline{L}_n$ is the toric variety associated with the polytope called the $(n-1)$-dimensional *permutohedron* studied by Kapranov [16, (4.3)]. These toric varieties form the A_n-family of the toric varieties associated with root systems: the Losev-Manin moduli space $\overline{L}_{n+1}$ coincides with the toric variety $X(A_n)$. Moreover, in [20] it was shown that the homology groups of $\overline{L}_{n+1}$ $(n \geq 0)$ together with the natural action of the Weyl group $W(A_n) \cong S_{n+1}$ are closely related to the so called *commutativity equations* (see also the recent paper of Shadrin and Zvonkine [22]). We remark that these varieties are special examples of toric varieties obtained as equivariant blowups of $\boldsymbol{P}^n$ considered recently by Bloch and Kreimer [5, §3].

The Losev-Manin moduli space $\overline{L}_n$ is an equivariant compactification of a maximal torus $T \cong (\boldsymbol{C}^*)^n/\boldsymbol{C}^* \subset \mathrm{PGL}(n, \boldsymbol{C})$, where the torus T can be identified as the moduli space of n points in $\boldsymbol{P}^1 \setminus \{0, \infty\}$ up to automorphisms of $\boldsymbol{P}^1$ fixing 0 and ∞. The boundary components of $\overline{L}_n$ parametrise certain types of n-pointed reducible rational curves. There are

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some similarities and relations between the Losev-Manin moduli spaces and the well-known Grothendieck-Knudsen moduli spaces. The Losev-Manin moduli spaces $\overline{L}_n$ parametrise isomorphism classes of chains of projective lines with two poles and n marked points that may coincide, whereas the Grothendieck-Knudsen moduli spaces $\overline{M}_{0,n+2}$ parametrise isomorphism classes of trees of projective lines with $n+2$ marked points that may not coincide. They are related by surjective birational morphisms $\overline{M}_{0,n+2} \to \overline{L}_n$ dependent on the choice of two different elements $i, j \in \{1, \ldots, n+2\}$. Both form a particular case of moduli spaces of weighted pointed stable curves as introduced by Hassett [15].

Our main objective was to generalise the result of Losev and Manin to other root systems R. This problem was mentioned by Losev and Manin in the introduction of their paper [20]. In [4] we present results in this direction for R a classical root system.

To investigate interpretations of the toric varieties $X(R)$ associated with root systems as moduli spaces, it is natural first to investigate their functors of points. The functor of toric varieties in general was described by Mumford in [1, Ch. I]; a different description was proposed by Cox [8] for smooth toric varieties. In the present paper we propose another description of the functor of the toric varieties $X(R)$ for root systems R, which is based on projection maps $X(R) \to \boldsymbol{P}^1$ and done with a view toward interpretations of these varieties as moduli spaces of pointed trees of projective lines.

OUTLINE OF THE PAPER. In the first section of this paper, we derive some general results about the toric varieties $X(R)$ associated with arbitrary root systems R. Important are functorial properties with respect to maps of root systems. For example, any pair of opposite roots $\{\pm\alpha\} \subset R$, i.e. a root subsystem of type A_1, gives rise to a projection $X(R) \to X(A_1) \cong \boldsymbol{P}^1$. Morphisms constructed this way appear in many variants in the following. As a main result, we give a description of the functor of the toric varieties $X(R)$ in terms of the root system R. We use the property of the spaces $X(R)$ that morphisms $Y \to X(R)$ are uniquely determined by their compositions with all the projection maps $X(R) \to \boldsymbol{P}^1$ given by the root subsystems $\{\pm\alpha\} \subset R$ of type A_1. Further, the relations between these morphisms are given by the root subsystems of type A_2 in R.

For the rest of the present paper, we are concerned with the toric varieties $X(A_n)$ associated with root systems of type A and their interpretation as Losev-Manin moduli spaces $\overline{L}_{n+1}$.

We consider the toric varieties $X(A_n)$ in Section 2. We review some results concerning the (co)homology of $X(A_n)$, we give a basis for the homology and, in a simple way, derive the relations between torus invariant cycles used in [20, Section 3]. Further, we comment on primitive collections and relations of the toric variety $X(A_n)$ and apply this to show that the anticanonical class of $X(A_n)$ is a semiample divisor. This implies that $X(A_n)$ is an almost Fano variety. The anticanonical divisor defines a birational toric morphism to the Gorenstein toric Fano variety $\boldsymbol{P}_{\Delta(A_n)}$ corresponding to the reflexive polytope $\Delta(A_n) =$ (convex hull of all roots of A_n).

In Section 3, we give a new proof of the fact that the toric varieties $X(A_n)$ are the fine moduli spaces $\overline{L}_{n+1}$ of $(n+1)$-pointed chains of projective lines introduced by Losev and Manin. We use the functorial properties of toric varieties associated with root systems developed in Subsection 1.2 to construct the universal curve $X(A_{n+1}) \to X(A_n)$ in Subsection 3.2. Our result about the functor of $X(R)$ (Subsection 1.3) is used in the case of root systems of type A in Subsection 3.3 to show that the functor of $X(A_n)$ is isomorphic to the moduli functor of $(n+1)$-pointed chains of projective lines. This provides an alternative proof of the fact that this moduli problem admits a fine moduli space $\overline{L}_{n+1}$ and furthermore shows that it coincides with the toric variety $X(A_n)$. We will see that the data describing morphisms $Y \to X(A_n)$ correspond in a natural way to parameters in equations describing stable $(n+1)$-pointed chains of projective lines over Y embedded in $(\boldsymbol{P}^1_Y)^{n+1}$.

1. Toric varieties associated with root systems and their functor.

1.1. The toric variety $X(R)$. Let R be a (reduced and crystallographic) root system in a Euclidean space E. With R we associate a toric variety $X(R)$ ([21], [11]).

Let $M(R)$ be the root lattice of R, i.e. the lattice in E generated by the roots of R, and let $N(R)$ be the lattice dual to $M(R)$. For any set of simple roots S, we have a cone $\sigma_S := S^\vee = \{v \in N(R)_{\boldsymbol{Q}}\,;\, \langle u, v\rangle \geq 0 \textit{ for all } u \in S\}$ in the vector space $N(R)_{\boldsymbol{Q}}$, the (closed) Weyl chamber corresponding to S.

DEFINITION 1.1. We define $\Sigma(R)$ to be the fan in the lattice $N(R)$ that consists of the Weyl chambers of the root system R and all their faces. Let $X(R)$ be the toric variety associated with the fan $\Sigma(R)$.

For $v \in N(R)_{\boldsymbol{Q}}$ let $\sigma_v \in \Sigma(R)$ be the cone minimal in $\Sigma(R)$ containing v. Equivalently, σ_v is the cone dual to the roots in $v^\vee = \{u \in M(R)_{\boldsymbol{Q}}\,;\, \langle u, v\rangle \geq 0\}$, i.e. $\sigma_v = (v^\vee \cap R)^\vee$. In particular, for a general choice of v this is the cone σ_S dual to the set of simple roots $S \subset R$ of the set of positive roots $v^\vee \cap R$ defined by v, i.e. the Weyl chamber for S. The Weyl chambers cover $N(R)_{\boldsymbol{Q}}$, so the fan $\Sigma(R)$ is complete. Note that each set of simple roots forms a basis of the root lattice $M(R)$ and $\sigma_S^\vee \cap M(R) = \langle S\rangle$ is the submonoid of $M(R)$ generated by S. $X(R)$ is covered by the open subvarieties $U_S := \operatorname{Spec} \boldsymbol{Z}[\sigma_S^\vee \cap M(R)] = \operatorname{Spec} \boldsymbol{Z}[\langle S\rangle] \cong \boldsymbol{A}^{\dim M(R)}$ for all the different sets of simple roots S.

The toric variety $X(R)$ is smooth and projective. It carries in a natural way the action of the Weyl group $W(R)$ of the root system R. The Weyl group permutes the sets of simple roots and this way it acts simply transitive on the set of Weyl chambers. The corresponding action on $X(R)$ permutes the open sets U_S, it is a simply transitive action on the set of torus fixed points of $X(R)$.

The root lattice $M(R)$ of R is the lattice of characters of the dense torus $T(R)$ in $X(R)$. This way, any element $u \in M(R)$ determines a character x^u of $T(R)$, i.e. a rational function on $X(R)$.

EXAMPLE 1.2. The toric variety $X(A_1)$ is isomorphic to $\boldsymbol{P}^1$.

REMARK 1.3. For two root systems R_1, R_2 there is an isomorphism of fans $\Sigma(R_1 \times R_2) \cong \Sigma(R_1) \times \Sigma(R_2)$ and thus an isomorphism of toric varieties $X(R_1 \times R_2) \cong X(R_1) \times X(R_2)$.

1.2. Morphisms for maps of root systems and closures of torus orbits. First, we show that maps between root systems coming from linear maps of the ambient vector spaces induce toric morphisms of the associated toric varieties.

PROPOSITION 1.4. *Let R, R' be root systems in Euclidean spaces E, E'. Then a map of vector spaces $\mu\colon E' \to E$ such that $\mu(R') \subset \{a\alpha\,;\ \alpha \in R, a \in \boldsymbol{Z}\}$ induces a toric morphism of the associated toric varieties $X(\mu)\colon X(R) \to X(R')$.*

PROOF. The map of vector spaces $\mu\colon E' \to E$ induces a map of the root lattices $\mu\colon M(R') \to M(R)$ because $\mu(R') \subset \{a\alpha\,;\alpha \in R, a \in \boldsymbol{Z}\}$. Let $\nu\colon N(R) \to N(R')$ be the dual map of the dual lattices.

We have to show that each cone of $\Sigma(R)$ is mapped by $\nu : N(R)_{\boldsymbol{Q}} \to N(R')_{\boldsymbol{Q}}$ into a cone of $\Sigma(R')$. Let $v \in N(R)_{\boldsymbol{Q}}$, we show that $\nu(\sigma_v) \subseteq \sigma_{\nu(v)}$ (where as above $\sigma_v = (v^\vee \cap R)^\vee$ is the cone minimal in $\Sigma(R)$ containing v; in the same way the cone $\sigma_{\nu(v)}$ of $\Sigma(R')$ is defined). It suffices to show that $\mu(\nu(v)^\vee \cap R') \subseteq \langle v^\vee \cap R\rangle$. This is true, since $\mu(R') \subset \{a\alpha\,;\alpha \in R, a \in \boldsymbol{Z}\}$ by assumption and $\mu(\nu(v)^\vee) \subseteq v^\vee$ because $\langle u', \nu(v)\rangle = \langle \mu(u'), v\rangle$ for any $u' \in M(R')_{\boldsymbol{Q}}$. □

We have two special cases:

(1) *Root subsystems induce proper surjective morphisms.* Let $R \subset E$ be a root system and $R' \subset E'$ a root system in a subspace $E' \subseteq E$ such that $R' \subseteq R$. Then $\mu\colon M(R') \to M(R)$ is injective, its dual $\nu\colon N(R) \to N(R')$ is surjective and we have a proper surjective morphism $X(R) \to X(R')$ which locally is given by inclusions of coordinate rings $\boldsymbol{Z}[\sigma_{\nu(v)}^\vee \cap M(R')] \to \boldsymbol{Z}[\sigma_v^\vee \cap M(R)]$.

(2) *Projections of root systems induce closed embeddings.* Let $R \subset E$, $R' \subset E'$ be root systems and $\mu\colon E' \to E$ a homomorphism of vector spaces such that $R \subseteq \mu(R') \subset \{a\alpha\,;\alpha \in R, a \in \boldsymbol{Z}\}$. Then $\mu\colon M(R') \to M(R)$ is surjective and for $v \in N(R)_{\boldsymbol{Q}}$ induces a surjection $\langle \nu(v)^\vee \cap R'\rangle \to \langle v^\vee \cap R\rangle$, the map $\nu\colon N(R) \to N(R')$ is injective and $\nu^{-1}(\sigma_{\nu(v)}) = \sigma_v$ for $v \in N(R)_{\boldsymbol{Q}}$. We have a closed embedding $X(R) \to X(R')$ which locally is given by surjective maps of coordinate rings $\boldsymbol{Z}[\sigma_{\nu(v)}^\vee \cap M(R')] \to \boldsymbol{Z}[\sigma_v^\vee \cap M(R)]$.

EXAMPLE 1.5. The first case in particular occurs if the root system R' is of the form $R' = R \cap E'$, i.e. cut out by a subspace $E' \subseteq E$. Then the morphism $X(R) \to X(R')$ is locally given by inclusions of coordinate rings $\boldsymbol{Z}[\sigma^\vee \cap M(R')] \to \boldsymbol{Z}[\sigma^\vee \cap M(R)]$ for $\sigma \in \Sigma(R)$. For example consider root subsystems $\{\pm\alpha\} \subseteq R$ consisting of two opposite roots, i.e. isomorphic to A_1. Each of these gives rise to a projection $\varphi_{\{\pm\alpha\}}\colon X(R) \to X(A_1) \cong \boldsymbol{P}^1$.

EXAMPLE 1.6. For any root system R the projections $X(R) \to \boldsymbol{P}^1$ for all root subsystems $A_1 \cong R' = \{\pm\alpha\} \subseteq R$ form a morphism $X(R) \to \prod_{A_1 \cong R' \subseteq R} \boldsymbol{P}^1$. This morphism is an instance of the second case: it corresponds to the projection of root systems $\prod_{A_1 \cong R' \subseteq R} R' \to R$. (A variant of this closed embedding has been considered in [7].)

In the second case we can describe the equations for $X(R)$ in $X(R')$.

PROPOSITION 1.7. *Let $R \subset E$, $R' \subset E'$ be root systems and $\mu\colon E' \to E$ a homomorphism of vector spaces such that $R \subseteq \mu(R') \subset \{a\alpha\,;\, \alpha \in R, a \in \boldsymbol{Z}\}$. Then the image of the closed embedding $X(\mu)\colon X(R) \to X(R')$ is determined by the equations $x^{u'} = 1$ for $u' \in \ker(\mu) \cap M(R')$. Locally, the subvariety $X(R) \cap U_{S'} \subseteq U_{S'}$ for any set of simple roots S' of R' is given by the equations $\prod_i x^{\alpha_i} = \prod_j x^{\beta_j}$ for collections of simple roots $\alpha_i, \beta_j \in S'$ such that $\sum_i \alpha_i - \sum_j \beta_j \in \ker(\mu)$.*

PROOF. Let $v \in N(R)_{\boldsymbol{Q}}$ be an element in the interior of some Weyl chamber, let S be the set of simple roots of R with respect to v and let S' be the set of simple roots of R' with respect to $\nu(v)$. Then $X(\mu)^{-1}(U_{S'}) = U_S$ and the inclusion $U_S \to U_{S'}$ corresponds to the surjective map of coordinate rings $\boldsymbol{Z}[\langle S'\rangle] \to \boldsymbol{Z}[\langle S\rangle]$ given by the surjection $\langle S'\rangle \to \langle S\rangle$ determined by μ. We have $\langle S\rangle \cong \langle S'\rangle/\sim$, where $\sim$ is the equivalence relation $s_1 \sim s_2 \Leftrightarrow s_1 - s_2 \in \ker(\mu)$. Thus $\boldsymbol{Z}[\langle S\rangle] \cong \boldsymbol{Z}[\langle S'\rangle]/I$, where I is the ideal generated by $x^{s_1} - x^{s_2}$ for $s_1, s_2 \in \langle S'\rangle$ such that $s_1 - s_2 \in \ker(\mu)$. We can write s_1 (resp. s_2) as sums $s_1 = \sum_i \alpha_i$ (resp. $s_2 = \sum_j \beta_j$) of simple roots $\alpha_i, \beta_j \in S'$. □

EXAMPLE 1.8. In the case of the embedding $X(R) \to \prod_{A_1 \cong R' \subseteq R} \boldsymbol{P}^1$, the equations come from the linear relations between the roots of the root system R. This will be discussed in detail in Subsection 1.3.

Next, we consider closures of torus orbits in $X(R)$. In general, such orbit closures are again toric varieties and are in bijection with the cones of the fan (see e.g. [13, 3.1]). We will see that in the case of toric varieties associated with root systems R the orbit closures are again toric varieties associated with certain root subsystems of R.

PROPOSITION 1.9. *The closure of the torus orbit $Z \subseteq X(R)$ corresponding to a cone $\tau \in \Sigma(R)$ is isomorphic to the toric variety $X(R')$ associated with the root subsystem $R' = R \cap E' \subset E'$ of R cut out by the subspace $E' = \tau^{\perp} \subseteq E$ with root lattice $M(R') = M(R) \cap E'$.*

Let S be a set of simple roots of R such that τ is contained in the Weyl chamber σ_S and put $S' = S \cap E'$. Then S' is a set of simple roots of the root system R'. The orbit closure Z is covered by the open sets $Z \cap U_S$ for such sets of simple roots S, the closed subvariety $Z \cap U_S \subseteq U_S$ is given by the equations $x^{\alpha} = 0$ for $\alpha \in S \setminus S'$.

PROOF. Let S be a set of simple roots of R such that τ is a face of σ_S. Then $E' = \tau^{\perp}$ cuts out a face of $\langle S\rangle_{\boldsymbol{Q}_{\geq 0}}$ generated by $S' = S \cap E'$, the set S' is a set of simple roots of the root system $R' = R \cap E'$ and $M(R') = M(R) \cap E'$.

By the general theory of toric varieties (see e.g. [13, 3.1]) the orbit closure corresponding to the cone τ is a toric variety covered by the affine charts $Z \cap \operatorname{Spec} \boldsymbol{Z}[\sigma_S^{\vee} \cap M(R)] =$

Spec $\boldsymbol{Z}[\sigma_S^\vee \cap M(R) \cap \tau^\perp]$ for the maximal cones $\sigma_S \in \Sigma(R)$ such that τ is a face of σ_S. In the present case $\sigma_S^\vee \cap M(R) \cap \tau^\perp = \langle S \rangle \cap \tau^\perp = \langle S' \rangle$, and so Z is covered by the open sets $Z \cap U_S = \operatorname{Spec} \boldsymbol{Z}[\langle S' \rangle]$ and isomorphic to the toric variety $X(R')$ associated with the root system R'. The inclusion $Z \cap U_S = \operatorname{Spec} \boldsymbol{Z}[\langle S' \rangle] \subseteq \operatorname{Spec} \boldsymbol{Z}[\langle S \rangle] = U_S$ is given by the homomorphism $\boldsymbol{Z}[\langle S \rangle] \to \boldsymbol{Z}[\langle S' \rangle]$, $x^u \mapsto x^u$ if $u \in \langle S' \rangle$ and $x^u \mapsto 0$ otherwise. Thus the closed subvariety $Z \cap U_S \subseteq U_S$ is determined by the equations $x^\alpha = 0$ for $\alpha \in S \setminus S'$. □

Concerning the situation of the proposition we have two further remarks.

REMARK 1.10. The Dynkin diagram of R' is the subdiagram of the Dynkin diagram of R formed with respect to the set of simple roots S, that arises after leaving out the vertices (and adjacent edges) corresponding to the roots $\alpha \in S \setminus S'$. Usually, the root system R' will be reducible and decompose as $R' \cong \prod_i R_i$ into a number of irreducible root systems R_i corresponding to the connected components of the Dynkin diagram of R'.

REMARK 1.11. Since the fan $\Sigma(R)$ is symmetric under reflection in the origin, also $-\tau$ is a cone of $\Sigma(R)$ and, apart from the inclusion $i^+ \colon X(R') \to X(R)$ of $X(R')$ as orbit closure corresponding to τ, there is another inclusion $i^- \colon X(R') \to X(R)$ that embeds $X(R')$ as orbit closure corresponding to $-\tau$. Any such root subsystem $R' \subset R$ comes with a proper surjective morphism $X(R) \to X(R')$, the two inclusions then are sections with respect to this morphism.

Consider the particular case of a one-dimensional cone $\tau = \langle v \rangle_{\boldsymbol{Q}_{\geq 0}}$: for $\tau, -\tau$ we have the two torus invariant divisors isomorphic to $X(R')$, $R' = R \cap v^\perp$, given as the images $i^\pm(X(R')) \subseteq X(R)$ and defined by the equations $x^\alpha = 0$ for $\alpha \in R$ such that $\langle \alpha, \pm v \rangle > 0$.

1.3. The functor of $X(R)$. We will give a description of the functor of the toric variety $X(R)$ in terms of the root system R. This is done via the proper surjective toric morphisms $X(R) \to \boldsymbol{P}^1$ for root subsystems isomorphic to A_1 forming the closed embedding $X(R) \to \prod_{A_1 \cong R' \subseteq R} \boldsymbol{P}^1$ and by the use of the functor of $\boldsymbol{P}^1$.

REMARK 1.12. We recall the well known description of the functor of $\boldsymbol{P}^1$. For any scheme Y a morphism $Y \to \boldsymbol{P}^1$ is uniquely determined (with respect to chosen coordinates on $\boldsymbol{P}^1$) by the data consisting of a line bundle $\mathscr{L}$ on Y together with two sections t_-, t_+ that generate $\mathscr{L}$ up to isomorphisms of line bundles with two sections (or equivalently by a line bundle $\mathscr{L}$ together with a surjective homomorphism $\mathcal{O}_Y^{\oplus 2} \to \mathscr{L}$ up to isomorphism). The functor that associates to a scheme Y such data and to a morphism $Y' \to Y$ the map given by pull-back of line bundles and sections then is isomorphic to the functor $\operatorname{Mor}(\,\cdot\,, \boldsymbol{P}^1)$. An isomorphism is determined by the universal data consisting of the twisting sheaf together with homogeneous coordinates $(\mathcal{O}_{\boldsymbol{P}^1}(1), x_0, x_1)$ on $\boldsymbol{P}^1$ corresponding to $\mathrm{id}_{\boldsymbol{P}^1} \in \operatorname{Mor}(\boldsymbol{P}^1, \boldsymbol{P}^1)$.

Another equivalent formulation is as follows: take as data open sets $U_-, U_+ \subseteq Y$ and regular functions $f_- \in \mathcal{O}_Y(U_-)$, $f_+ \in \mathcal{O}_Y(U_+)$ such that $Y = U_- \cup U_+$, $f_- f_+ = 1$ on $U_- \cap U_+$ and $\{f_- \neq 0\} = U_- \cap U_+ = \{f_+ \neq 0\}$. Such data (U_-, U_+, f_-, f_+) corresponds to a line bundle with two generating sections $(\mathscr{L}, t_-, t_+)$ uniquely determined up to isomorphism by $U_- = \{t_+ \neq 0\}$, $U_+ = \{t_- \neq 0\}$ and $f_- = t_-/t_+$ on U_-, $f_+ = t_+/t_-$ on U_+.

By Subsection 1.2, root subsystems of a root system R isomorphic to A_1 define toric morphisms $X(R) \to \boldsymbol{P}^1$. These morphisms can also be described in terms of the preceding remark.

EXAMPLE 1.13. Any root subsystem of R isomorphic to A_1, i.e. an unordered pair of opposite roots $\{\pm\alpha\}$ in R, defines a morphism $X(R) \to \boldsymbol{P}^1$. The data $\{(U_\alpha, f_\alpha), (U_{-\alpha}, f_{-\alpha})\}$ are defined in terms of the rational functions $x^\alpha, x^{-\alpha}$ associated with the roots $\alpha, -\alpha$: let U_α be the open subset of $X(R)$ where x^α is regular, $f_\alpha := x^\alpha|_{U_\alpha}$ and the same for $-\alpha$. The rational functions $x^\alpha, x^{-\alpha}$ have no common zeros or poles because any half-space in $M(R)$ contains at least one of the roots $\alpha, -\alpha$.

We will denote these morphisms by

$$\varphi_{\{\pm\alpha\}}\colon\ X(R) \to \boldsymbol{P}^1_{\{\pm\alpha\}}.$$

We consider $\boldsymbol{P}^1_{\{\pm\alpha\}}$ as a copy of $\boldsymbol{P}^1$ with chosen homogeneous coordinates $z_\alpha, z_{-\alpha}$. The toric morphism $\varphi_{\{\pm\alpha\}}$ corresponds to a map of lattices $\boldsymbol{Z}u_\alpha \to M(R)$, $u_\alpha \mapsto \alpha$ and hence to a homomorphism of algebras $\boldsymbol{Z}[y_\alpha^{\pm}] \to \boldsymbol{Z}[M(R)]$, $y_\alpha \mapsto x^\alpha$, where $y_\alpha = z_\alpha/z_{-\alpha}$, i.e. the pull-back of the rational function $y_\alpha = z_\alpha/z_{-\alpha}$ on $\boldsymbol{P}^1$ via $\varphi_{\{\pm\alpha\}}$ is the rational function x^α on $X(R)$.

Also by the preceding subsection, the collection of these morphisms defines a closed embedding

$$\varphi\colon\ X(R) \to \prod_{A_1 \cong R' \subseteq R} \boldsymbol{P}^1_{R'}.$$

We choose a set of positive roots R^+ of R. This toric morphism corresponds to a surjective map of lattices $\mu\colon \bigoplus_{\alpha\in R^+} \boldsymbol{Z}u_\alpha \to M(R)$, $u_\alpha \mapsto \alpha$ or of algebras $\bigotimes_{\alpha\in R^+} \boldsymbol{Z}[y_\alpha^{\pm}] \to \boldsymbol{Z}[M(R)]$, $y_\alpha \mapsto x^\alpha$. The equations describing $X(R)$ in $\prod_{A_1\cong R'\subseteq R} \boldsymbol{P}^1_{R'}$ come from elements in the kernel of μ, and such an element $u = \sum_i l_i u_{\alpha_i} \in \ker(\mu)$ corresponds to a linear relation $\sum_i l_i\alpha_i = 0$ among the positive roots of the root system R. For any such element we have an equation $\prod_i y_{\alpha_i}^{l_i} = 1$ or equivalently a homogeneous equation $\prod_i z_{\alpha_i}^{l_i} = \prod_i z_{-\alpha_i}^{l_i}$.

EXAMPLE 1.14. Consider the toric variety $X(A_2)$ associated with the root system $A_2 = \{\pm\alpha, \pm\beta, \pm(\gamma = \alpha + \beta)\}$ and its embedding $X(A_2) \to \boldsymbol{P}^1_{\{\pm\alpha\}} \times \boldsymbol{P}^1_{\{\pm\beta\}} \times \boldsymbol{P}^1_{\{\pm\gamma\}}$. There is a one-dimensional space of linear relations generated by the relation $\alpha + \beta = \gamma$, so $X(A_2) \subset \boldsymbol{P}^1_{\{\pm\alpha\}} \times \boldsymbol{P}^1_{\{\pm\beta\}} \times \boldsymbol{P}^1_{\{\pm\gamma\}}$ is determined by the homogeneous equation $z_\alpha z_\beta z_{-\gamma} = z_{-\alpha} z_{-\beta} z_\gamma$.

In general, for any root subsystem in R isomorphic to A_2, there is a linear relation of the form $\alpha + \beta = \gamma$. We show that these generate the space of all linear relations.

PROPOSITION 1.15. *Let R be a root system. Then the space of linear relations between the positive roots of R is generated by the relations $\alpha + \beta = \gamma$ for root subsystems $\{\pm\alpha, \pm\beta, \pm(\gamma = \alpha + \beta)\}$ of R isomorphic to A_2.*

PROOF. We show that the kernel of the map of lattices $\mu\colon \bigoplus_{\alpha\in R^+} \mathbf{Z}u_\alpha \to M(R)$ is generated by elements of the form $u_\alpha + u_\beta - u_\gamma$.

Since the simple roots α_i form a basis of the lattice $M(R)$, the lattice $\ker(\mu)$ is generated by elements of the form $u_\beta - \sum_i l_i u_{\alpha_i}$, where β is a positive root and thus a linear combination of simple roots $\beta = \sum_i l_i\alpha_i$ with $l_i \in \mathbf{Z}_{\geq 0}$.

The statement now follows from the fact that starting with the set of simple roots $S_0 := S$ one obtains all positive roots by successively adding roots that are sums of two roots already obtained, i.e. $S_{i+1} = S_i \cup \{\gamma \in R;\ \gamma = \alpha + \beta \text{ for some } \alpha, \beta \in S_i\}$ (see [6, Ch. 4, §1.6, Prop. 19] or [12, §21.3]). □

COROLLARY 1.16. *The image of the closed embedding* $\varphi\colon X(R) \to \prod_{A_1\cong R'\subseteq R} \boldsymbol{P}^1_{R'}$ *is determined by the homogeneous equations* $z_\alpha z_\beta z_{-\gamma} = z_{-\alpha} z_{-\beta} z_\gamma$ *for root subsystems* $\{\pm\alpha, \pm\beta, \pm(\gamma = \alpha + \beta)\}$ *of* R *isomorphic to* A_2.

With a view to the closed embedding $\varphi\colon X(R) \to \prod_{A_1\cong R'\subseteq R} \boldsymbol{P}^1_{R'}$, we formulate a description of the functor of $X(R)$ by characterising a morphism $Y \to X(R)$ in terms of the family of morphisms $Y \to \boldsymbol{P}^1_{\{\pm\alpha\}}$ for all root subsystems of R isomorphic to A_1 that satisfy compatibility conditions coming from the root subsystems of R isomorphic to A_2.

DEFINITION 1.17. Let R be a root system. We define a contravariant functor $F_R\colon$ (schemes) $\to$ (sets) that associates to a scheme Y the following data, called R-data: a family $(U_\alpha, f_\alpha)_{\alpha\in R}$ consisting of open sets $U_\alpha \subseteq Y$ and regular functions $f_\alpha \in \mathcal{O}_Y(U_\alpha)$ that satisfy the conditions,

(i) *for all* $\alpha \in R$: $Y = U_\alpha \cup U_{-\alpha}$, $\{f_\alpha \neq 0\} = U_\alpha \cap U_{-\alpha}$ *and* $f_\alpha f_{-\alpha} = 1$ *on* $U_\alpha \cap U_{-\alpha}$,

(ii) *for all* $\alpha, \beta, \gamma \in R$: *if* $\gamma = \alpha + \beta$, *then* $U_\alpha \cap U_\beta \subseteq U_\gamma$ *and* $f_\alpha f_\beta = f_\gamma$ *on* $U_\alpha \cap U_\beta$,

or, equivalently, a family $(\mathscr{L}_{\{\pm\alpha\}}, \{t_\alpha, t_{-\alpha}\})_{\{\pm\alpha\}\subseteq R}$ of line bundles with two generating sections that satisfy

(ii)′ *for all* $\alpha, \beta, \gamma \in R$: *if* $\gamma = \alpha + \beta$, *then* $t_\alpha t_\beta t_{-\gamma} = t_{-\alpha} t_{-\beta} t_\gamma$,

up to isomorphism of line bundles with a pair of sections. To a morphism $h\colon Y' \to Y$ we associate the map $F_R(h)\colon F_R(Y) \to F_R(Y')$ given by pull-back of open sets and functions or line bundles with sections.

EXAMPLE 1.18. R-data over a field K can be written as a collection $((t_\alpha : t_{-\alpha}))_{\{\pm\alpha\}\subseteq R}$ of ratios of elements of K (such that for any α not both $t_\alpha, t_{-\alpha}$ are zero) that satisfy the equations $t_\alpha t_\beta t_{-\gamma} = t_{-\alpha} t_{-\beta} t_\gamma$ for $\gamma = \alpha + \beta$.

REMARK 1.19. On $X(R)$ we have the following R-data, called the universal R-data, coming from the morphisms $\varphi_{\{\pm\alpha\}}\colon X(R) \to \boldsymbol{P}^1_{\{\pm\alpha\}}$, i.e. for $\alpha \in R$ consider the rational function x^α, define U_α as the open set where x^α is regular and put $f_\alpha := x^\alpha|_{U_\alpha}$.

THEOREM 1.20. *The toric variety* $X(R)$ *associated with the root system* R *together with the universal* R*-data represents the functor* F_R.

PROOF. We show that there is an isomorphism of functors $\mathrm{Mor}(\,\cdot\,, X(R)) \cong F_R$ such that the identity in $\mathrm{Mor}(X(R), X(R))$ corresponds to the universal R-data on $X(R)$ denoted by $D_0 \in F_R(X(R))$.

By $\Phi(Y)\colon \mathrm{Mor}(Y, X(R)) \to F_R(Y)$, $h \mapsto h^*(D_0)$, we have defined a morphism of functors $\Phi\colon \mathrm{Mor}(\,\cdot\,, X(R)) \to F_R$.

On the other hand, for R-data D on Y we have a morphism $\varphi_D\colon Y \to X(R) \subseteq \prod_{\{\pm\alpha\}} \boldsymbol{P}^1_{\{\pm\alpha\}}$, where $X(R)$ is considered as a closed subvariety of $\prod_{\{\pm\alpha\}} \boldsymbol{P}^1_{\{\pm\alpha\}}$ via the embedding $\varphi\colon X(R) \to \prod_{\{\pm\alpha\}} \boldsymbol{P}^1_{\{\pm\alpha\}}$. In particular it is $\varphi_{D_0} = \mathrm{id}_{X(R)}$. The maps $\Psi(Y)\colon F_R(Y) \to \mathrm{Mor}(Y, X(R))$, $D \mapsto \varphi_D$ form a morphism of functors $\Psi\colon F_R \to \mathrm{Mor}(\,\cdot\,, X(R))$, because for any morphism of schemes $h\colon Y' \to Y$ it is $\varphi_{h^*(D)} = \varphi_D \circ h$. This is true since the maps $\varphi_{D,\{\pm\alpha\}}\colon Y \to \boldsymbol{P}^1_{\{\pm\alpha\}}$ associated with the part of R-data for pairs of roots $\{\pm\alpha\}$ satisfy $\varphi_{h^*(D),\{\pm\alpha\}} = \varphi_{D,\{\pm\alpha\}} \circ h$.

Thus, we have two morphisms of functors Φ and Ψ; these are inverse to each other. □

REMARK 1.21. The universal R-data on $X(R)$ gives rise to R-data $((t_\alpha : t_{-\alpha}))_{\{\pm\alpha\}\subseteq R}$ over points of $X(R)$ having the following properties:

- Over the affine chart U_S for a set of simple roots S of R, we have $(t_\alpha : t_{-\alpha}) \neq (1 : 0)$ if $\alpha \in \langle S\rangle$, and over the torus fixed point of U_S, we have $(t_\alpha : t_{-\alpha}) = (0 : 1)$ for $\alpha \in \langle S\rangle$.
- Over the torus invariant divisor corresponding to a one-dimensional cone generated by v, we have $(t_\alpha : t_{-\alpha}) = (0 : 1)$ if $\langle \alpha, v\rangle > 0$ (cf. Remark 1.11).

2. Toric varieties associated with root systems of type A.

2.1. Toric varieties $X(A_n)$. Consider an $(n+1)$-dimensional Euclidean vector space with basis $u_1, \ldots, u_{n+1}$. The root system A_n in the n-dimensional subspace $E = \{\sum_i a_i u_i\,;\ \sum_i a_i = 0\}$ consists of the $n(n+1)$ roots

$$u_i - u_j \quad \textit{for } i, j \in \{1, \ldots, n+1\}, \quad i \neq j\,.$$

The lattice $N(A_n) \cong \boldsymbol{Z}^n$ dual to the root lattice $M(A_n) \cong \boldsymbol{Z}^n$ has a generating system $v_1, \ldots, v_{n+1}$ with one relation $\sum_i v_i = 0$, it is a quotient of the lattice dual to $\bigoplus_{i=1}^{n+1} \boldsymbol{Z} u_i$ with basis $v_1, \ldots, v_{n+1}$ dual to $u_1, \ldots, u_{n+1}$.

The sets of simple roots of the root system A_n are of the form

$$S = \{u_{i_1} - u_{i_2}, u_{i_2} - u_{i_3}, \ldots, u_{i_n} - u_{i_{n+1}}\}$$

for some ordering $i_1, \ldots, i_{n+1}$ of the set $\{1, \ldots, n+1\}$. The maximal cone $\sigma_S = S^\vee$ of $\Sigma(A_n)$, i.e. the Weyl chamber corresponding to S, consists of those elements $\sum_i a_i v_i \in N(A_n)$ that satisfy $a_{i_1} \geq a_{i_2}, \ldots, a_{i_n} \geq a_{i_{n+1}}$ or equivalently of non-negative linear combinations of

$$v_{i_1},\ v_{i_1} + v_{i_2}, \ldots, v_{i_1} + \cdots + v_{i_n}\,.$$

So, the fan $\Sigma(A_n)$ can be described as follows (and coincides with that of [20, (2.5), (2.6)]): there are $2^{n+1} - 2$ one-dimensional cones of the fan $\Sigma(A_n)$, and these are generated

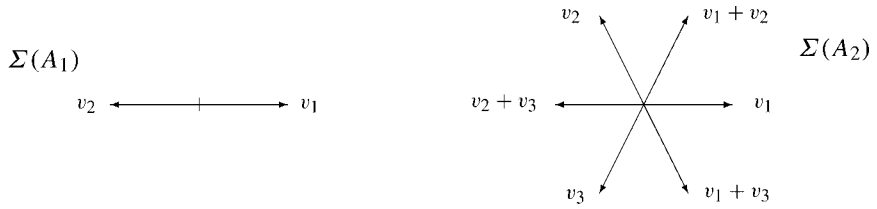

FIGURE 1.

by the elements $v_A = \sum_{i \in A} v_i \in N(A_n)$ for $A \in \mathcal{A}$ where

$$\mathcal{A} = \{A\,;\, \emptyset \neq A \subsetneq \{1, \ldots, n+1\}\}\,.$$

A family $(v_{A^{(i)}})_{i=1,\ldots,k}$ corresponding to a collection of pairwise different sets $A^{(1)}, \ldots, A^{(k)} \in \mathcal{A}$ generates a k-dimensional cone of $\Sigma(A_n)$ whenever these sets can be ordered such that $A^{(i_1)} \subset A^{(i_2)} \subset \cdots \subset A^{(i_k)}$.

The fan $\Sigma(A_n)$ defines an n-dimensional smooth projective toric variety $X(A_n)$. It is covered by the $(n+1)!$ open subvarieties $U_S = \operatorname{Spec} \boldsymbol{Z}[\sigma_S^\vee \cap M(A_n)]$ for the sets of simple roots S corresponding to strict orderings of the set $\{1, \ldots, n+1\}$. If $S = \{u_{i_1} - u_{i_2}, u_{i_2} - u_{i_3}, \ldots, u_{i_n} - u_{i_{n+1}}\}$, then $\boldsymbol{Z}[\sigma_S^\vee \cap M(A_n)] = \boldsymbol{Z}[x_{i_1}/x_{i_2}, \ldots, x_{i_n}/x_{i_{n+1}}]$. The Weyl group of the root system A_n is the symmetric group S_{n+1}; it acts on $\Sigma(A_n)$ and on $X(A_n)$ by permuting the Weyl chambers and the open sets U_S.

By results of the last subsection (Proposition 1.9 and Remark 1.10), the closures of torus orbits in $X(A_n)$ are isomorphic to products $\prod_i X(A_{n_i})$. In particular, the torus invariant divisor corresponding to the cone generated by $v_{i_1} + \cdots + v_{i_k}$ is of the form $X(A_{n-k}) \times X(A_{k-1})$.

EXAMPLE 2.1. The fans $\Sigma(A_1)$ and $\Sigma(A_2)$ are as Figure 1.

There are other descriptions of the toric variety $X(A_n)$:

$X(A_n)$ as blow-up of $\boldsymbol{P}^n$. $X(A_n)$ can be constructed by a sequence of toric blow-ups starting with $\boldsymbol{P}^n$ by first blowing up the $n+1$ torus fixed points, then blowing up the strict transform of the lines joining two of these points, then the strict transform of the planes through any three of these points and so on (see [21, Ch. 3], [16, (4.3.13)], [11, (5.1)]).

$X(A_n)$ as toric variety associated with the lattice polytope called permutohedron. The n-dimensional permutohedron is defined as the convex hull in $\boldsymbol{Q}^{n+1}$ of the S_{n+1}-orbit of the point $(1, 2, \ldots, n+1)$ where the symmetric group S_{n+1} acts by permuting coordinates. One can show that the fan for this polytope is the fan $\Sigma(A_n)$. The toric variety $X(A_n)$ also appears as "permutohedral space Π^n" in [16, (4.3.10) through (4.3.13)].

2.2. (Co)homology of $X(A_n)$. We know that the integral cohomology is torsion-free and confined to the even degrees. Standard methods from toric geometry (see e.g. [10, (10.8)]) furthermore imply the description of the cohomology ring of the toric variety $X(A_n)$ over the complex numbers (see also [20, (2.7)]) as

$$H^*(X(A_n), \boldsymbol{Z}) \;\cong\; \boldsymbol{Z}[\,l_A;\, A \in \mathcal{A}\,]/(R_1 + R_2)\,,$$

where R_1 is the ideal generated by the elements $r_{i,j} = \sum_{i\in A} l_A - \sum_{j\in A} l_A$ for $i, j \in \{1, \ldots, n+1\}, i \neq j$, and R_2 the ideal generated by the elements $r_{A,A'} = l_A l_{A'}$ for $A, A' \in \mathcal{A}$ such that $A \not\subseteq A'$, $A' \not\subseteq A$ (these correspond to the primitive collections of the fan $\Sigma(A_n)$, see next subsection).

The Betti numbers and Poincaré polynomials of the varieties $X(A_n)$ over the complex numbers are calculated in [20, (2.3)]; see also [23, Section 6], [11, Section 4] for a description in terms of the Eulerian numbers and see [17], [18], [23] for the general case of toric varieties associated with root systems. In particular, we know that the rank of $H^*(X(A_n), \mathbf{Z})$ is $(n+1)!$, i.e. the number of maximal cones of $\Sigma(A_n)$.

The $\mathbf{Z}$-module $\mathbf{Z}[\, l_A : A \in \mathcal{A}\,]/(R_1 + R_2)$ is generated by the classes of square-free monomials (see [10, (10.7.1)]). We can restrict to monomials each of which has only factors corresponding to one-dimensional faces of one maximal cone. Such a monomial $\prod_{i=1}^{m} l_{A^{(i)}}$ corresponds to an m-dimensional face of the respective maximal cone and, on the other hand, to a collection $A^{(1)} \subsetneq \cdots \subsetneq A^{(m)}$ of elements of $\mathcal{A}$. We denote by G the $\mathbf{Z}$-submodule of $\mathbf{Z}[\, l_A; A \in \mathcal{A}\,]$ generated by these monomials (called "good monomials" in [20, (2.8)]). We have the canonical isomorphism of $\mathbf{Z}$-modules $G/U \cong \mathbf{Z}[\, l_A; A \in \mathcal{A}\,]/(R_1 + R_2)$ where $U = (R_1 + R_2) \cap G$. The module G/U can be identified with the homology module $H_*(X(A_n), \mathbf{Z})$ (cf. [20, (2.9.2)]).

In [20, (2.8.2)] the following generators of the module of relations U are given. For a collection $A^{(1)} \subsetneq \cdots \subsetneq A^{(m)}$ of elements of $\mathcal{A}$ and $k \in \{1, \ldots, m+1\}$, $i, j \in A^{(k)} \setminus A^{(k-1)}$ (put $A^{(0)} = \emptyset$, $A^{(m+1)} = \{1, \ldots, n+1\}$), $i \neq j$, let

$$r_{i,j}((A^{(h)})_h, k) = \Bigg(\sum_{\substack{i\in A \\ j\notin A}} l_A - \sum_{\substack{j\in A \\ i\notin A}} l_A \Bigg) \prod_{h=1}^{m} l_{A^{(h)}}$$

where the sums run over sets $A \in \mathcal{A}$ such that $A^{(k-1)} \subsetneq A \subsetneq A^{(k)}$.

The maximal cones of the fan $\Sigma(A_n)$ correspond to collections $A^{(1)} \subsetneq \cdots \subsetneq A^{(n)}$ of elements of $\mathcal{A}$ and these correspond to permutations $\sigma \in S_{n+1}$ via $\{\sigma(1), \ldots, \sigma(k)\} = A^{(k)}$ for $k = 1, \ldots, n$. The descent set of a permutation $\sigma \in S_{n+1}$ is the set

$$\mathrm{Desc}(\sigma) = \{k \in \{1, \ldots, n\};\ \sigma(k) > \sigma(k+1)\}\,.$$

For any $\sigma \in S_{n+1}$ we define a monomial in G by

$$l^{\sigma} = \prod_{k\notin \mathrm{Desc}(\sigma)} l_{\{\sigma(1),\ldots,\sigma(k)\}}\,.$$

PROPOSITION 2.2. *The classes of the monomials l^{σ} for $\sigma \in S_{n+1}$ form a basis of the homology module $G/U = H_*(X(A_n), \mathbf{Z})$. The module of relations U is generated by the elements $r_{i,j}((A^{(h)})_h, k)$ for collections $A^{(1)} \subsetneq \ldots \subsetneq A^{(m)}$ of elements of $\mathcal{A}$ and $k \in \{1, \ldots, m+1\}$, $i, j \in A^{(k)} \setminus A^{(k-1)}$ (put $A^{(0)} = \emptyset$, $A^{(m+1)} = \{1, \ldots, n+1\}$), $i \neq j$.*

PROOF. We have $(n+1)!$ distinct monomials l^σ, and this number coincides with the rank of G/U. Thus it remains to show that every monomial in G via the relations $r_{i,j}((A^{(h)})_h, k)$ is equivalent to a linear combination of the monomials l^σ.

For a monomial $\prod_{k=1}^m l_{A^{(k)}}$ corresponding to a collection $A^{(1)} \subsetneq \cdots \subsetneq A^{(m)}$ we define the number $d(\prod_{k=1}^m l_{A^{(k)}}) := |\{k \in \{1,\dots,m\};\ \min P_k > \max P_{k+1}\}| \in \boldsymbol{Z}_{\geq 0}$ in terms of the associated partition $P_1 = A^{(1)}$, $P_2 = A^{(2)} \setminus A^{(1)}$, $\dots$, $P_m = A^{(m)} \setminus A^{(m-1)}$, $P_{m+1} = \{1,\dots,n+1\} \setminus A^{(m)}$ of the set $\{1,\dots,n+1\}$. The monomials $y \in G$ satisfying $d(y) = 0$ are exactly the monomials of the form l^σ. On the other hand we have an ordering $\prec$ of the monomials of G: take the partition $(P_i)_{i=1,\dots,m+1}$ associated with a monomial and consider the sequence (corresponding to a permutation of the set $\{1,\dots,n+1\}$) that arises by taking first the elements of P_{m+1} then those of P_m and so on and by ordering the elements of each P_i according to their size, and on these sequences we take the lexicographic order.

We show that every monomial in G modulo U is equivalent to a linear combination of the monomials l^σ, $\sigma \in S_{n+1}$, by showing that every monomial $y \in G$ with $d(y) > 0$ modulo a relation is equivalent to a linear combination of monomials y' with $y \prec y'$. In fact, let $A^{(1)} \subsetneq \cdots \subsetneq A^{(m)}$ be a collection of elements of $\mathcal{A}$ with associated partition $(P_k)_{k=1,\dots,m+1}$ such that the corresponding monomial $y = \prod_{k=1}^m l_{A^{(k)}}$ satisfies $d(y) > 0$. Take $k \in \{1,\dots,m\}$ such that $i := \min P_k > \max P_{k+1} =: j$, then

$$r_{i,j}((A^{(h)})_{h\neq k}, k) = \Bigg(\sum_{\substack{i \in A \\ j \notin A}} l_A - \sum_{\substack{j \in A \\ i \notin A}} l_A \Bigg) \prod_{h \neq k} l_{A^{(h)}},$$

where the sums run over sets $A \in \mathcal{A}$ such that $A^{(k-1)} \subsetneq A \subsetneq A^{(k+1)}$, is a relation that contains y as the monomial minimal with respect to $\prec$. □

REMARK 2.3. The above set of generators for the module of relations is used in [20, Section 3] in the context of solutions to the commutativity equations; the respective statement [20, (2.9)] is proven in a different way in that paper. Our statement involves a basis of homology which, in the general case of toric varieties associated with root systems, is given in [17], [18] (see [4, Rem. 5.4] for more explanations).

2.3. Primitive collections and the morphism $X(A_n) \to \boldsymbol{P}_{\Delta(A_n)}$. It was observed in [2] that any n-dimensional smooth projective toric variety X corresponding to a fan Σ can be described by the set of primitive collections among the generators of the one-dimensional cones of Σ together with the corresponding primitive relations. A primitive collection of the fan Σ is a set of generators of one-dimensional cones that does not generate a cone of Σ, but all of its proper subsets generate a cone of Σ.

THEOREM 2.4. ([2, Thm. 2.15], see also [9, Thm. 1.4, Prop. 1.10]). *The Mori cone* $\mathrm{NE}(X) \subset A_1(X) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ *of effective* 1*-cycles is generated by the primitive relations. Equivalently, the nef-cone* $\mathrm{Nef}(X) \subset \mathrm{Pic}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ (*which coincides with the closure of the ample cone*) *is given by line bundles corresponding to piecewise linear functions* φ *satisfying*

$\varphi(w_1) + \cdots + \varphi(w_k) \geq \varphi(w_1 + \cdots + w_k)$ *for all primitive collections* $\{w_1, \ldots, w_k\}$ *of the fan* Σ.

In the case $\Sigma = \Sigma(A_n)$, the primitive collections consist of two elements $v_A, v_{A'}$ (again put $v_A = \sum_{i \in A} v_i$) corresponding to non comparable elements of $\mathcal{A}$, i.e. elements $A, A' \in \mathcal{A}$ such that $A \not\subseteq A'$, $A' \not\subseteq A$. The corresponding primitive relation has one of the following four forms (put $I = \{1, \ldots, n+1\}$):

(1) $v_A + v_{A'} = 0$, *if* $A \cup A' = I$ *and* $A \cap A' = \emptyset$;
(2) $v_A + v_{A'} = v_{A \cup A'}$, *if* $A \cup A' \neq I$ *and* $A \cap A' = \emptyset$;
(3) $v_A + v_{A'} = v_{A \cap A'}$, *if* $A \cup A' = I$ *and* $A \cap A' \neq \emptyset$;
(4) $v_A + v_{A'} = v_{A \cap A'} + v_{A \cup A'}$, *if* $A \cup A' \neq I$ *and* $A \cap A' \neq \emptyset$.

COROLLARY 2.5. *Let* $D = \sum_{A \in \mathcal{A}} a_A D_A$ *be a torus-invariant divisor in* $X(A_n)$, *where* D_A *is the torus-invariant prime divisor corresponding to* v_A. *We put* $a_I = a_\emptyset = 0$. *Then* D *is ample* (*resp. semiample*, *or equivalently nef*) *if and only if*

$$a_A + a_{A'} > a_{A \cap A'} + a_{A \cup A'}$$

(*resp.* $\geq$) *for all primitive collections* $\{v_A, v_{A'}\}$.

COROLLARY 2.6. *The anticanonical class of* $X(A_n)$ *is semiample*, *or equivalently nef*; $X(A_n)$ *is an almost Fano variety.*

PROOF. By corollary 2.5 $-K_{X(A_n)}$ is semiample (and ample if there is no primitive relation of type (4), i.e. if $n \leq 2$). That $X(A_n)$ is an almost Fano variety means that $-K_{X(A_n)}$ is semiample and big, where the second property follows from the fact that the corresponding polytope is full-dimensional. □

Being semiample and big, the anticanonical class defines a toric morphism $X(A_n) \to \boldsymbol{P}(H^0(\mathcal{O}_{X(A_n)}(-K_{X(A_n)})))$, which is birational onto its image, but not necessarily an embedding.

PROPOSITION 2.7. *The anticanonical class defines a birational toric morphism* $X(A_n) \to \boldsymbol{P}_{\Delta(A_n)}$, *where* $\boldsymbol{P}_{\Delta(A_n)}$ *is the Gorenstein toric Fano variety associated with the reflexive polytope*

$$\Delta(A_n) = \{m \in M_{\boldsymbol{Q}} \,;\, \langle m, v_A \rangle \geq -1 \text{ for } A \in \mathcal{A}\} = \operatorname{conv}\{\alpha_{ij} \,;\, i, j \in I, i \neq j\}$$

and $\alpha_{ij} = u_i - u_j$ *are the roots of the root system* A_n.

PROOF. In general, the image of the morphism given by a semiample divisor $\sum_i a_i D_i$ (sum over the torus invariant prime divisors D_i) is the polarised toric variety corresponding to the polytope $\{m \in M_{\boldsymbol{Q}} \,;\, \langle m, v_i \rangle \geq -a_i \text{ for all } i\}$ ($v_i \in N$ being the lattice generators of the one dimensional cones corresponding to D_i). In the present case, we have the anticanonical divisor $-K_{X(A_n)} = \sum_{A \in \mathcal{A}} D_A$ on $X(A_n)$ giving rise to the polytope $\Delta(A_n) = \{m \in M_{\boldsymbol{Q}} \,;\, \langle m, v_A \rangle \geq -1 \text{ for } A \in \mathcal{A}\}$. This polytope $\Delta(A_n)$ coincides with the convex hull of the roots of A_n. Clearly, every root α_{ij} satisfies $\langle m, v_A \rangle \geq -1$ for all $A \in \mathcal{A}$. On the other hand, a given element $m = (m_1, \ldots, m_{n+1}) \in \Delta(A_n)$ we can write as $m = \sum_{i \in B,\, j \in C} a_{ij} \alpha_{ij}$ with

$a_{ij} \geq 0$ and $B = \{i\,;\, m_i > 0\}$, $C = \{i\,;\, m_i < 0\}$, and the condition $\langle m, v_C\rangle \geq -1$ gives $\sum a_{ij} \leq 1$, so $m \in \operatorname{conv}\{\alpha_{ij}\,;\ i \neq j\}$. The polytope $\Delta(A_n)$ is reflexive (0 being its only inner lattice point, the other lattice points in $\Delta(A_n)$ are the roots), equivalently $\boldsymbol{P}_{\Delta(A_n)}$ is a Gorenstein toric Fano variety. □

REMARK 2.8. The fan $\Sigma_{\Delta(A_n)}$ determined by $\Delta(A_n)$ consists of the cones

$$\sigma_{B_1,B_2} = \langle v_A\,;\ B_1 \subseteq A \subseteq B_2\rangle$$

for $B_1, B_2 \in \mathcal{A}$, $B_1 \subseteq B_2$. The cone σ_{B_1,B_2} has dimension $|B_2 \setminus B_1| + 1$ and is generated by $2^{|B_2\setminus B_1|}$ elements v_A. We see that $\boldsymbol{P}_{\Delta(A_n)}$ is singular for $n \geq 3$, the singular locus consisting of the torus orbits of codimension at least 3. The morphism $X(A_n) \to \boldsymbol{P}_{\Delta(A_n)}$ is a crepant resolution (an MPCP-desingularisation in the sense of [3]). It is given by subdividing each d-dimensional cone σ_{B_1,B_2} into $(d-1)!$ d-dimensional cones generated by $v_{B_1}, v_{B_1\cup\{a_1\}}, \ldots, v_{B_1\cup\{a_1,\ldots,a_{d-1}\}} = v_{B_2}$ corresponding to permutations of the set $B_2 \setminus B_1$. A set $\{v_{A^{(1)}}, \ldots, v_{A^{(k)}}\}$ is contained in a single cone of $\Sigma_{\Delta(A_n)}$, if $\bigcap_i A^{(i)} \neq \emptyset$ and $\bigcup_i A^{(i)} \neq I$. In particular, the primitive collections of types (1), (2) and (3) survive.

3. Losev-manin moduli spaces.

3.1. The moduli functor of A_n-curves. We begin with a presentation of the Losev-Manin moduli spaces $\overline{L}_n$ introduced in [20] and the corresponding moduli functor of stable n-pointed chains of projective lines (here called A_{n-1}-curves).

Consider a complex projective line $\boldsymbol{P}^1_{\boldsymbol{C}}$ with two distinct closed points $s_-, s_+ \in \boldsymbol{P}^1_{\boldsymbol{C}}$ called poles. The moduli space of n distinguishable points in $\boldsymbol{P}^1_{\boldsymbol{C}} \setminus \{s_-, s_+\}$ is a torus $(\boldsymbol{C}^*)^n/\boldsymbol{C}^* \cong (\boldsymbol{C}^*)^{n-1}$, here we divide out the automorphism group $\boldsymbol{C}^*$ of the projective line with two poles $(\boldsymbol{P}^1_{\boldsymbol{C}}, s_-, s_+)$. This space has a natural compactification $\overline{L}_n$ such that its boundary parametrises isomorphism classes of certain types of reducible n-pointed curves.

DEFINITION 3.1. A *chain of projective lines* of length m over an algebraically closed field K is a projective curve $C = C_1 \cup \cdots \cup C_m$ over K such that each irreducible component C_j of C is a projective line with poles p_j^-, p_j^+ and these components intersect as follows: different components C_i and C_j intersect only if $|i-j| = 1$ and in this case C_j, C_{j+1} intersect transversally at the single point $p_j^+ = p_{j+1}^-$. For $p_1^- \in C_1$ (resp. $p_m^+ \in C_m$) we write s_- (resp. s_+). Two chains of projective lines (C, s_-, s_+) and (C', s'_-, s'_+) are called isomorphic if there is an isomorphism $\varphi\colon C \to C'$ such that $\varphi(s_-) = s'_-$ and $\varphi(s_+) = s'_+$.

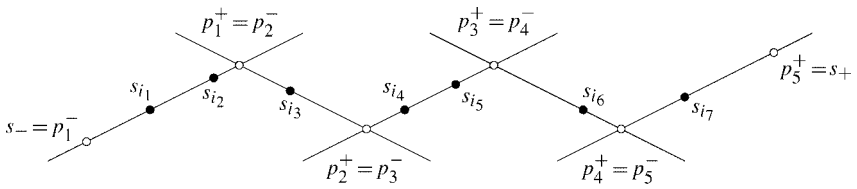

FIGURE 2.

DEFINITION 3.2. An *n-pointed chain of projective lines* $(C, s_-, s_+, s_1, \ldots, s_n)$ is a chain of projective lines together with closed points $s_i \in C$ different from s_-, s_+ and the intersection points of components (see Figure 2). Two n-pointed chains of projective lines $(C, s_-, s_+, s_1, \ldots, s_n)$ and $(C', s'_-, s'_+, s'_1, \ldots, s'_n)$ are called isomorphic if there is an isomorphism $\varphi\colon (C, s_-, s_+) \to (C', s'_-, s'_+)$ of the underlying chains of projective lines such that $\varphi(s_j) = s'_j$ for all $j \in \{1, \ldots, n\}$. An n-pointed chain of projective lines $(C, s_-, s_+, s_1, \ldots, s_n)$ is called *stable* if each component of C contains at least one of the points s_j. An *A_n-curve* over an algebraically closed field K is a stable $(n+1)$-pointed chain of projective lines over K.

DEFINITION 3.3. An A_n-curve of length m decomposes into irreducible components $C = C_1 \cup \cdots \cup C_m$ with $s_- \in C_1$, $s_+ \in C_m$. We will call the resulting decomposition

$$\{1, \ldots, n+1\} = P_1 \sqcup \cdots \sqcup P_m$$

such that $i \in P_k$ if and only if $s_i \in C_k$ the *combinatorial type* of the A_n-curve C. We will also write this in the form $s_{i_1} \ldots s_{i_l} | s_{i_{l+1}} \ldots | \ldots$ with the sections for the different sets P_k separated by the symbol "|" starting on the left with P_1.

DEFINITION 3.4. Let Y be a scheme. An *A_n-curve over Y* is a collection $(\pi : C \to Y, s_-, s_+, s_1, \ldots, s_{n+1})$, where C is a scheme, π is a proper flat morphism of schemes and $s_-, s_+, s_1, \ldots, s_{n+1} : Y \to C$ are sections such that for any geometric point y of Y the collection $(C_y, (s_-)_y, (s_+)_y, (s_1)_y, \ldots, (s_{n+1})_y)$ is an A_n-curve over y. An isomorphism of A_n-curves over Y is an isomorphism of Y-schemes that is compatible with the sections. We define the *moduli functor of A_n-curves* as the contravariant functor

$$\begin{aligned} \overline{L}_{n+1} : (\text{schemes}) &\to (\text{sets}) \\ Y &\mapsto \{A_n\text{-curves over } Y\} / \sim \end{aligned}$$

that associates to a scheme Y the set of isomorphism classes of A_n-curves over Y and to a morphism of schemes the map obtained by pulling back A_n-curves.

It is shown in [20, (2.2)] that the functor $\overline{L}_{n+1}$ is represented by a smooth projective variety, denoted by the same symbol $\overline{L}_{n+1}$, that is, we have a fine moduli space of A_n-curves. Further it is argued in [20, (2.6.3)] that $\overline{L}_{n+1}$ is isomorphic to the toric variety $X(A_n)$ associated with the root system A_n.

In [20, (2.1)] an inductive construction of $\overline{L}_{n+1}$ together with the universal curve $C_{n+1} \to \overline{L}_{n+1}$ is given using arguments of [19]. As in the case of the similar moduli spaces $\overline{M}_{0,n}$, the universal curve C_{n+1} over $\overline{L}_{n+1}$ is isomorphic to the next moduli space $\overline{L}_{n+2}$.

EXAMPLE 3.5. (1) $C_1 \to \overline{L}_1$ is isomorphic to $\boldsymbol{P}^1 \to pt$ with $0, \infty$ and 1-section. This reflects the fact that any A_0-curve over a scheme Y is isomorphic to the trivial projective bundle $\boldsymbol{P}^1_Y$ with $0, \infty$ and 1-section.

(2) $\overline{L}_2$ is isomorphic to $\boldsymbol{P}^1$. The fibres of the universal curve C_2 over $\boldsymbol{P}^1 \setminus \{(0:1), (1:0)\}$ are pointed $(\boldsymbol{P}^1, s_-, s_+)$, and over $(0:1)$ and $(1:0)$ the fibres are pointed chains consisting of two components (see Figure 3).

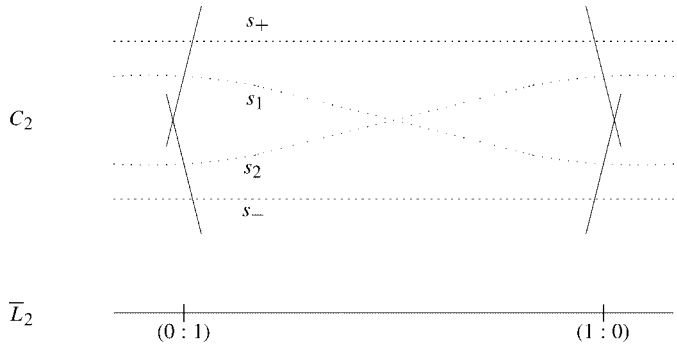

FIGURE 3.

3.2. The universal curve. We construct the universal A_n-curve $X(A_{n+1}) \to X(A_n)$ with its sections, which later will be seen to coincide with the universal curve over the Losev-Manin moduli space $\overline{L}_{n+1}$, in terms of the functorial properties of toric varieties associated with root systems developed in Subsection 1.2.

CONSTRUCTION 3.6 (The universal A_n-curve). Consider the root subsystem $A_n \subset A_{n+1}$ consisting of the roots $\beta_{ij} = u_i - u_j$ for $i, j \in \{1, \ldots, n+1\}$. The inclusion of root systems $A_n \subset A_{n+1}$ determines a proper surjective morphism $X(A_{n+1}) \to X(A_n)$.

There are the $n+1$ additional pairs of opposite roots $\pm\alpha_i$, $\alpha_i = u_i - u_{n+2}$ for $i \in \{1, \ldots, n+1\}$. For each of the root subsystems $\{\pm\alpha_i\} \cong A_1$ in A_{n+1} not contained in A_n we have a section $s_i : X(A_n) \to X(A_{n+1})$ corresponding to the projection of the root system A_{n+1} onto the root subsystem A_n with kernel generated by α_i. The image of s_i can be described by the equation $x^{\alpha_i} = 1$. On the other hand, the inclusion $\{\pm\alpha_i\} \subset A_{n+1}$ defines a projection $X(A_{n+1}) \to X(A_1) \cong \boldsymbol{P}^1_{\{\pm\alpha_i\}}$. If we choose coordinates of $\boldsymbol{P}^1_{\{\pm\alpha_i\}}$ as in Subsection 1.3, then the image of the section s_i is given by the preimage of the point $(1:1) \in \boldsymbol{P}^1_{\{\pm\alpha_i\}}$.

Further, we have two sections $s_\pm : X(A_n) \to X(A_{n+1})$ which are inclusions of $X(A_n)$ into $X(A_{n+1})$ as torus invariant divisors (cf. Remark 1.11). The image of s_- (resp. s_+) corresponds to the ray of the fan $\Sigma(A_{n+1})$ generated by $-v_{n+2}$ (resp. v_{n+2}) and is given by the equations $x^{\alpha_i} = 0$ (resp. $x^{-\alpha_i} = 0$).

PROPOSITION 3.7. *The collection* $(X(A_{n+1}) \to X(A_n), s_-, s_+, s_1, \ldots, s_{n+1})$ *is an* A_n*-curve over* $X(A_n)$.

PROOF. The morphism $X(A_{n+1}) \to X(A_n)$ is proper. To show flatness, consider the covering by affine toric charts: for any set of simple roots S of the root system A_n there are $n+2$ affine spaces $U_{S_j} \cong \boldsymbol{A}^{n+1}$ lying over $U_S \cong \boldsymbol{A}^n$ corresponding to $n+2$ sets of simple roots S_j of the root system A_{n+1} such that the submonoid $\langle S_j \rangle$ of the root lattice

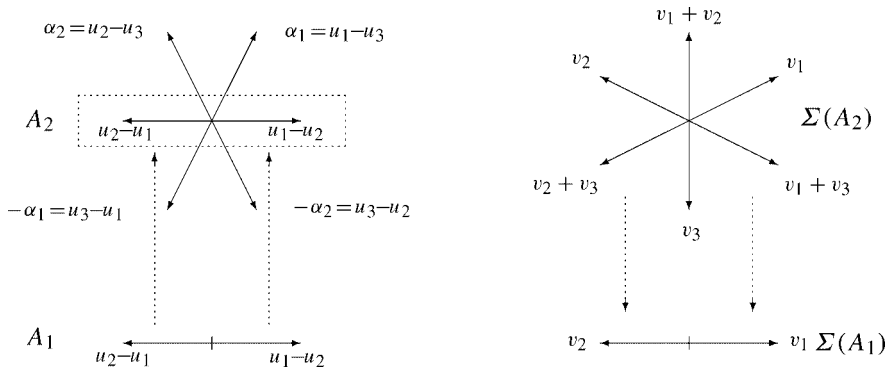

FIGURE 4.

$M(A_{n+1})$ generated by S_j contains S. For example, if $S = \{u_1 - u_2, \ldots, u_n - u_{n+1}\}$ we have $S_j = \{\ldots, u_{j-1} - u_j, u_j - u_{n+2}, u_{n+2} - u_{j+1}, u_{j+1} - u_{j+2}, \ldots\}$ for $j = 0, \ldots, n+1$ (in particular $S_0 = \{u_{n+2} - u_1, u_1 - u_2, \ldots\}$ and $S_{n+1} = \{\ldots, u_n - u_{n+1}, u_{n+1} - u_{n+2}\}$). The maps $U_{S_j} \to U_S$ are flat morphisms $\boldsymbol{A}^{n+1} \to \boldsymbol{A}^n$. Indeed, they are given by the identity on a polynomial ring in $n-1$ variables tensored with a map of the form $\boldsymbol{Z}[z] \to \boldsymbol{Z}[x, y]$, $z \mapsto xy$ or $z \mapsto y$.

By what is shown below, the fibres are A_n-curves: Remark 3.10 describes the universal A_n-curve in terms of equations given by A_n-data, and Proposition 3.12 shows that such equations define an A_n-curve once it is known that any A_n-data over a field arises as in Proposition 3.12 from an A_n-curve. This is shown in Lemma 3.13. □

DEFINITION 3.8. We call the object $(X(A_{n+1}) \to X(A_n), s_-, s_+, s_1, \ldots, s_{n+1})$ of Construction 3.6 the universal A_n-curve over $X(A_n)$.

EXAMPLE 3.9. The universal curve C_2 over $\overline{L}_2$ is pictured as Figure 3 and later seen to coincide with $X(A_2)$ over $X(A_1)$. Here, in Figure 4, we draw the corresponding inclusion of root systems (left) and the map of fans $\Sigma(A_2) \to \Sigma(A_1)$ (right).

We apply the embedding into a product of projective lines

$$X(R) \to \prod_{A_1 \cong R' \subseteq R} \boldsymbol{P}^1_{R'} =: P(R)$$

considered in Subsections 1.2 and 1.3 to the situation of Construction 3.6.

REMARK 3.10. Consider $X(A_{n+1})$ (resp. $X(A_n)$) as embedded into the product of projective lines $P(A_{n+1})$ (resp. $P(A_n)$). Then the morphism $X(A_{n+1}) \to X(A_n)$ is induced by the projection onto the subproduct $P(A_{n+1}) \to P(A_n)$. The subvarieties $X(A_{n+1}) \subseteq P(A_{n+1})$ (resp. $X(A_n) \subseteq P(A_n)$) are determined by the homogeneous equations $z_\alpha z_\beta z_{-\gamma} = z_{-\alpha} z_{-\beta} z_\gamma$ for roots α, β, γ such that $\alpha + \beta = \gamma$, i.e. root subsystems of type A_2 in A_{n+1} (resp. A_n). If we first consider the product $P(A_{n+1})$ and the equations coming from root subsystems

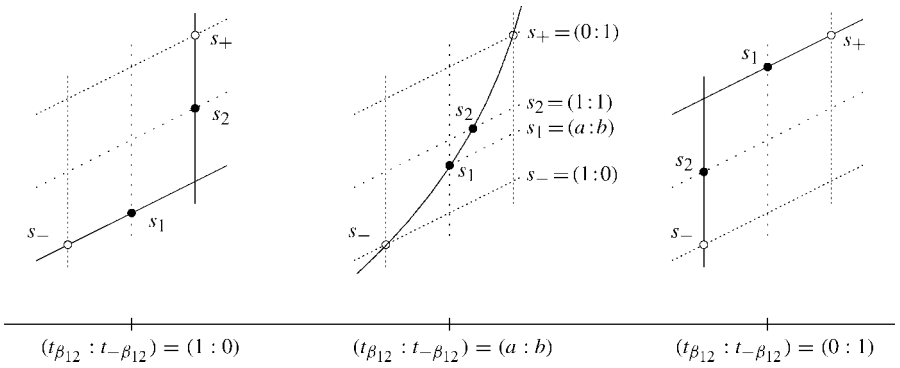

FIGURE 5.

of type A_2 in A_n, this gives

$$P(A_{n+1}/A_n)_{X(A_n)} = \left(\prod_{A_1 \cong R \subseteq A_{n+1} \setminus A_n} \boldsymbol{P}^1_R \right)_{X(A_n)} = \left(\prod_{i=1}^{n+1} \boldsymbol{P}^1_{\{\pm\alpha_i\}} \right)_{X(A_n)} .$$

Therein, $X(A_{n+1})$ is the closed subscheme given by the homogeneous equations

$$t_{\beta_{ij}} z_{\alpha_j} z_{-\alpha_i} = t_{-\beta_{ij}} z_{-\alpha_j} z_{\alpha_i} , \quad i, j \in \{1, \ldots, n+1\} , \quad i \neq j , \quad \beta_{ij} = \alpha_i - \alpha_j \tag{1}$$

where $t_{\pm\beta_{ij}}$ are the homogeneous coordinates of $\boldsymbol{P}^1_{\{\pm\beta_{ij}\}}$ (consider $X(A_n)$ as embedded into $\prod_R \boldsymbol{P}^1_R$) or equivalently the two generating sections of the line bundle $\mathscr{L}_{\{\pm\beta_{ij}\}}$ being part of the universal A_n-data on $X(A_n)$. The sections $s_i : X(A_n) \to X(A_{n+1})$ for $i \in \{1, \ldots, n+1\}$ are given by the additional equations $z_{\alpha_i} = z_{-\alpha_i}$. The sections s_- (resp. s_+) are given by $z_{\alpha_1} = \cdots = z_{\alpha_{n+1}} = 0$ (resp. $z_{-\alpha_1} = \cdots = z_{-\alpha_{n+1}} = 0$).

EXAMPLE 3.11. The universal A_1-curve $X(A_2) \subset (\boldsymbol{P}^1_{\{\pm\alpha_1\}} \times \boldsymbol{P}^1_{\{\pm\alpha_2\}})_{X(A_1)}$ over $X(A_1)$ is given by the homogeneous equation

$$t_{\beta_{12}} z_{\alpha_2} z_{-\alpha_1} = t_{-\beta_{12}} z_{-\alpha_2} z_{\alpha_1}$$

where $\beta_{12} = u_1 - u_2$ and $(\mathscr{L}_{\{\pm\beta_{12}\}}, \{t_{\beta_{12}}, t_{-\beta_{12}}\})$ is the universal A_1-data on $X(A_1) \cong \boldsymbol{P}^1$. We picture in Figure 5 the A_1-curves in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ defined by this equation over the points given by $(t_{\beta_{12}} : t_{-\beta_{12}}) = (0:1), (a:b), (1:0)$. In the second case, we write the marked points in terms of the homogeneous coordinates $(z_{-\alpha_2} : z_{\alpha_2})$.

By Remark 3.10 the universal A_n-curve over $X(A_n)$ can be embedded into a product $P(A_{n+1}/A_n)_{X(A_n)} \cong (\boldsymbol{P}^1)^{n+1}_{X(A_n)}$ and the embedded curve is given by equations (1) determined by the universal A_n-data. We show that any A_n-curve C over a field can be embedded into a product $(\boldsymbol{P}^1)^{n+1}$ and extract A_n-data such that C is described by the same equations as the universal curve.

PROPOSITION 3.12. *Let $(C, s_-, s_+, s_1, \ldots, s_{n+1})$ be an A_n-curve over a field. For $i \in \{1, \ldots, n+1\}$ let $z_{\alpha_i}, z_{-\alpha_i}$ be a basis of $H^0(C, \mathcal{O}_C(s_i))$ such that $z_{\alpha_i}(s_-) = 0$, $z_{-\alpha_i}(s_+) = 0$,*

$z_{\alpha_i}(s_i) = z_{-\alpha_i}(s_i) \neq 0$. *We will write* $\boldsymbol{P}_{\{\pm\alpha_i\}}$ *for* $\boldsymbol{P}(H^0(C, \mathcal{O}_C(s_i)))$. *Then by*

$$(t_{\beta_{ij}} : t_{-\beta_{ij}}) = (z_{-\alpha_j}(s_i) : z_{\alpha_j}(s_i))$$

for $\beta_{ij} = \alpha_i - \alpha_j$, $i \neq j$, *we can define* A_n*-data* $(t_{\beta_{ij}} : t_{-\beta_{ij}})_{\{\pm\beta_{ij}\}\subseteq A_n}$, *and the morphism*

$$C \;\to\; \prod_{i=1}^{n+1} \boldsymbol{P}^1_{\{\pm\alpha_i\}} = P(A_{n+1}/A_n)$$

is an isomorphism onto the closed subvariety $C' \subseteq P(A_{n+1}/A_n)$ *determined by the homogeneous equations*

$$t_{\beta_{ij}} z_{\alpha_j} z_{-\alpha_i} = t_{-\beta_{ij}} z_{-\alpha_j} z_{\alpha_i}\,, \quad i, j \in \{1, \ldots, n+1\}, i \neq j\,, \;\; \beta_{ij} = \alpha_i - \alpha_j\,. \tag{2}$$

Furthermore, C' *together with the marked points* s'_i *defined by the additional equations* $z_{\alpha_i} = z_{-\alpha_i}$ *and* s'_- *(resp.* s'_+*) defined by* $z_{\alpha_i} = 0$ *for* $i = 1, \ldots, n+1$ *(resp.* $z_{-\alpha_i} = 0$ *for* $i = 1, \ldots, n+1$*) is an* A_n*-curve and we have an isomorphism of* A_n*-curves* $(C, s_-, s_+, s_1, \ldots, s_{n+1}) \to (C', s'_-, s'_+, s'_1, \ldots, s'_{n+1})$.

PROOF. The data $(t_{\beta_{ij}} : t_{-\beta_{ij}})$ is defined as the position of a marked point s_i relative to another marked point s_j of C. Note that the two ways to define $(t_{\beta_{ij}} : t_{-\beta_{ij}})$ are equivalent because $(z_{\alpha_i}(s_j) : z_{-\alpha_i}(s_j)) = (z_{-\alpha_j}(s_i) : z_{\alpha_j}(s_i))$.

The data $(t_{\beta_{ij}} : t_{-\beta_{ij}})$ relates the bases $z_{\alpha_i}, z_{-\alpha_i}$ and $z_{\alpha_j}, z_{-\alpha_j}$ via the equation $t_{\beta_{ij}} z_{\alpha_j} z_{-\alpha_i} = t_{-\beta_{ij}} z_{-\alpha_j} z_{\alpha_i}$. If the corresponding marked points are not contained in the same component of C, both sides of the equation are zero. Otherwise, both are homogeneous coordinates of the same component of C and we have the formula $(t_{\beta_{ij}} z_{-\alpha_i} : t_{-\beta_{ij}} z_{\alpha_i}) = (z_{-\alpha_j} : z_{\alpha_j})$ which can be checked at the two poles and the marked point s_i. It follows that the morphism $C \to \boldsymbol{P}(A_{n+1}/A_n)$ maps C to the subscheme C' defined by the equations in (2).

The collection $(t_{\beta_{ij}} : t_{-\beta_{ij}})_{\{\pm\beta_{ij}\}\subseteq A_n}$ is A_n-data: we have to show that the equations $t_{\beta_{ij}} t_{\beta_{jk}} t_{-\beta_{ik}} = t_{-\beta_{ij}} t_{-\beta_{jk}} t_{\beta_{ik}}$ are satisfied. If s_i, s_j, s_k are not contained in the same component, then both sides are zero. If s_i, s_j, s_k are in the same component, then $(t_{\beta_{ik}} : t_{-\beta_{ik}}) = (z_{-\alpha_k}(s_i) : z_{\alpha_k}(s_i)) = (t_{\beta_{jk}} z_{-\alpha_j}(s_i) : t_{-\beta_{jk}} z_{\alpha_j}(s_i)) = (t_{\beta_{jk}} t_{\beta_{ij}} : t_{-\beta_{jk}} t_{-\beta_{ij}})$ making use of $(z_{-\alpha_k} : z_{\alpha_k}) = (t_{\beta_{jk}} z_{-\alpha_j} : t_{-\beta_{jk}} z_{\alpha_j})$.

We show that $C \to C' \subseteq P(A_{n+1}/A_n)$ is an isomorphism. The curve C decomposes into irreducible components $C = C_1 \cup \cdots \cup C_m$. Let $\{1, \ldots, n+1\} = P_1 \sqcup \cdots \sqcup P_m$ be the combinatorial type of C (see Definition 3.3). For $k \in \{1, \ldots, m\}$ the morphism $C_k \to \prod_{i\in P_k} \boldsymbol{P}^1_{\{\pm\alpha_i\}} =: P(P_k)$ is an isomorphism onto $C''_k \subseteq P(P_k)$ given by the equations in (2) involving only coordinates $z_{\pm\alpha_i}$ for $i \in P_k$. The equations in (2) involving coordinates for roots α_i, α_j for $i \in P_k$, $j \in P_{k'}$, $k \neq k'$, are of the form $z_{\alpha_j} z_{-\alpha_i} = 0$ if $k < k'$ and $z_{-\alpha_j} z_{\alpha_i} = 0$ if $k' < k$. So, the equations in (2) containing coordinates for some $i \in P_k$ define a subvariety of $P(A_{n+1}/A_n) = \prod_{i\in P_j, j<k} \boldsymbol{P}^1_{\{\pm\alpha_i\}} \times P(P_k) \times \prod_{i\in P_j, k<j} \boldsymbol{P}^1_{\{\pm\alpha_i\}}$ consisting of the irreducible components $C'_k = \prod_{i\in P_{k'}, k'<k}\{z_{-\alpha_i} = 0\} \times C''_k \times \prod_{i\in P_{k'}, k<k'}\{z_{\alpha_i} = 0\}$, $\prod_{i\in P_j, j<k} \boldsymbol{P}^1_{\{\pm\alpha_s\}} \times \prod_{i\in P_j, k\leq j}\{z_{\alpha_i} = 0\}$, $\prod_{i\in P_j, j\leq k}\{z_{-\alpha_i} = 0\} \times \prod_{i\in P_j, k<j} \boldsymbol{P}^1_{\{\pm\alpha_s\}}$. For each k we have an isomorphism $C_k \to C'_k$, and these form the isomorphism $C \to C'$.

The sections of C' were defined such that the isomorphism $C \to C'$ is an isomorphism of A_n-curves. □

LEMMA 3.13. *Any A_n-data over a field arises as A_n-data extracted from an A_n-curve by the method of Proposition* 3.12.

PROOF. Let $(t_{\beta_{ij}} : t_{-\beta_{ij}})_{\{\pm\beta_{ij}\}\subseteq A_n}$ be A_n-data over a field.

We define an ordering $\prec$ on the set $\{1, \ldots, n+1\}$. For $i \neq j$ define $i \prec j$ (resp. $i \preceq j$) if $(t_{\beta_{ij}} : t_{-\beta_{ij}}) = (1 : 0)$ (resp. $(t_{\beta_{ij}} : t_{-\beta_{ij}}) \neq (0 : 1)$) where $\beta_{ij} = \alpha_i - \alpha_j$. This ordering defines a decomposition $\{1, \ldots, n+1\} = P_1 \sqcup \cdots \sqcup P_m$ into nonempty equivalence classes such that $i \prec j$ if and only if $i \in P_k,\ j \in P_{k'}$ for $k < k'$.

We construct an A_n-curve such that the A_n-data extracted from it by the method of Proposition 3.12 is the given A_n-data. Take a chain of projective lines (C, s_-, s_+) of length m. For each component C_k we can choose for each $i \in P_k$ a point $s_i \in C_k$ different from the poles $p_k^{\pm}$, such that their relative positions are given by the data $(t_{\beta_{ij}} : t_{-\beta_{ij}})$, i.e. $s_i = (t_{\beta_{ij}} : t_{-\beta_{ij}})$ with respect to coordinates of C_k such that $p_k^- = (1 : 0)$, $p_k^+ = (0 : 1)$, $s_j = (1 : 1)$. This is possible, the compatibility is assured by the conditions of A_n-data. □

Considering the universal A_n-curve, the combinatorial types of the geometric fibres determine a stratification of $X(A_n)$ which coincides with the stratification of this toric variety into torus orbits.

PROPOSITION 3.14. *Over the torus orbit in $X(A_n)$ corresponding to the one-dimensional cone generated by $v_{i_1} + \cdots + v_{i_k}$ we have the combinatorial type*

$$s_{i_{n+1}} \cdots s_{i_{k+1}} | s_{i_k} \cdots s_{i_1} .$$

PROOF. The universal A_n-data over each point of the closure of the orbit corresponding to a generator of a one-dimensional cone generated by v has the property $(t_\beta : t_{-\beta}) = (0 : 1)$ if $\langle \beta, v \rangle > 0$ (see Remark 1.21). For $v = v_{i_1} + \cdots + v_{i_k}$ this in particular implies $(t_{\beta_{ij}} : t_{-\beta_{ij}}) = (0 : 1)$ if $i \in \{i_1, \ldots, i_k\}$ and $j \in \{i_{k+1}, \ldots, i_{n+1}\}$. We obtain for points in this torus orbit the above combinatorial type. □

The proposition describes the combinatorial types over the torus orbits in $X(A_n)$ of codimension one. The combinatorial type over a lower-dimensional torus orbit is given by the partition that arises as common refinement of the partitions for the torus orbits of codimension one that contain the respective orbit in their closure.

3.3. Isomorphism between the functor of $X(A_n)$ and the moduli functor. We show that the moduli functor of A_n-curves is isomorphic to the functor of $X(A_n)$ of Subsection 1.3. This implies that the toric variety $X(A_n)$ is a fine moduli space of A_n-curves and hence coincides with the Losev-Manin moduli space $\overline{L}_{n+1}$.

To relate A_n-curves to A_n-data, we consider an embedding of arbitrary A_n-curves over a scheme Y into a product $(\boldsymbol{P}^1)_Y^{n+1}$ that generalises the embedding in Proposition 3.12 to

the relative situation. The main tools are the following contraction morphisms. The technical details are standard, compare to similar constructions in [19], [14].

CONSTRUCTION 3.15. Let $(\pi\colon C \to Y, s_-, s_+, s_1, \dots, s_{n+1})$ be an A_n-curve over Y. Each subset of sections $\{s_{i_1}, \dots, s_{i_k}\} \subseteq \{s_1, \dots, s_n\}$ defines a line bundle $\mathcal{O}_C(s_{i_1} + \cdots + s_{i_k})$ on C and a locally free sheaf $\pi_*\mathcal{O}_C(s_{i_1} + \cdots + s_{i_k})$ of rank $k+1$ on Y. The natural surjective homomorphism $\pi^*\pi_*\mathcal{O}_C(s_{i_1}+\cdots+s_{i_k}) \to \mathcal{O}_C(s_{i_1}+\cdots+s_{i_k})$ induces a morphism $p_{\{i_1,\dots,i_k\}}\colon C \to C_{\{i_1,\dots,i_k\}} \subseteq \boldsymbol{P}_Y(\pi_*\mathcal{O}_C(s_{i_1} + \cdots + s_{i_k}))$.

This construction commutes with base change. In particular, for each fibre C_y, $y \in Y$, the morphism $(p_{\{i_1,\dots,i_k\}})_y\colon C_y \to \boldsymbol{P}_Y(\pi_*\mathcal{O}_C(s_{i_1} + \cdots + s_{i_k}))_y$ arises by applying the above construction to the line bundle $\mathcal{O}_{C_y}(s_{i_1}(y) + \cdots + s_{i_k}(y))$ on C_y; it is an isomorphism on the components containing at least one of the marked points $s_{i_j}(y)$ and contracts all other components. The image of $p_{\{i_1,\dots,i_k\}}$ with the sections $p_{\{i_1,\dots,i_k\}} \circ s_\pm$, $p_{\{i_1,\dots,i_k\}} \circ s_{i_j}$ is an A_{k-1}-curve.

These morphisms are functorial with respect to multiple inclusions of sets of sections.

We will make use of the following particular cases:

(1) *Contraction with respect to one section onto a $\boldsymbol{P}^1$-bundle.* For each section s_i of C there is a contraction morphism $p_i\colon C \to C_i = \boldsymbol{P}_Y(\pi_*\mathcal{O}_C(s_i))$. Since the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}_Y(\pi_*\mathcal{O}_C(s_i))$ has three disjoint sections $p_i \circ s_-$, $p_i \circ s_+$, $p_i \circ s_i\colon Y \to \boldsymbol{P}_Y(\pi_*\mathcal{O}_C(s_i))$, there is an isomorphism $\boldsymbol{P}_Y(\pi_*\mathcal{O}_C(s_i)) \cong \boldsymbol{P}^1_Y$ that identifies $p_i \circ s_-$, $p_i \circ s_+$, $p_i \circ s_i$ with the $(1:0)$, $(0:1)$, $(1:1)$-section of $\boldsymbol{P}^1_Y$ (see Figure 6).

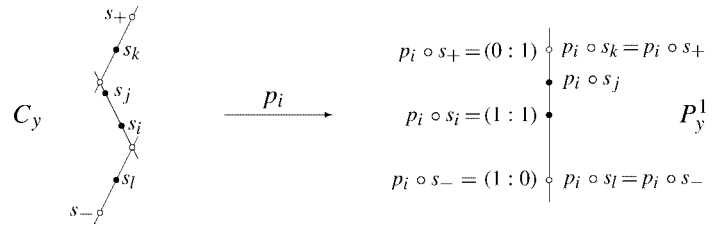

FIGURE 6.

(2) *Contraction with respect to two sections onto an A_1-curve.* Let s_{i_1}, s_{i_2} be two of the sections of C. Then there is a contraction morphism $p_{\{i_1,i_2\}}\colon C \to C_{\{i_1,i_2\}}$ and $C_{\{i_1,i_2\}}$ is an A_1-curve over Y. The curve $C_{\{i_1,i_2\}}$ contains the information about the relative positions of the sections s_{i_1}, s_{i_2} in C. This data relates the two contraction morphisms with respect to the sections s_{i_1} and s_{i_2}.

CONSTRUCTION 3.16. Let $(C \to Y, s_-, s_+, s_1, s_2)$ be an A_1-curve. Then we have the morphism $p_2\colon C \to \boldsymbol{P}^1_Y$ for the section s_2 such that, with respect to homogeneous coordinates $z_0^{(2)}, z_1^{(2)}$, the sections s_-, s_+, s_2 become the $(1:0)$, $(0:1)$, $(1:1)$-section of $\boldsymbol{P}^1_Y$. The section s_1 determines $(t_{1,2} : t_{2,1}) := p_2 \circ s_1$. This can be rewritten as a line bundle with two generating sections $(\mathscr{L}_{\{1,2\}}, \{t_{1,2}, t_{2,1}\})$ up to isomorphism.

Equivalently, we can consider the morphism $p_1 \colon C \to \boldsymbol{P}^1_Y$ for the section s_1. Then s_-, s_+, s_1 become the $(1:0)$, $(0:1)$, $(1:1)$-section with respect to homogeneous coordinates $z_0^{(1)}, z_1^{(1)}$ and $p_1 \circ s_2 = (t_{2,1} : t_{1,2})$.

LEMMA 3.17. *The morphism $p_1 \times p_2 \colon C \to \boldsymbol{P}^1_Y \times \boldsymbol{P}^1_Y$ maps C to the closed subvariety given by the homogeneous equation*

$$t_{1,2} z_1^{(2)} z_0^{(1)} = t_{2,1} z_0^{(2)} z_1^{(1)} .$$

PROOF. It suffices to show the lemma for the strata of Y corresponding to the three possible combinatorial types of A_1-curves. On the strata corresponding to reducible curves the above equation becomes $z_1^{(2)} z_0^{(1)} = 0$ (resp. $z_0^{(2)} z_1^{(1)} = 0$). These equations are satisfied, e.g. in the first case $z_1^{(2)} = 0$ on the component containing s_-, s_1 and $z_0^{(1)} = 0$ on the component containing s_+, s_2. On the remaining stratum $Y' \subseteq Y$ both $(z_1^{(1)} : z_0^{(1)})$ and $(z_1^{(1)} : z_0^{(1)})$ are homogeneous coordinates of $\boldsymbol{P}^1_{Y'}$ and related by $(t_{1,2} z_1^{(2)} : t_{2,1} z_0^{(2)}) = (z_1^{(1)} : z_0^{(1)})$. □

(3) *Contraction with respect to three sections onto an A_2-curve.* Let $s_{i_1}, s_{i_2}, s_{i_3}$ be three of the sections of C. Then there is a contraction morphism $p_{\{i_1,i_2,i_3\}} \colon C \to C_{\{i_1,i_2,i_3\}}$ and $C_{\{i_1,i_2,i_3\}}$ is an A_2-curve over Y. This curve contains the information about the relative positions of the pairs of two sections in a set of three sections $s_{i_1}, s_{i_2}, s_{i_3}$ of C. These data are related by one equation:

LEMMA 3.18. *Let $(C, s_-, s_+, s_1, s_2, s_3)$ be an A_2-curve over Y. Then the collection of A_1-data $\{(\mathscr{L}_{\{1,2\}}, \{t_{1,2}, t_{2,1}\}), (\mathscr{L}_{\{2,3\}}, \{t_{2,3}, t_{3,2}\}), (\mathscr{L}_{\{3,1\}}, \{t_{3,1}, t_{1,3}\})\}$ extracted by the method of Construction* 3.16 *from the A_1-curves $C_{\{i_1,i_2\}}, C_{\{i_2,i_3\}}, C_{\{i_3,i_1\}}$ is A_2-data, i.e. the sections satisfy the equation*

$$t_{1,2} t_{2,3} t_{3,1} = t_{2,1} t_{3,2} t_{1,3}$$

in $\mathscr{L}_{\{1,2\}} \otimes \mathscr{L}_{\{2,3\}} \otimes \mathscr{L}_{\{3,1\}}$.

PROOF. Again, this equation is satisfied on a closed subset and it suffices to consider the situation for the strata corresponding to the different combinatorial types. If the sections are not contained in the same component, then both sides vanish. Over the remaining stratum $Y' \subseteq Y$ we have a bundle $\boldsymbol{P}^1_{Y'}$. Using the formula $(t_{i,j} z_1^{(j)} : t_{j,i} z_0^{(j)}) = (z_1^{(i)} : z_0^{(i)})$ for the homogeneous coordinates we obtain $(z_1^{(1)} : z_0^{(1)}) = (t_{1,2} z_1^{(2)} : t_{2,1} z_0^{(2)}) = (t_{1,2} t_{2,3} z_1^{(3)} : t_{2,1} t_{3,2} z_0^{(3)}) = (t_{1,2} t_{2,3} t_{3,1} z_1^{(1)} : t_{2,1} t_{3,2} t_{1,3} z_0^{(1)})$, so $(t_{1,2} t_{2,3} t_{3,1} : t_{2,1} t_{3,2} t_{1,3}) = (1:1)$. □

For an A_n-curve $(C \to Y, s_-, s_+, s_1, \ldots, s_n)$ we have the contraction morphisms $p_i \colon C \to \boldsymbol{P}_Y(\pi_* \mathcal{O}_C(s_i)) \cong (\boldsymbol{P}^1_{\{\pm\alpha_i\}})_Y$ where on $(\boldsymbol{P}^1_{\{\pm\alpha_i\}})_Y$ we have homogeneous coordinates $z_{-\alpha_i}, z_{\alpha_i}$ such that, in these coordinates, s_-, s_+, s_i become the $(1:0)$, $(0:1)$, $(1:1)$-section of $\boldsymbol{P}^1_Y$. We have the roots $\alpha_i = u_i - u_{n+2}$ and $\beta_{ij} = \alpha_i - \alpha_j = u_i - u_j$.

THEOREM 3.19. *There is an isomorphism between the functor F_{A_n} and the moduli functor of A_n-curves $\overline{L}_{n+1}$ such that the universal A_n-data on $X(A_n)$ is mapped to the universal A_n-curve over $X(A_n)$.*

PROOF. Let Y be a scheme. For A_n-data on Y we construct an A_n-curve C over Y via equations in $P(A_{n+1}/A_n)_Y = \prod_{i=1}^{n+1}(\boldsymbol{P}^1_{\{\pm\alpha_i\}})_Y$ as in Remark 3.10 with the given A_n-data on Y replacing the universal A_n-data on $X(A_n)$. This is an A_n-curve since all A_n-data are pull-back of the universal A_n-data on $X(A_n)$ and so the constructed curve is a pull-back of the universal A_n-curve over $X(A_n)$.

In the other direction, given an A_n-curve on Y we extract A_n-data. For each pair of distinct sections s_i, s_j we have a contraction morphism $C \to C_{\{i,j\}}$ onto an A_1-curve over Y. From $(C_{\{i,j\}}, s_i, s_j)$ we extract A_1-data $(\mathscr{L}_{\{\pm\beta_{ij}\}}, t_{\beta_{ij}}, t_{-\beta_{ij}}) := (\mathscr{L}_{\{i,j\}}, t_{i,j}, t_{j,i})$ as in Construction 3.16. That the collection of all these data forms A_n-data follows from Lemma 3.18.

Both constructions commute with base-change and thus define morphisms of functors $F_{A_n} \to \overline{L}_{n+1}$ and $\overline{L}_{n+1} \to F_{A_n}$. We show that they are inverse to each other.

Starting with an A_n-curve C over Y, we extract data $(\mathscr{L}_{\{\pm\beta_{ij}\}}, \{t_{-\beta_{ij}}, t_{\beta_{ij}}\})_{\{\pm\beta_{ij}\}\subseteq A_n}$ and from these A_n-data we construct an A_n-curve $C' \subseteq P(A_{n+1}/A_n)_Y$ as in Remark 3.10. We show that the product of the contraction morphisms p_i defines an isomorphism of A_n-curves $C \to C' \subseteq P(A_{n+1}/A_n)_Y$. The morphism $\prod_i p_i \colon C \to P(A_{n+1}/A_n)_Y$ factors through the inclusion $C' \subseteq P(A_{n+1}/A_n)_Y$ by Lemma 3.17. The morphism $C \to C'$ is an isomorphism, because it is an isomorphism on the fibres by Proposition 3.12 and the curves are flat over Y. The sections and the involution of C' were defined such that $C \to C'$ is an isomorphism of A_n-curves.

Starting with A_n-data we construct an A_n-curve and extract A_n-data from it. It is easily verified that this A_n-data coincides with the original A_n-data, since the contraction morphism with respect to a section of an embedded A_n-curve $C' \subseteq P(A_{n+1}/A_n)_Y$ is induced by projection onto the corresponding factor. □

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