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CLASSIFICATION OF SIMPLE $\mathfrak{q}_2$-SUPERMODULES

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Abstract. We classify all simple supermodules over the queer Lie superalgebra $\mathfrak{q}_2$ up to classification of equivalence classes of irreducible elements in a certain Euclidean ring.

1. Introduction and description of the results. The Lie algebra $\mathfrak{sl}_2(\boldsymbol{C})$ is the only complex semisimple finite dimensional Lie algebra for which all simple (not necessarily finite dimensional) modules are classified (by Block, see [Bl]). This classification was later extended to all generalized Weyl algebras by Bavula (see [Ba1]). The approach via generalized Weyl algebra further allowed to obtain a classification of all supermodules over the Lie superalgebra $\mathfrak{osp}(1,2)$ (see [BO]). For the Lie superalgebra $\mathfrak{p}(2)$ a classification of simple modules can be deduced from [Se] and [Bl]. All the above classifications are given up to classification of equivalence classes of irreducible elements in a certain Euclidean ring (see [Ba1] or [Ma, Chapter 6] for details).

The queer Lie superalgebra $\mathfrak{q}_2$ is another Lie superalgebra, which is closely related to the Lie algebra $\mathfrak{sl}_2(\boldsymbol{C})$. It would be more correct to say that $\mathfrak{q}_2$ is closely related to the Lie algebra $\mathfrak{gl}_2(\boldsymbol{C})$. Namely, the superalgebra $\mathfrak{q}_2$ can be regarded as a kind of a "super-doubling" of the algebra $\mathfrak{gl}_2(\boldsymbol{C})$ in the sense that $\mathfrak{q}_2$ is a direct sum of one even and one odd copy of $\mathfrak{gl}_2(\boldsymbol{C})$. Although all queer Lie superalgebras $\mathfrak{q}_n$ are classical, their properties are rather different from the properties of other classical Lie superalgebras. For example, the Cartan subsuperalgebra of $\mathfrak{q}_n$ (in particular, of $\mathfrak{q}_2$) is not commutative, which makes the corresponding theory of weight supermodules more complicated, but also more interesting. In comparison to Lie algebras, there are several types of degenerations in the representation theory of Lie superalgebras, for example there are atypical, typical and strongly typical types of supermodules, which are also subdivided into regular and singular subtypes. All these degenerations make representation theory of Lie superalgebras much more complicated, but again, more interesting. Various classes of representations of $\mathfrak{q}_n$ were studied in [Pe2, PS, BK, Br, Go, Fr, FM, Or].

As $\mathfrak{gl}_2(\boldsymbol{C})$ is a direct sum of $\mathfrak{sl}_2(\boldsymbol{C})$ and a one-dimensional central subalgebra, the classification of simple $\mathfrak{sl}_2(\boldsymbol{C})$-modules extends to a classification of simple $\mathfrak{gl}_2(\boldsymbol{C})$-modules in the natural way. This naturally raises the question whether the classification of simple $\mathfrak{gl}_2(\boldsymbol{C})$-modules can be extended to a classification of simple $\mathfrak{q}_2$-supermodules. A strong evidence for a close connection between simple (super)modules over (any classical) Lie superalgebra and its even Lie subalgebra was obtained by Penkov in [Pe1]. In the present paper we address this

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question and prove the following main result, which, in particular, reduces the classification of all simple $\mathfrak{q}_2$-supermodules to that of all simple $\mathfrak{gl}_2$-modules (see Section 2 for the setup):

THEOREM 1. (i) *Every simple $\mathfrak{q}_2$-supermodule has finite length as a $\mathfrak{gl}_2$-module.*

(ii) *For every primitive ideal $\mathcal{I}$ of $U(\mathfrak{q}_2)$ there is an explicitly given primitive ideal $\mathbf{I}$ of $U(\mathfrak{gl}_2)$ and an explicitly given $U(\mathfrak{q}_2)$-$U(\mathfrak{gl}_2)$-bimodule B such that the functor $B\otimes_{U(\mathfrak{gl}_2)}-$ induces a bijection from the set of isomorphism classes of simple $\mathfrak{gl}_2$-modules annihilated by $\mathbf{I}$ to the set of isomorphism classes* (*up to parity change*) *of simple $\mathfrak{q}_2$-supermodules annihilated by $\mathcal{I}$.*

(iii) *For every strongly typical or atypical simple $\mathfrak{q}_2$-supermodule N we have $N\not\cong\Pi N$, where Π denotes the parity change functor. For every typical simple $\mathfrak{q}_2$-supermodule N, which is not strongly typical, we have $N\cong\Pi N$.*

The $U(\mathfrak{q}_2)$-$U(\mathfrak{gl}_2)$-bimodule B, which appears in the formulation of Theorem 1 is a Harish-Chandra $U(\mathfrak{q}_2)$-$U(\mathfrak{gl}_2)$-bimodule. Such bimodules are our main technical tool and a substantial part of the paper is devoted to developing the corresponding techniques. This also makes most of our arguments quite homological. In addition to Theorem 1 we also explicitly describe the *rough structure* of all simple $\mathfrak{q}_2$-supermodules, considered as $\mathfrak{gl}_2$-modules, in the sense of [KM, MS]. In the atypical and regular typical cases the rough structure coincides with the actual $\mathfrak{gl}_2$-module structure, whereas for singular typical supermodules the difference between these two structures is a potential direct sum of finitely many copies of one-dimensional $\mathfrak{gl}_2$-modules, which we are not able to determine explicitly.

The structure of the paper is as follows: We collect all necessary preliminaries, in particular on primitive ideals in $U(\mathfrak{q}_2)$ and on Harish-Chandra bimodules, in Section 2. In Section 3 we prove Theorem 1 and describe the rough structure of simple $\mathfrak{q}_2$-supermodules. We also outline an alternative approach to classification of simple weight supermodules via a localized superalgebra. In Subsections 3.10 through 3.12 we extend Theorem 1 to superalgebras $\mathfrak{pq}_2$, $\mathfrak{sq}_2$ and $\mathfrak{psq}_2$.

2. The superalgebra $\mathfrak{q}_2$ and $\mathfrak{q}_2$-supermodules.

2.1. The superalgebra $\mathfrak{q}_2$. For all undefined notions we refer the reader to [Fr]. Let $\boldsymbol{Z}$, $\boldsymbol{N}$ and $\boldsymbol{N}_0$ denote the sets of all, positive and nonnegative integers, respectively. Let $\boldsymbol{k}$ be an uncountable algebraically closed field of characteristic zero. Denote by $\mathbf{i}\in\boldsymbol{k}$ a fixed square root of -1. Let $\mathfrak{g}=\mathfrak{gl}_2(\boldsymbol{k})$ denote the general linear Lie algebra of 2×2 matrices over $\boldsymbol{k}$. The *queer Lie superalgebra* $\mathfrak{q}=\mathfrak{q}_2$ over $\boldsymbol{k}$ consists of all block matrices of the form

$$\mathbf{M}(A,B)=\begin{pmatrix} A & B\\ B & A\end{pmatrix},\quad A,B\in\mathfrak{gl}_2\,.$$

The even and the odd spaces $\mathfrak{q}_{\overline{0}}$ and $\mathfrak{q}_{\overline{1}}$ consist of the matrices $\mathbf{M}(A,0)$ and $\mathbf{M}(0,B)$, respectively, and we have $\mathfrak{q}=\mathfrak{q}_{\overline{0}}\oplus\mathfrak{q}_{\overline{1}}$. For a homogeneous element $X\in\mathfrak{q}$ we denote by $\overline{X}\in\boldsymbol{Z}/2\boldsymbol{Z}$ the degree of X. Then the Lie superbracket in $\mathfrak{q}$ is given by $[X,Y]=XY-(-1)^{\overline{X}\,\overline{Y}}YX$, where $X,Y\in\mathfrak{q}$ are homogeneous.

For $i, j \in \{1, 2\}$ let $E_{ij} \in \mathfrak{gl}_2$ denote the corresponding matrix unit. Set

$$E = \mathbf{M}(E_{12}, 0)\,,\quad F = \mathbf{M}(E_{21}, 0)\,,\quad H_1 = \mathbf{M}(E_{11}, 0)\,,\quad H_2 = \mathbf{M}(E_{22}, 0)\,;$$
$$\overline{E} = \mathbf{M}(0, E_{12})\,,\quad \overline{F} = \mathbf{M}(0, E_{21})\,,\quad \overline{H}_1 = \mathbf{M}(0, E_{11})\,,\quad \overline{H}_2 = \mathbf{M}(0, E_{22})\,.$$

We have the *Cartan subsuperalgebra* $\mathfrak{h}$ of $\mathfrak{q}$, which is the linear span of H_1, H_2, $\overline{H}_1$ and $\overline{H}_2$. The superalgebra $\mathfrak{h}$ inherits from $\mathfrak{q}$ the decomposition $\mathfrak{h} = \mathfrak{h}_{\overline{0}} \oplus \mathfrak{h}_{\overline{1}}$. Similarly we define the subsuperalgebra $\mathfrak{n} = \mathfrak{n}_{\overline{0}} \oplus \mathfrak{n}_{\overline{1}}$, which is generated by E (it spans $\mathfrak{n}_{\overline{0}}$) and $\overline{E}$ (it spans $\mathfrak{n}_{\overline{1}}$); and the subsuperalgebra $\mathfrak{m} = \mathfrak{m}_{\overline{0}} \oplus \mathfrak{m}_{\overline{1}}$, which is generated by F (it spans $\mathfrak{m}_{\overline{0}}$) and $\overline{F}$ (it spans $\mathfrak{m}_{\overline{1}}$). This leads to the standard triangular decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{n}$. The even Lie subalgebra $\mathfrak{q}_{\overline{0}}$, which is the linear span of E, F, H_1 and H_2, is identified with the Lie algebra $\mathfrak{g}$ in the obvious way.

Let $U(\mathfrak{q})$ and $U(\mathfrak{g})$ denote the universal enveloping (super)algebras of $\mathfrak{q}$ and $\mathfrak{g}$, respectively. Let $Z(\mathfrak{q})$ and $Z(\mathfrak{g})$ denote the centers of the algebras $U(\mathfrak{q})$ and $U(\mathfrak{g})$, respectively. The PBW theorem for Lie superalgebras (see [Ro]) asserts that $U(\mathfrak{q})$ is free of finite rank over $U(\mathfrak{g})$ both as a right and as a left module with the basis

$$\{a^{\varepsilon_1} b^{\varepsilon_2} b^{\varepsilon_3} d^{\varepsilon_4};\, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}\}\,,$$

where $\{a, b, c, d\} = \{\overline{E}, \overline{F}, \overline{H}_1, \overline{H}_2\}$.

2.2. Supermodules. If $\mathfrak{a} = \mathfrak{a}_{\overline{0}} \oplus \mathfrak{a}_{\overline{1}}$ is a Lie superalgebra over $\boldsymbol{k}$, then an $\mathfrak{a}$ *supermodule* is a $\boldsymbol{k}$-vector superspace $V = V_{\overline{0}} \oplus V_{\overline{1}}$ with a Lie superalgebra homomorphism from $\mathfrak{a}$ to the Lie superalgebra of all linear operators on V.

A homomorphism $\varphi : V \to W$ of two $\mathfrak{a}$-supermodules V and W is a homogeneous linear map *of degree zero* from V to W, which intertwines the actions of $\mathfrak{a}$ on V and W. We denote by $\mathfrak{a}$-sMod the category of all $\mathfrak{a}$-supermodules with the above defined morphisms. We will use the standard notation and denote morphisms in $\mathfrak{a}$-sMod by $\mathrm{Hom}_{\mathfrak{a}}$.

The category $\mathfrak{a}$-sMod is abelian with usual kernels and cokernels. Simple objects in $\mathfrak{a}$-sMod are *simple* $\mathfrak{a}$-supermodules, that is $\mathfrak{a}$-supermodules, which do not have proper subsupermodules. An example of a simple $\mathfrak{a}$-supermodule is the *trivial* supermodule $\boldsymbol{k} = \boldsymbol{k}_{\overline{0}}$, which is defined using the zero action of $\mathfrak{a}$.

Let Π denote the endofunctor of $\mathfrak{a}$-sMod, which changes the parity. For example, if $\boldsymbol{k}$ is the trivial $\mathfrak{a}$-supermodule from the previous paragraph (which is purely even), then $\Pi\boldsymbol{k}$ is purely odd. In particular, $\boldsymbol{k}$ and $\Pi\boldsymbol{k}$ are not isomorphic in $\mathfrak{a}$-sMod.

2.3. $\mathfrak{h}$-supermodules. Elements in $\mathfrak{h}_{\overline{0}}^*$ are called *weights* and are written $\lambda = (\lambda_1, \lambda_2)$ with respect to the basis $(\varepsilon_1, \varepsilon_2)$ of $\mathfrak{h}_{\overline{0}}^*$, which is dual to the basis (H_1, H_2) of $\mathfrak{h}_{\overline{0}}$. We set $\alpha = (1, -1) \in \mathfrak{h}_{\overline{0}}^*$ (the positive root of $\mathfrak{q}$).

For every $z \in \boldsymbol{k}$ we fix some $\sqrt{z}$. For $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{h}_{\overline{0}}^*$ we define the $\mathfrak{h}$-supermodule $\mathcal{V}(\lambda)$ as follows: The supermodule $\mathcal{V}(0)$ is the trivial supermodule $\boldsymbol{k}$. If $\lambda \neq 0$, then the supermodule $\mathcal{V}(\lambda)$ has a one-dimensional even space, spanned by v, and a one-dimensional odd space, spanned by $\overline{v}$, such that the action of the elements H_1, H_2, $\overline{H}_1$, $\overline{H}_1$ in the basis

$\{v, \overline{v}\}$ is given by the following formulae:

$$\begin{aligned} &H_1(v) = \lambda_1 v\,; \quad H_2(v) = \lambda_2 v\,; \quad \overline{H}_1(v) = \sqrt{\lambda_1}\,\overline{v}\,; \quad \overline{H}_2(v) = -\mathbf{i}\sqrt{\lambda_2}\,\overline{v}\,;\\ &H_1(\overline{v}) = \lambda_1 \overline{v}\,; \quad H_2(\overline{v}) = \lambda_2 \overline{v}\,; \quad \overline{H}_1(\overline{v}) = \sqrt{\lambda_1}\,v\,; \quad \overline{H}_2(\overline{v}) = \mathbf{i}\sqrt{\lambda_2}\,v\,. \end{aligned}$$

This is usually depicted as follows:

$$\begin{tikzcd} v \arrow[loop left, "{\substack{H_1 = \lambda_1 \\ H_2 = \lambda_2}}"] \arrow[r, bend left, "{\overline{H}_1 = \sqrt{\lambda_1},\ \overline{H}_2 = -\mathbf{i}\sqrt{\lambda_2}}"] & \overline{v} \arrow[loop right, "{\substack{H_1 = \lambda_1 \\ H_2 = \lambda_2}}"] \arrow[l, bend left, "{\overline{H}_1 = \sqrt{\lambda_1},\ \overline{H}_2 = \mathbf{i}\sqrt{\lambda_2}}"] \end{tikzcd}$$

LEMMA 2. (i) *Every simple* $\mathfrak{h}$*-supermodule is isomorphic to either* $\mathcal{V}(\lambda)$ *or* $\Pi\mathcal{V}(\lambda)$ *for some* $\lambda \in \mathfrak{h}_{\overline{0}}^*$.

(ii) *For* $\lambda \in \mathfrak{h}_{\overline{0}}^*$ *we have* $\mathcal{V}(\lambda) \cong \Pi\mathcal{V}(\lambda)$ *if and only if* $\lambda_1\lambda_2 = 0$ *and* $\lambda \neq 0$.

PROOF. This is well-known and follows directly from the theory of Clifford algebras. See for example [Go, Appendix A] or, alternatively, [Or] for full details and a direct computational approach. □

2.4. $\mathfrak{q}_2$-supermodules. A $\mathfrak{q}$-supermodule V is called a *weight* $\mathfrak{q}$-supermodule provided that the action of $\mathfrak{h}_{\overline{0}}$ on V is diagonalizable. This means that

$$V = \bigoplus_{\lambda \in \mathfrak{h}_{\overline{0}}^*} V_\lambda\,, \qquad \text{where} \qquad V_\lambda = \{v \in V;\, H_1(v) = \lambda_1 v,\ H_2(v) = \lambda_2 v\}\,.$$

Each V_λ is obviously an $\mathfrak{h}$-subsupermodule. If V_λ is finite dimensional, then from Lemma 2 it follows that V_λ has a finite composition series with subquotients isomorphic to either $\mathcal{V}(\lambda)$ or $\Pi\mathcal{V}(\lambda)$. The category of all weight $\mathfrak{q}$-supermodules is obviously closed with respect to taking any subquotients and direct sums.

For a simple $\mathfrak{h}$-supermodule V set $\mathfrak{n}V = 0$ and define the *Verma* $\mathfrak{q}$-supermodule $M(V)$ by

$$M(V) = U(\mathfrak{q}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n})} V\,.$$

The supermodule $M(V)$ is a weight supermodule with a unique simple top, which we will denote by $L(V)$. From the definitions it follows that $\Pi M(V) \cong M(\Pi V)$ and $\Pi L(V) \cong L(\Pi V)$.

A weight λ is called

- *integral* provided that $\lambda_1 - \lambda_2 \in \boldsymbol{Z}$;
- *strongly integral* provided that $\lambda_1, \lambda_2 \in \boldsymbol{Z}$;
- *dominant* provided that $\lambda_1 - \lambda_2 \in \boldsymbol{N}$;
- *regular* provided that $\lambda_1 \neq \lambda_2$;
- *typical* provided that $\lambda_1 + \lambda_2 \neq 0$;
- *strongly typical* provided that it is typical and $\lambda_1, \lambda_2 \neq 0$.

LEMMA 3. *The supermodules* $L(\mathcal{V}(\lambda))$ *and* $\Pi L(\mathcal{V}(\lambda))$, *where* $\lambda \in \mathfrak{h}_{\overline{0}}^*$ *is either zero or dominant, constitute an exhaustive list of all simple finite dimensional* $\mathfrak{h}$*-supermodules.*

PROOF. See, for example, [Pe2] or, alternatively, [Or] for full details. □

The inclusion $\mathfrak{g} \subset \mathfrak{q}$ of superalgebras (here $\mathfrak{g}$ is considered as a purely even superalgebra) gives rise to the usual restriction functor $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} : \mathfrak{q}\text{-sMod} \to \mathfrak{g}\text{-sMod}$. We identify $\mathfrak{g}$-Mod with the full subcategory of $\mathfrak{g}$-sMod, consisting of even supermodules. Hence we can compose $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}}$ with the projection on $\mathfrak{g}$-Mod and get the restriction functor $\mathrm{Res} : \mathfrak{q}\text{-sMod} \to \mathfrak{g}\text{-Mod}$. We denote by $M^{\mathfrak{g}}(\lambda)$ and $L^{\mathfrak{g}}(\lambda)$ the Verma $\mathfrak{g}$-module corresponding to λ and its unique simple quotient, respectively (see for example [Ma, Chapter 3]). Note that from the definitions it follows directly that $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} M(V) \cong \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} M(\Pi V)$ and $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} L(V) \cong \mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} L(\Pi V)$.

LEMMA 4. *For any simple* $\mathfrak{h}$*-supermodule* $V \not\cong \boldsymbol{k}, \Pi\boldsymbol{k}$ *we have isomorphisms* $\mathrm{Res}\, M(V) \cong \mathrm{Res}\, M(\Pi V)$ *and* $\mathrm{Res}\, L(V) \cong \mathrm{Res}\, L(\Pi V)$.

PROOF. Assume that $V = \mathcal{V}(\lambda)$ for a weight $\lambda \neq 0$. We consider the element $\overline{H}_1 + \overline{H}_2 \in U(\mathfrak{q})$. This element commutes with every element in $U(\mathfrak{g})$ and squares to $H_1 + H_2$. Hence for typical λ the multiplication with $\overline{H}_1 + \overline{H}_2$ defines a $\mathfrak{g}$-isomorphism between $M(V)_{\overline{0}}$ and $M(V)_{\overline{1}}$ (resp. $L(V)_{\overline{0}}$ and $L(V)_{\overline{1}}$) in both directions, and the claim follows.

For atypical $\lambda \neq 0$ from the definition of $V(\lambda)$ it follows that the multiplication with $\overline{H}_1 + \overline{H}_2$ defines a $\mathfrak{g}$-isomorphism from either $V(\lambda)_{\overline{0}}$ to $V(\lambda)_{\overline{1}}$ or vice versa. Using the definition of the Verma supermodule this isomorphism lifts to an isomorphism from $M(V)_{\overline{0}}$ to $M(V)_{\overline{1}}$ (or vice versa, respectively) and then induces an isomorphism from $L(V)_{\overline{0}}$ to $L(V)_{\overline{1}}$ (or vice versa, respectively). Hence the claim follows in this case as well. □

Later on we will need the following explicit description of the restrictions of the supermodules $M(V)$ and $L(V)$.

LEMMA 5. (i) *Let* $\lambda \in \mathfrak{h}_{\overline{0}}^*$, $\lambda \neq 0$, *and* $V \in \{\mathcal{V}(\lambda), \Pi\mathcal{V}(\lambda)\}$. *Then we have an isomorphism* $\mathrm{Res}\, M(V) \cong X$, *where for the* $\mathfrak{g}$*-module* X *there is a short exact sequence*

$$0 \to M^{\mathfrak{g}}(\lambda) \to X \to M^{\mathfrak{g}}(\lambda - \alpha) \to 0 .$$

(ii) *We have* $\mathrm{Res}\, M(\boldsymbol{k}) \cong M^{\mathfrak{g}}(0)$ *and* $\mathrm{Res}\, \Pi M(\boldsymbol{k}) \cong M^{\mathfrak{g}}(-\alpha)$.

PROOF. This follows from the definitions and the PBW theorem. □

LEMMA 6. *Let* $\lambda \in \mathfrak{h}_{\overline{0}}^*$ *and* $V \in \{\mathcal{V}(\lambda), \Pi\mathcal{V}(\lambda)\}$.

(i) $L(\mathcal{V}(0)) \cong \boldsymbol{k}$, $L(\Pi\mathcal{V}(0)) \cong \Pi\boldsymbol{k}$.

(ii) *If* $\lambda \neq 0$ *is atypical, then* $\mathrm{Res}\, L(V) \cong L^{\mathfrak{g}}(\lambda)$.

(iii) *If* $\lambda \neq 0$ *is typical and* $\lambda_1 - \lambda_2 = 1$, *then* $\mathrm{Res}\, L(V) \cong L^{\mathfrak{g}}(\lambda)$.

(iv) *If* $\lambda \neq 0$ *is typical, dominant and* $\lambda_1 - \lambda_2 \neq 1$, *then we have* $\mathrm{Res}\, L(V) \cong L^{\mathfrak{g}}(\lambda) \oplus L^{\mathfrak{g}}(\lambda - \alpha)$.

(v) *In all other cases we have* $L(V) \cong M(V)$.

PROOF. See, for example, [Pe2] or, alternatively, [Or] for full details. □

2.5. Induction and coinduction. The restriction functor Res defined in the previous subsection is obviously exact and hence admits both a left and a right adjoint. As usual, the left adjoint of Res is the induction functor

$$\mathrm{Ind} = U(\mathfrak{q}) \otimes_{U(\mathfrak{g})} - : \mathfrak{g}\text{-Mod} \to \mathfrak{a}\text{-sMod}\,.$$

As usual, the right adjoint of Res is the coinduction functor, however, by [Fr, Proposition 22], it is isomorphic to the induction functor Ind. In particular, Ind is exact (which also follows from the PBW theorem). From the PBW theorem it follows that the composition Res $\circ$ Ind is isomorphic to the endofunctor $\left(\bigwedge \mathfrak{q}_{\bar{1}} \otimes_{\boldsymbol{k}} -\right)_{\bar{0}}$ of $\mathfrak{g}$-Mod (here $\mathfrak{q}_{\bar{1}}$ is considered as a purely odd $\mathfrak{g}$-supermodule in the natural way).

2.6. Localization of $U(\mathfrak{q})$. Consider the multiplicative subset $X = \{1, F, F^2, F^3, \dots\}$ of $U(\mathfrak{q})$. The adjoint action of F on $U(\mathfrak{q})$ is obviously locally nilpotent. Hence X is an Ore subset of $U(\mathfrak{q})$ (see [Mat, Lemma 4.2]), and we denote by U' the Ore localization of $U(\mathfrak{q})$ with respect to X. According to [Mat, Lemma 4.3], there exists a unique family θ_z, $z \in \boldsymbol{k}$, of automorphisms of U', which are polynomials in z and satisfy the condition $\theta_z(u) = F^z u F^{-z}$, $u \in U'$, provided that $z \in \boldsymbol{Z}$. From the PBW theorem we have that F is not a zero divisor in $U(\mathfrak{q})$ and hence the canonical map $U(\mathfrak{q}) \to U'$ is injective. For $z \in \boldsymbol{k}$ denote by Θ_z the endofunctor of $\mathfrak{q}$-sMod defined for $M \in \mathfrak{q}$-sMod by

$$\Theta_z(M) = \{v \in U' \otimes_{U(\mathfrak{q})} M\,;\ E^k(v) = 0 \text{ for some } k \in \boldsymbol{N}\}\,,$$

where the left action of $U(\mathfrak{q})$ on U' is given by the multiplication with $\theta_z(u)$, $u \in U(\mathfrak{q})$. The functor Θ_z is defined on morphisms in the natural way.

2.7. Primitive ideals. An important rough invariant of a simple $\mathfrak{q}$-supermodule L is its annihilator $\mathrm{Ann}_{U(\mathfrak{q})}(L)$, which is a graded primitive ideal in $U(\mathfrak{q})$. Hence it is important to know all primitive ideals in $U(\mathfrak{q})$. For $\lambda \in \mathfrak{h}_{\bar{0}}^*$ we set

$$\begin{aligned}
\mathcal{I}_\lambda &= \mathrm{Ann}_{U(\mathfrak{q})}(L(\mathcal{V}(\lambda))) = \mathrm{Ann}_{U(\mathfrak{q})}(L(\Pi\mathcal{V}(\lambda)))\,,\\
\mathcal{J}_\lambda &= \mathrm{Ann}_{U(\mathfrak{q})}(L(\mathcal{V}(\lambda))_{\bar{0}}) = \mathrm{Ann}_{U(\mathfrak{q})}(L(\Pi\mathcal{V}(\lambda))_{\bar{1}})\,,\\
\mathcal{J}'_\lambda &= \mathrm{Ann}_{U(\mathfrak{q})}(L(\mathcal{V}(\lambda))_{\bar{1}}) = \mathrm{Ann}_{U(\mathfrak{q})}(L(\Pi\mathcal{V}(\lambda))_{\bar{0}})\,.
\end{aligned}$$

By definition, $\mathcal{I}_\lambda$ is an ideal of $U(\mathfrak{q})$, while $\mathcal{J}_\lambda$ and $\mathcal{J}'_\lambda$ are $U(\mathfrak{q})$-$U(\mathfrak{g})$-subbimodules of $U(\mathfrak{q})$. Obviously, we have $\mathcal{I}_\lambda \subset \mathcal{J}_\lambda$, $\mathcal{I}_\lambda \subset \mathcal{J}'_\lambda$, and $\mathcal{I}_\lambda = \mathcal{J}_\lambda \cap \mathcal{J}'_\lambda$.

LEMMA 7 ([Mu]). *The map $\lambda \mapsto \mathcal{I}_\lambda$ is a surjection from $\mathfrak{h}_{\bar{0}}^*$ onto the set of graded primitive ideals of $U(\mathfrak{q})$.*

We would need to know the fibers of the map from Lemma 7. They are given by the following:

PROPOSITION 8. *Let $\lambda, \mu \in \mathfrak{h}_{\bar{0}}^*$. Then the equality $\mathcal{I}_\lambda = \mathcal{I}_\mu$ holds only in the case when $\lambda = \mu$ or in the following cases*:

(a) $\lambda_1 + \lambda_2 \neq 0$, λ *is not integral and* $(\mu_1, \mu_2) = (\lambda_2, \lambda_1)$;

(b) $\lambda_1 + \lambda_2 = 0$, λ *is not integral and* $(\mu_1, \mu_2) = (\lambda_2 - 1, \lambda_1 + 1)$.

PROOF. The element $H_1 + H_2$ acts on $L(\mathcal{V}(\lambda))$ via the scalar $\lambda_1 + \lambda_2$ and on $L(\mathcal{V}(\mu))$ via the scalar $\mu_1 + \mu_2$. Therefore $H_1 + H_2 - \lambda_1 - \lambda_2 \in \mathcal{I}_\lambda$ and $H_1 + H_2 - \mu_1 - \mu_2 \in \mathcal{I}_\mu$. Hence the equality $\mathcal{I}_\lambda = \mathcal{I}_\mu$ implies $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$.

Assume now that $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 \neq 0$. Consider the quadratic Casimir element $\mathbf{c} = (H_1 - H_2 + 1)^2 + 4FE$. For $\nu \in \mathfrak{h}_{\bar{0}}^*$ we have $(\mathbf{c} - (\nu_1 - \nu_2 + 1)^2)L^{\mathfrak{g}}(\nu) = 0$ (see e.g. [Ma, Chapter 3]). Hence from Lemmata 6 and 4 it follows that the ideal $\mathcal{I}_\mu$ contains the element $(\mathbf{c} - (\nu_1 - \nu_2 + 1)^2)(\mathbf{c} - (\nu_1 - \nu_2 - 1)^2)$. This means that the equality $\mathcal{I}_\lambda = \mathcal{I}_\mu$ implies $(\mu_1, \mu_2) = (\lambda_2, \lambda_1)$ provided that $\lambda \neq \mu$.

Assume that $(\mu_1, \mu_2) = (\lambda_2, \lambda_1)$, $\lambda \neq \mu$, and both λ and μ are integral. Then without loss of generality we may assume that λ is dominant and regular. In this case $U(\mathfrak{q})/\mathcal{I}_\lambda$ is finite dimensional while $U(\mathfrak{q})/\mathcal{I}_\mu$ is not (by Lemma 3). Therefore the equality $\mathcal{I}_\lambda = \mathcal{I}_\mu$ implies that λ is not integral.

Assume now that $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = 0$. Then from Lemmata 6 and 4 it follows that $\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2$ and $\mathbf{c} - (\mu_1 - \mu_2 + 1)^2$ belong to $\mathcal{I}_\lambda$ and $\mathcal{I}_\mu$, respectively. Hence the equality $\mathcal{I}_\lambda = \mathcal{I}_\mu$ implies the equality $(\mu_1, \mu_2) = (\lambda_2 - 1, \lambda_1 + 1)$. Similarly to the previous paragraph one also show that the equality $\mathcal{I}_\lambda = \mathcal{I}_\mu$ implies that λ is not integral. This proves necessity of the conditions (a) and (b).

Let us now prove sufficiency of the condition (b). Assume that the weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$ is atypical and not integral. Set $z = -2(\lambda_1 - \lambda_2) - 2$ and $(\mu_1, \mu_2) = (\lambda_2 - 1, \lambda_1 + 1)$. Consider the $U(\mathfrak{q})$-supermodule $M = \Theta_z(L(\mathcal{V}(\lambda)))$. Then a direct calculation (see e.g. [Ma, Section 3.5]) implies that $\operatorname{Res} M \cong L^{\mathfrak{g}}(\mu)$ and hence we have either $M \cong L(\mathcal{V}(\mu))$ or $M \cong \Pi L(\mathcal{V}(\mu))$. In particular, $\mathcal{I}_\mu$ coincides with the annihilator of the supermodule M.

Let $N = U' \otimes_{U(\mathfrak{q})} L(\mathcal{V}(\lambda))$ be the usual induced U'-supermodule. Since U' is an Ore localization of U, we have $\operatorname{Ann}_{U'} N = U'\mathcal{I}_\lambda U' =: I$. As F acts injectively on $L(\mathcal{V}(\lambda))$ (since λ is not integral, see Lemma 6), the supermodule N contains $L(\mathcal{V}(\lambda))$ as a subspace and hence is not trivial. Thus I is a proper ideal of U', which is obviously stable with respect to all inner automorphisms of U' (i.e., $\theta_z(I) = F^z I F^{-z} \subset I$ for all $z \in \boldsymbol{Z}$).

Now let $u \in I$. The polynomiality of θ_z, $z \in \boldsymbol{k}$, says that there exist $k \in \boldsymbol{N}$ and $a_0, \ldots, a_k \in U'$ such that for all $z \in \boldsymbol{k}$

$$\theta_z(u) = \sum_{i=0}^{k} z^i a_i \, .$$

Set $b_j := \theta_j(u)$, $j = 0, 1, \ldots, k$. Then $b_j \in I$ by the previous paragraph. Inverting the nondegenerate Vandermonde matrix we can express all a_i's as linear combinations of b_j's showing that all a_i are in I. But then $\theta_z(u) \in I$ for all $z \in \boldsymbol{k}$. Thus $\theta_z(I) \subset I$ for all $z \in \boldsymbol{k}$. Hence if we twist the U'-action on N by θ_z (as in the definition of the functor Θ_z), the ideal I will still annihilate the resulting U'-supermodule. This yields the inclusion $\mathcal{I}_\lambda \subset \mathcal{I}_\mu$.

Because of the symmetry of μ and λ we obtain $\mathcal{I}_\lambda = \mathcal{I}_\mu$. This proves sufficiency of the condition (b). Sufficiency of the condition (a) is proved similarly. This completes the proof. $\square$

For $\lambda \in \mathfrak{h}_{\bar{0}}^*$ we denote by Irr_λ the set of classes of simple $U(\mathfrak{q})$-supermodules with annihilator $\mathcal{I}_\lambda$ up to isomorphism and parity change. We also denote by $\mathbf{I}_\lambda$ the annihilator (in $U(\mathfrak{g})$) of the module $L^{\mathfrak{g}}(\lambda)$ and by $\mathbf{Irr}_\lambda^{\mathfrak{g}}$ the set of isomorphism classes of simple $U(\mathfrak{g})$-modules with annihilator $\mathbf{I}_\lambda$. We refer the reader to [Bl], [Ba1] or [Ma, Chapter 6] for descriptions of $\mathbf{Irr}_\lambda^{\mathfrak{g}}$.

2.8. Category $\mathcal{O}$. Denote by $\mathcal{O}$ the full subcategory of $\mathfrak{q}$-sMod consisting of all finitely generated weight supermodules, the action of E on which is locally nilpotent. Denote also by $\mathbf{O}$ the corresponding category of $\mathfrak{g}$-modules. As $U(\mathfrak{q})$ is a finite extension of $\mathfrak{g}$ it follows that $\mathrm{Res}\,\mathcal{O} \subset \mathbf{O}$. It then follows that $\mathrm{Ind}\,\mathbf{O} \subset \mathcal{O}$. As every object in $\mathbf{O}$ has finite length (see [Ma, Chapter 5]), every object in $\mathcal{O}$ has finite length as well.

For $\lambda \in \mathfrak{h}_{\bar{0}}^*$ we denote by $\mathcal{O}_\lambda$ and $\mathbf{O}_\lambda$ the full subcategories of $\mathcal{O}$ and $\mathbf{O}$, respectively, consisting of all (super)modules M satisfying the condition that $M_\mu \neq 0$, $\mu \in \mathfrak{h}_{\bar{0}}^*$, implies $(\lambda_1 - \lambda_2) - (\mu_1 - \mu_2) \in \mathbf{Z}$. The categories $\mathcal{O}$ and $\mathbf{O}$ then decompose into a direct sum of $\mathcal{O}_\lambda$'s and $\mathbf{O}_\lambda$'s, respectively.

2.9. Harish-Chandra bimodules. Let $\mathfrak{q}$-Mod-$\mathfrak{g}$ denote the category of all $U(\mathfrak{q})$-$U(\mathfrak{g})$-(super)bimodules. A $U(\mathfrak{q})$-$U(\mathfrak{g})$-bimodule is called a *Harish-Chandra* bimodule if it is finitely generated and decomposes into a direct sum of simple finite dimensional $\mathfrak{g}$-modules (each occurring with finite multiplicity) with respect to the adjoint action of $\mathfrak{g}$. We denote by $\mathcal{H}$ the full subcategory of $\mathfrak{q}$-Mod-$\mathfrak{g}$ consisting of all Harish-Chandra bimodules. As any Harish-Chandra $U(\mathfrak{q})$-$U(\mathfrak{g})$-bimodule is a Harish-Chandra $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule by restriction, it has finite length (even as $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule), see [BG, 5.7].

A typical way to produce Harish-Chandra bimodules is the following: Let X be a $\mathfrak{g}$-module and Y be a $\mathfrak{q}$-supermodule. Then $\mathrm{Hom}_{\Bbbk}(X, Y)$ has a natural structure of a $U(\mathfrak{q})$-$U(\mathfrak{g})$-(super)bimodule. Denote by $\mathcal{L}(X, Y)$ the subspace of $\mathrm{Hom}_{\Bbbk}(X, Y)$, consisting of all elements, on which the adjoint action of $U(\mathfrak{g})$ is locally finite. As $U(\mathfrak{q})$ is a finite extension of $U(\mathfrak{g})$, from [Ja, Kapitel 6] it follows that $\mathcal{L}(X, Y)$ is in fact a $U(\mathfrak{q})$-$U(\mathfrak{g})$-subbimodule of $\mathrm{Hom}_{\Bbbk}(X, Y)$. In particular, $\mathcal{L}(X, Y)$ is a $U(\mathfrak{g})$-bimodule by restriction. Under some natural conditions (for examples when both X and Y are simple (super)modules) one easily verifies that $\mathcal{L}(X, Y)$ is a Harish-Chandra bimodule. For $\lambda \in \mathfrak{h}_{\bar{0}}^*$ denote by $\mathcal{H}_\lambda^1$ the full subcategory of $\mathcal{H}$, which consists of all bimodules B satisfying $B(\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2) = 0$. Obviously, $\mathcal{H}_\lambda^1$ is an abelian category with finite direct sums. We will need the following generalization of [BG, Theorem 5.9]:

PROPOSITION 9. *Let $\lambda \in \mathfrak{h}_{\bar{0}}^*$ be such that $\lambda_1 - \lambda_2 \notin \{-1, -2, -3, \dots\}$. Then the functors*

$$\mathcal{H}_\lambda^1 \underset{\mathrm{G}:=\mathcal{L}(M^{\mathfrak{g}}(\lambda),-)}{\overset{\mathrm{F}:=-\otimes_{U(\mathfrak{g})} M^{\mathfrak{g}}(\lambda)}{\rightleftarrows}} \mathcal{O}_\lambda$$

are mutually inverse equivalences of categories.

PROOF. From [Ja, 6.22] it follows that (F, G) is an adjoint pair of functors. As every object in both $\mathcal{H}_\lambda^1$ and $\mathcal{O}_\lambda$ has finite length (see above) by standard arguments (see e.g. proof of [Ma, Theorem 3.7.3]) it follows that it is enough to check that both functors F and G send simple objects to simple objects.

Under our assumptions we have that $M^{\mathfrak{g}}(\lambda)$ is a dominant regular projective module in **O** (see [Ma, Chapter 5]), in particular, on the level of $\mathfrak{g}$ modules the functors $\mathcal{L}(M^{\mathfrak{g}}(\lambda), -)$ and $-\otimes_{U(\mathfrak{g})} M^{\mathfrak{g}}(\lambda)$ are mutually inverse equivalences of categories by [BG, Theorem 5.9]. This means, in particular, that if $L \in \mathcal{O}_\lambda$ is simple, then the character of the supermodule FG L coincides with the character of L. From the natural transformation FG $\to \mathrm{Id}_{\mathcal{O}_\lambda}$ (given by adjunction) we thus obtain that FG $L \cong L$. Similarly one shows that GF sends simple objects from $\mathcal{H}^1_\lambda$ to simple objects in $\mathcal{O}_\lambda$. It follows that both F and G send simple objects to simple objects, which implies the claim of the proposition. □

From Proposition 9 we immediately get the following corollary.

COROLLARY 10. *Let $\lambda, \mu \in \mathfrak{h}^*_{\bar{0}}$ be such that $\mu_1 - \mu_2 \notin \{-1, -2, \dots\}$ and B be a simple Harish-Chandra bimodule satisfying the conditions $\mathcal{I}_\lambda B = B(\mathbf{c} - (\mu_1 - \mu_2 + 1)^2) = 0$. Then we have $(\lambda_1 - \lambda_2) - (\mu_1 - \mu_2) \in \mathbf{Z}$ and $B \cong \mathcal{L}(M^{\mathfrak{g}}(\mu), L(\mathcal{V}(\lambda)))$ or $B \cong \mathcal{L}(M^{\mathfrak{g}}(\mu), L(\Pi\mathcal{V}(\lambda)))$.*

REMARK 11. Let $\lambda = (-1/2, 1/2)$ and B be a simple Harish-Chandra bimodule such that $\mathcal{I}_\lambda B = B(\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2) = 0$. Then from Lemma 6 and [BG, Theorem 5.9] it follows that $B \cong \mathcal{L}(M^{\mathfrak{g}}(\lambda), L(\mathcal{V}(\lambda)))$ or $B \cong \mathcal{L}(M^{\mathfrak{g}}(\lambda), L(\Pi\mathcal{V}(\lambda)))$ as well.

For $\lambda \in \mathfrak{h}^*_{\bar{0}}$ define

$$\mathcal{L}_\lambda = \mathcal{L}(L^{\mathfrak{g}}(\lambda), L(\mathcal{V}(\lambda)))\,, \quad \mathcal{M}_\lambda = \mathcal{L}(L^{\mathfrak{g}}(\lambda - \alpha), L(\mathcal{V}(\lambda)))$$

and

$$\mathcal{L}'_\lambda = \begin{cases} \mathcal{L}(M^{\mathfrak{g}}(\lambda), L(\mathcal{V}(\lambda)))\,, & \lambda_1 - \lambda_2 \notin \{-2, -3, -4, \dots\}\,; \\ \mathcal{L}(M^{\mathfrak{g}}(-\lambda - 2), L(\mathcal{V}(\lambda)))\,, & \text{otherwise}\,. \end{cases}$$

The following lemma is one of our most important technical tools.

LEMMA 12. *Let $\lambda \in \mathfrak{h}^*_{\bar{0}}$.*

(i) *If $\lambda_1 - \lambda_2 \notin \mathbf{Z} \setminus \{-1\}$, then $\mathcal{L}_\lambda \cong \mathcal{L}'_\lambda$.*

(ii) *If $\lambda_1 - \lambda_2 \in \{-2, -3, -4, \dots\}$, then for any simple infinite dimensional $\mathfrak{g}$-module X there is a short exact sequence*

(1) $$0 \to \mathrm{Ker} \to \mathcal{L}'_\lambda \otimes_{U(\mathfrak{g})} X \to \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} X \to 0$$

of $\mathfrak{q}$-supermodules, where Ker *is finite dimensional.*

PROOF. The statement (i) is trivial for in this case we have an isomorphism $M^{\mathfrak{g}}(\lambda) \cong L^{\mathfrak{g}}(\lambda)$ (see [Ma, Chapter 3]).

To prove (ii) we assume that $\lambda_1 - \lambda_2 \in \{-2, -3, -4, \dots\}$. Applying the left exact functor $\mathcal{L}(-, \mathcal{V}(\lambda))$ to the short exact sequence

(2) $$0 \to L^{\mathfrak{g}}(\lambda) \to M^{\mathfrak{g}}(-\lambda - \alpha) \to L^{\mathfrak{g}}(-\lambda - \alpha) \to 0\,,$$

we obtain the exact sequence

$$0 \to \mathcal{L}(L^{\mathfrak{g}}(-\lambda-\alpha), \mathcal{V}(\lambda)) \to \mathcal{L}'_\lambda \to \mathcal{L}_\lambda\,.$$

The socle of the $\mathfrak{g}$-module $\mathrm{Res}\,\mathcal{V}(\lambda)$ does not contain finite dimensional submodules because of our choice of λ and Lemma 6. As $L^{\mathfrak{g}}(-\lambda-\alpha)$ is finite dimensional, for every finite dimensional $\mathfrak{g}$-module V we thus have

$$\mathrm{Hom}_{\mathfrak{g}}(L^{\mathfrak{g}}(-\lambda-\alpha)\otimes_{\boldsymbol{k}} V, \mathrm{Res}\,\mathcal{V}(\lambda)) = 0\,. \tag{3}$$

Hence $\mathcal{L}(L^{\mathfrak{g}}(-\lambda-\alpha), \mathcal{V}(\lambda)) = 0$ by [Ja, 6.8]. This equality implies the existence of a natural inclusion $\mathcal{L}'_\lambda \hookrightarrow \mathcal{L}_\lambda$. Let us denote by B the cokernel of this inclusion.

Let $\mu \in \mathfrak{h}_{\bar{0}}^*$ be such that $L^{\mathfrak{g}}(\mu)$ is finite dimensional. If $\mu_1+\mu_2 \neq 0$, then the action of the central element H_1+H_2 on the modules $L^{\mathfrak{g}}(\lambda)$ and $M^{\mathfrak{g}}(-\lambda-\alpha)$ on one side and the module $V \otimes_{\boldsymbol{k}} \mathrm{Res}\,\mathcal{V}(\lambda)$ on the other side do not agree, and hence such V occurs as a direct summand (with respect to the adjoint action of $\mathfrak{g}$) in neither $\mathcal{L}'_\lambda$ nor $\mathcal{L}_\lambda$.

If $\mu_1+\mu_2 = 0$ and $\mu_1-\mu_2 = m \in \boldsymbol{N}_0$ is big enough, then the module $L^{\mathfrak{g}}(\mu)\otimes_{\boldsymbol{k}} L^{\mathfrak{g}}(-\lambda-\alpha)$ decomposes into a direct sum of $L^{\mathfrak{g}}(\nu)$, where $\nu \in \{-\lambda-\alpha+\mu, -\lambda-2\alpha+\mu, \dots, \mu+\lambda+\alpha\}$ (see e.g. [Ma, Section 1.4]). For all μ big enough the central characters of all these $L^{\mathfrak{g}}(\nu)$ are different from the central characters of all simple subquotients in $\mathrm{Res}\,\mathcal{V}(\lambda)$. Hence from [Ja, 6.8] it follows that $L^{\mathfrak{g}}(\mu)$ occurs with the same multiplicity in $\mathcal{L}'_\lambda$ and $\mathcal{L}_\lambda$ with respect to the adjoint action. This implies that the bimodule B is finite dimensional.

Now we claim that

$$B \otimes_{U(\mathfrak{g})} X = 0\,. \tag{4}$$

Assume that this is not the case. Since X is simple and B is finite dimensional, the module $B \otimes_{U(\mathfrak{g})} X$ is holonomic and hence has finite length. Let N be some simple quotient of $B \otimes_{U(\mathfrak{g})} X$. Then, by adjunction, we have

$$0 \neq \mathrm{Hom}_{\mathfrak{q}}(B \otimes_{U(\mathfrak{g})} X, N) = \mathrm{Hom}_{\mathfrak{g}}(X, \mathrm{Hom}_{\mathfrak{q}}(B, N))\,.$$

If N is finite dimensional, then $\mathrm{Hom}_{\mathfrak{q}}(B, N)$ is finite dimensional and thus $\mathrm{Hom}_{\mathfrak{g}}(X, \mathrm{Hom}_{\mathfrak{q}}(B, N)) = 0$, which is a contradiction. If N is infinite dimensional, then $\mathrm{Hom}_{\mathfrak{q}}(B, N) = 0$ as B is finite dimensional, which is again a contradiction.

Applying $-\otimes_{U(\mathfrak{g})} X$ to the short exact sequence of bimodules

$$0 \to \mathcal{L}'_\lambda \to \mathcal{L}_\lambda \to B \to 0$$

and using (4) and the right exactness of the tensor product, we obtain an exact sequence

$$\mathrm{Tor}_1^{U(\mathfrak{g})}(B, X) \to \mathcal{L}'_\lambda \otimes_{U(\mathfrak{g})} X \to \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} X \to 0\,.$$

Now we just recall that $U(\mathfrak{g})$ has finite global dimension. So we can take a minimal finite free resolution of X (where each component will be finitely generated as $U(\mathfrak{g})$ is noetherian), tensor it with our finite dimensional bimodule B and obtain a finite complex of finite dimensional vector spaces. All torsion groups from B to X are homologies of this complex, hence finite dimensional. The claim (ii) follows, which completes the proof of the proposition. $\square$

3. Classification of simple $\mathfrak{q}_2$-supermodules. In this section we prove Theorem 1 and some related results. All this is divided into separate steps, organized as subsections.

3.1. All simple $\mathfrak{q}$-supermodules are subsupermodules of some induced supermodules. We start with the following observation:

PROPOSITION 13. *Let N be a simple $\mathfrak{q}$-supermodule. Then there exists a simple $\mathfrak{g}$-module L such that either N or ΠN is a subsupermodule of the supermodule* $\mathrm{Ind}\, L$.

PROOF. The supermodule N is simple, in particular, it is finitely generated. As $U(\mathfrak{q})$ is a finite extension of $U(\mathfrak{g})$, it follows that the $\mathfrak{g}$-module $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N$ is finitely generated as well. Let $\{v_1, \dots, v_k\}$ be some minimal generating system of $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N$ and

$$M = \mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N/(U(\mathfrak{g})\{v_2, \dots, v_k\}) .$$

Then the module M is non-zero because of the minimality of the system $\{v_1, \dots, v_k\}$. Moreover, M is generated by one element v (the image of v_1 in M). Let $I = \{u \in U(\mathfrak{g}); u(v) = 0\}$. Then I is a left ideal of $U(\mathfrak{g})$ different from $U(\mathfrak{g})$ and $M \cong U(\mathfrak{g})/I$. Let J be some maximal left ideal of $U(\mathfrak{g})$, containing I (such J exists by Zorn's lemma). Consider the module $L = M/JM$, which is a simple quotient of M by construction. Changing, if necessary, the parity of N, we may assume that L is even. As M is a quotient of $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N$, we obtain that L is a simple quotient of either $\mathrm{Res}\, N$ or $\mathrm{Res}\, \Pi N$. We consider the first case and the second one is dealt with by similar arguments.

Using the adjunction between Res and Ind (see Subsection 2.5) we have

$$0 \neq \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Res}\, N, L) = \mathrm{Hom}_{\mathfrak{q}}(N, \mathrm{Ind}\, L) .$$

The claim of the proposition follows. □

3.2. Finite length of the restriction. Now we are ready to prove the first part of Theorem 1:

PROPOSITION 14. *Let N be a simple $\mathfrak{q}$-supermodule. Then the $\mathfrak{g}$-module* $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N$ *has finite length.*

PROOF. By Proposition 13, there exists a simple $\mathfrak{g}$-module L such that either N or ΠN is a subsupermodule of the supermodule $\mathrm{Ind}\, L$. Changing, if necessary, the parity of N we may assume that N is a subsupermodule of $\mathrm{Ind}\, L$. Hence $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N$ is a submodule of the $\mathfrak{g}$-module $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}} \circ \mathrm{Ind}\, L$. The latter is isomorphic to the $\mathfrak{g}$-module $\bigwedge \mathfrak{q}_{\bar{1}} \otimes_{\mathbb{k}} L$ (see Subsection 2.5).

From the classification of all simple $\mathfrak{g}$-modules (see [Bl, Ba1] or [Ma, Chapter 6]) we obtain that the module L is holonomic in the sense of [Ba3]. As tensoring with the finite dimensional module $\bigwedge \mathfrak{q}_{\bar{1}}$ cannot increase the Gelfand-Kirillov dimension of the module L, it follows that the module $\bigwedge \mathfrak{q}_{\bar{1}} \otimes_{\mathbb{k}} L$ is holonomic as well. Hence it has finite length (see [Ba3, Section 3]). This implies that the submodule $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, N$ has finite length as well. The proof is complete. □

3.3. Finite-dimensional supermodules. From Lemma 7 and Proposition 8 we have a complete description of all different primitive ideals in $U(\mathfrak{q})$. We approach our classification via a case-by-case analysis and start with the easiest case of finite dimensional supermodules.

LEMMA 15. *Let $\lambda \in \mathfrak{h}_{\overline{0}}^*$ be either zero or dominant. Then $L(\mathcal{V}(\lambda))$ is a unique (up to isomorphism and parity change) simple supermodule with annihilator $\mathcal{I}_\lambda$.*

PROOF. From Lemma 3 it follows that under our assumptions we have $\dim U(\mathfrak{q})/\mathcal{I}_\lambda < \infty$. Therefore any simple supermodule L with annihilator $\mathcal{I}_\lambda$ must be finite dimensional. Now the claim follows from Lemma 3 and Proposition 8. □

3.4. Atypical supermodules. Surprisingly enough the second easiest case is that of atypical supermodules. This is due to the easiest possible structure of atypical simple highest weight supermodules (see Lemma 6). Note that the case of all (in particular, atypical) finite dimensional supermodules has already been taken care of in Subsection 3.3. In the present subsection we consider the case of atypical infinite dimensional supermodules and classify such supermodules via certain simple $U(\mathfrak{g})$-modules. We also explicitly describe the restriction of each simple infinite dimensional atypical $U(\mathfrak{q})$-supermodule to $U(\mathfrak{g})$.

PROPOSITION 16. *Let $t \in \boldsymbol{k} \setminus \frac{1}{2}\boldsymbol{N}_0$ and $\lambda = (t, -t)$.*

(i) *The following correspondence is a bijection between $\mathbf{Irr}_\lambda^{\mathfrak{g}}$ and Irr_λ:*

$$\begin{aligned} \mathbf{Irr}_\lambda^{\mathfrak{g}} &\leftrightarrow \mathrm{Irr}_\lambda \\ L &\mapsto \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \end{aligned}$$

(ii) *For any $L \in \mathbf{Irr}_\lambda^{\mathfrak{g}}$ the module $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L$ is semi-simple and we have $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \cong L \oplus L$ and $\mathrm{Res}\, \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \cong L$.*

PROOF. We start with the statement (ii). We observe that $L^{\mathfrak{g}}(\lambda) = M^{\mathfrak{g}}(\lambda)$ because of our restrictions on λ (see e.g. [Ma, Chapter 3]). Consider $\mathcal{L}_\lambda$ as a $U(\mathfrak{g})$-bimodule. We have

$$\begin{aligned} \mathcal{L}_\lambda &:= \mathcal{L}(L^{\mathfrak{g}}(\lambda), L(\mathcal{V}(\lambda))) \\ &\cong \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda) \oplus L^{\mathfrak{g}}(\lambda)) && \text{(Lemmata 6 and 4)} \\ &\cong \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda)) \oplus \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda)) && \text{([Ja, 6.8])} \\ &\cong U(\mathfrak{g})/\mathbf{I}_\lambda \oplus U(\mathfrak{g})/\mathbf{I}_\lambda && \text{([Ja, 7.25])}. \end{aligned} \tag{5}$$

As for any $L \in \mathbf{Irr}_\lambda^{\mathfrak{g}}$ we have $\mathbf{I}_\lambda L = 0$, we deduce that

$$\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \cong (U(\mathfrak{g})/\mathbf{I}_\lambda \oplus U(\mathfrak{g})/\mathbf{I}_\lambda) \otimes_{U(\mathfrak{g})} L \cong L \oplus L,$$

proving (ii).

To prove (i) we first take some $L \in \mathbf{Irr}_\lambda^{\mathfrak{g}}$ and consider the supermodule $N = \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L$. Then from the definition of $\mathcal{L}_\lambda$ we obtain $\mathcal{I}_\lambda N = 0$. If N is not simple, then from (ii) it follows that either $N_{\overline{0}}$ or $N_{\overline{1}}$ is a simple submodule of N. Assume that $N_{\overline{0}}$ is a submodule (for $N_{\overline{1}}$ the arguments are similar). Then $U(\mathfrak{q})N_{\overline{0}} \subset N_{\overline{0}}$, which implies that $N_{\overline{0}}$ is annihilated by $U(\mathfrak{q})_{\overline{1}}$.

If $\mu \in \mathfrak{h}_{\overline{0}}^*$ is nonzero, then from Subsection 2.3 it follows that at least one of the elements $\overline{H}_1$ or $\overline{H}_2$ does not annihilate $L(\mathcal{V}(\lambda))$. Therefore $\mathcal{I}_0$ is the only primitive ideal of $U(\mathfrak{q})$, which contains $U(\mathfrak{q})_{\overline{1}}$. From Lemma 15 it thus follows that $N_{\overline{0}}$ must be the trivial $U(\mathfrak{q})$-supermodule, which is impossible as $\lambda \neq 0$. This contradiction shows that the supermodule N is simple and hence $N \in \mathrm{Irr}_\lambda$. In particular, the correspondence from (i) is well-defined. From (ii) it follows immediately that it is even injective.

So, to complete the proof we have to show that every $N \in \mathrm{Irr}_\lambda$ has the form $\mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L$ for some $L \in \mathbf{Irr}_\lambda^{\mathfrak{g}}$. By Proposition 14, the $U(\mathfrak{g})$-module $\mathrm{Res}\, N$ has finite length. Let L be a simple submodule of this module $\mathrm{Res}\, N$.

We have $\mathcal{I}_\lambda L = 0$. From Lemma 6 and [Ma, Chapter 3] it follows that $\mathbf{c}-(\lambda_1-\lambda_2+1)^2$ is in $\mathcal{I}_\lambda$ and thus $\mathbf{c}-(\lambda_1-\lambda_2+1)^2$ annihilates L, which yields that $\mathbf{I}_\lambda L = 0$.

Now we claim that L cannot be finite dimensional. Indeed, if L would be finite dimensional, using the adjunction between Ind and Res we would have

$$0 \neq \mathrm{Hom}_{\mathfrak{g}}(L, \mathrm{Res}\, N) = \mathrm{Hom}_{\mathfrak{q}}(\mathrm{Ind}\, L, N)\,.$$

The supermodule $\mathrm{Ind}\, L$, which is isomorphic to the module $\bigwedge \mathfrak{q}_{\overline{1}} \otimes_{\boldsymbol{k}} L$ as $\mathfrak{g}$-module, would thus be finite dimensional, which would imply that N is finite dimensional as well. This contradicts Lemma 15 and thus L is infinite dimensional. This implies $L \in \mathbf{Irr}_\lambda^{\mathfrak{g}}$ (see [Ma, Chapter 3]).

Consider the bimodule $\mathcal{L}(L, N)$, which is a Harish-Chandra bimodule by [Ja, 6.8] and [BG, 5.7]. The inclusion $L \hookrightarrow N$ is a $\mathfrak{g}$-homomorphism and hence is annihilated by the adjoint action of $\mathfrak{g}$. Hence this inclusion is a nontrivial element of $\mathcal{L}(L, N)$, which shows that $\mathcal{L}(L, N) \neq 0$.

From the above and $\mathcal{I}_\lambda N = 0$ we have $\mathcal{I}_\lambda \mathcal{L}(L, N) = \mathcal{L}(L, N)\mathbf{I}_\lambda = 0$. As $\mathcal{L}(L, N)$ has finite length, it thus must contain a simple Harish-Chandra subbimodule B such that $\mathcal{I}_\lambda B = B\mathbf{I}_\lambda = 0$. Changing, if necessary, the parity of N and using Corollary 10 and Remark 11 we get that $B \cong \mathcal{L}'_\lambda$.

This implies the following:

$$\begin{aligned}
0 &\neq \mathrm{Hom}_{U(\mathfrak{q})\text{-}U(\mathfrak{g})}(\mathcal{L}'_\lambda, \mathcal{L}(L, N)) \\
&\subset \mathrm{Hom}_{U(\mathfrak{q})\text{-}U(\mathfrak{g})}(\mathcal{L}'_\lambda, \mathrm{Hom}_{\boldsymbol{k}}(L, N)) \quad && \text{(definition of } \mathcal{L}(L, N)) \\
&\cong \mathrm{Hom}_{\mathfrak{q}}(\mathcal{L}'_\lambda \otimes_{U(\mathfrak{g})} L, N) && \text{(adjunction)}\,.
\end{aligned}$$

At the same time, applying $\mathrm{Hom}_{\mathfrak{q}}(-, N)$ to the short exact sequence (1) we obtain the exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{q}}(\mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L, N) \to \mathrm{Hom}_{\mathfrak{q}}(\mathcal{L}'_\lambda \otimes_{U(\mathfrak{g})} L, N) \to \mathrm{Hom}_{\mathfrak{q}}(\mathrm{Ker}, N)\,.$$

As Ker is finite dimensional, while N is simple infinite dimensional, we get $\mathrm{Hom}_{\mathfrak{q}}(\mathrm{Ker}, N) = 0$ and hence

$$\mathrm{Hom}_{\mathfrak{q}}(\mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L, N) \cong \mathrm{Hom}_{\mathfrak{q}}(\mathcal{L}'_\lambda \otimes_{U(\mathfrak{g})} L, N) \neq 0\,.$$

As we already know that the supermodule $\mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L$ is simple (see the first part of the proof above), we conclude that $\mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \cong N$ using Schur's lemma. This completes the proof of the claim (i) and of the whole proposition. $\square$

After Proposition 16 it is natural to ask whether the bimodule $\mathcal{L}_\lambda$ can be described explicitly. We will actually need this description later on. This is done in the following:

LEMMA 17. *Let* λ *be as in Proposition* 16. *The* $U(\mathfrak{q})$-$U(\mathfrak{g})$-*bimodule* $\mathcal{L}_\lambda$ *is isomorphic* (*as a bimodule*) *either to the* $U(\mathfrak{q})$-$U(\mathfrak{g})$-*bimodule* $U(\mathfrak{q})/\mathcal{J}_\lambda$ *or to the* $U(\mathfrak{q})$-$U(\mathfrak{g})$-*bimodule* $U(\mathfrak{q})/\mathcal{J}'_\lambda$ (*in which the right* $U(\mathfrak{g})$-*structure is naturally given by the right multiplication*).

PROOF. We have either $\sqrt{t}-\mathbf{i}\sqrt{-t}\neq 0$ or $\sqrt{t}+\mathbf{i}\sqrt{-t}\neq 0$ (for otherwise $t=0$, which contradicts our choice of λ). We consider the first case $\sqrt{t}-\mathbf{i}\sqrt{-t}\neq 0$.

By Lemma 6 there exists a $\mathfrak{g}$-monomorphism $\varphi : L^{\mathfrak{g}}(\lambda)\to L(\mathcal{V}(\lambda))_{\overline{0}}$. As φ commutes with all elements from $\mathfrak{g}$, the adjoint action of $\mathfrak{g}$ on φ is zero and hence $\varphi\in\mathcal{L}_\lambda$. Observe that $\overline{H}_1+\overline{H}_2$ commutes with all elements of $\mathfrak{g}$. As $\sqrt{t}-\mathbf{i}\sqrt{-t}\neq 0$, from Subsection 2.3 it follows that multiplication with the element $\overline{H}_1+\overline{H}_2$ defines a nonzero homomorphism from $L(\mathcal{V}(\lambda))_{\overline{0}}$ to $L(\mathcal{V}(\lambda))_{\overline{1}}$. From Lemma 6 we even get that this homomorphism is an isomorphism. We have

$$\begin{aligned}
\mathcal{L}_\lambda &\supset U(\mathfrak{q})\varphi \\
&\supset U(\mathfrak{g})\varphi\oplus U(\mathfrak{g})(\overline{H}_1+\overline{H}_2)\varphi &&\text{(using grading)}\\
&= U(\mathfrak{g})\varphi\oplus(\overline{H}_1+\overline{H}_2)U(\mathfrak{g})\varphi &&\text{(by the above)}\\
&\cong \mathcal{L}(L^{\mathfrak{g}}(\lambda),L^{\mathfrak{g}}(\lambda))\oplus\mathcal{L}(L^{\mathfrak{g}}(\lambda),L^{\mathfrak{g}}(\lambda)) &&\text{([Ja, 7.25])}\\
&\cong \mathcal{L}_\lambda &&\text{(Lemma 6)}\,.
\end{aligned}$$

Hence $U(\mathfrak{q})\varphi\cong U(\mathfrak{q})/\mathcal{J}_\lambda\cdot\varphi\cong\mathcal{L}_\lambda$ (note that $\mathcal{J}_\lambda\varphi=0$ by the definition of $\mathcal{J}_\lambda$). Thus we obtain that the map

$$\begin{aligned}
U(\mathfrak{q})/\mathcal{J}_\lambda &\to \mathcal{L}_\lambda\\
u+\mathcal{J}_\lambda &\mapsto u\cdot\varphi
\end{aligned}$$

is bijective. Since φ is a $\mathfrak{g}$-homomorphism, this map is a homomorphism of $U(\mathfrak{q})$-$U(\mathfrak{g})$-bimodules.

The case $\sqrt{t}+\mathbf{i}\sqrt{-t}\neq 0$ reduces to the case $\sqrt{t}-\mathbf{i}\sqrt{-t}\neq 0$ by changing parity (by Subsection 2.3) and hence leads to the appearance of the ideal $\mathcal{J}'_\lambda$ instead of $\mathcal{J}_\lambda$. This completes the proof. □

LEMMA 18. *Let* λ *be as in Proposition* 16.

(i) *If* $\sqrt{t}-\mathbf{i}\sqrt{-t}\neq 0$, *then the ideal* $\mathcal{J}_\lambda$ *is generated* (*as a* $U(\mathfrak{q})$-$U(\mathfrak{g})$-*bimodule*) *by* $\mathbf{I}_\lambda$ *and the element* $\overline{H}_1-\overline{H}_2$.

(ii) *If* $\sqrt{t}+\mathbf{i}\sqrt{-t}\neq 0$, *then the ideal* $\mathcal{J}'_\lambda$ *is generated* (*as a* $U(\mathfrak{q})$-$U(\mathfrak{g})$-*bimodule*) *by* $\mathbf{I}_\lambda$ *and the element* $\overline{H}_1-\overline{H}_2$.

PROOF. The statement (ii) reduces to (i) by parity change, so we prove the statement (i). First let us show that $\overline{H}_1-\overline{H}_2$ belongs to $\mathcal{J}_\lambda$.

If $\sqrt{t}-\mathbf{i}\sqrt{-t}\neq 0$, then $\overline{H}_1+\overline{H}_2$ induces an isomorphism from $L(\mathcal{V}(\lambda))_{\overline{0}}$ to $L(\mathcal{V}(\lambda))_{\overline{1}}$ (see proof of Lemma 17). From Subsection 2.3 it then follows that $(\overline{H}_1-\overline{H}_2)L(\mathcal{V}(\lambda))_{\overline{0}}=0$ and hence $\overline{H}_1-\overline{H}_2\in\mathcal{J}_\lambda$. Let J denote the $U(\mathfrak{q})$-$U(\mathfrak{g})$-subbimodule of $\mathcal{J}_\lambda$ generated by $\mathbf{I}_\lambda$ and the element $\overline{H}_1-\overline{H}_2$.

Applying to $\overline{H}_1 - \overline{H}_2$ the adjoint action of $\mathfrak{g}$ we obtain $\overline{F}, \overline{E} \in J$. Multiplying these with elements of $U(\mathfrak{q})$ from the left we get that J also contains the following elements:

$$\overline{H}_1\overline{H}_2\overline{FE}\,, \quad \overline{H}_2\overline{FE}\,, \quad \overline{H}_1\overline{FE}\,, \quad \overline{H}_1\overline{H}_2\overline{E}\,, \quad \overline{H}_1\overline{H}_2\overline{F}\,,$$
$$\overline{H}_1\overline{E},\ \overline{H}_2\overline{E}\,, \quad \overline{FE}\,, \quad \overline{H}_1\overline{F},\ \overline{H}_2\overline{F}\,, \quad H_1 - \overline{H}_1\overline{H}_2\,.$$

Now from the PBW theorem it follows that the quotient $U(\mathfrak{q})/J$, as a right $U(\mathfrak{g})$-module, is a quotient of $U(\mathfrak{g})/\mathbf{I}_\lambda + (\overline{H}_1 + \overline{H}_2)U(\mathfrak{g})/\mathbf{I}_\lambda$. From Lemma 17 it thus follows that $J = \mathcal{J}_\lambda$. □

3.5. Typical regular nonintegral supermodules. Now we move to the easiest typical case, that is the case of typical regular nonintegral λ. In this case all Verma supermodules are irreducible and this substantially simplifies arguments. For generic supermodules a stronger result can be deduced from [Pe1].

Set $\lambda' = \lambda - \alpha$. Let for the moment V denote the 3-dimensional simple $\mathfrak{g}$-module with the trivial action of $H_1 + H_2$. Let $\mathfrak{C}_\lambda$ and $\mathfrak{C}_{\lambda'}$ denote the full subcategories of the category $\mathfrak{g}$-Mod, consisting of all modules, on which the action of the elements $\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2$ and $\mathbf{c} - (\lambda_1 - \lambda_2 - 1)^2$, respectively, is locally finite. Recall (see e.g. [BG, 4.1]) that the translation functor

$$\mathrm{T}_\lambda^{\lambda'} = \mathrm{proj}_{\mathfrak{C}_{\lambda'}} \circ V \otimes_{\boldsymbol{k}} - : \mathfrak{C}_\lambda \to \mathfrak{C}_{\lambda'}$$

is an equivalence of categories (here it is important that λ is not integral). In particular, it sends simple modules to simple modules. We also denote by $\mathrm{T}_{\lambda'}^{\lambda}$ the translation functor from $\mathfrak{C}_{\lambda'}$ to $\mathfrak{C}_\lambda$ (which is the inverse of $\mathrm{T}_\lambda^{\lambda'}$).

PROPOSITION 19. *Assume that λ is typical, regular and nonintegral.*

(i) *The correspondence*

$$\begin{aligned} \mathbf{Irr}_\lambda^{\mathfrak{g}} &\leftrightarrow \mathrm{Irr}_\lambda \\ L &\mapsto \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \end{aligned}$$

is a bijection.

(ii) *For every simple $L \in \mathbf{Irr}_\lambda^{\mathfrak{g}}$ the module $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L$ is semi-simple and we have*

$$\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \cong L \oplus \mathrm{T}_\lambda^{\lambda'} L \oplus L \oplus \mathrm{T}_\lambda^{\lambda'} L\,,$$

where $\mathrm{T}_\lambda^{\lambda'} L$ is a simple module. Moreover, we also have the isomorphism $\mathrm{Res}\, \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L \cong L \oplus \mathrm{T}_\lambda^{\lambda'} L$.

PROOF. We again prove the claim (19) first. From Lemmata 6 and 4 we obtain that, after restriction of the left action to $U(\mathfrak{g})$, the $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule $\mathcal{L}_\lambda$ decomposes as follows:

$$\begin{aligned} \mathcal{L}_\lambda \cong\ & \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda)) \oplus \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda)) \\ & \oplus \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \oplus \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda'))\,. \end{aligned} \tag{6}$$

The first two direct summands (one even and one odd) are isomorphic to $U(\mathfrak{g})/\mathbf{I}_\lambda$ and result into the direct summand $L \oplus L$ of $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_\lambda \otimes_{U(\mathfrak{g})} L$ (the even one also gives the direct summand L of $\mathrm{Res}\, L$) similarly to the proof of Proposition 16(ii).

As both $L^{\mathfrak{g}}(\lambda)$ and $L^{\mathfrak{g}}(\lambda')$ are projective in $\mathbf{O}$ because of our choice of λ, from [BG, 3.3] we derive that the functor $\mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \otimes_{U(\mathfrak{g})} -$ is a projective functor isomorphic to $\mathrm{T}_{\lambda}^{\lambda'}$. The claim (ii) follows.

Now let us prove that for any $L \in \mathbf{Irr}_{\lambda}^{\mathfrak{g}}$ the $U(\mathfrak{q})$-supermodule $\mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L$ is simple. As was already mentioned in the proof of Proposition 16, the element $\overline{H}_1 + \overline{H}_2$ commutes with all element of $U(\mathfrak{g})$. As $(\overline{H}_1 + \overline{H}_2)^2 = H_1 + H_2$ and our λ is now typical, we deduce that for any simple $U(\mathfrak{q})$-supermodule $N \in \mathrm{Irr}_{\lambda}$ the multiplication with $\overline{H}_1 + \overline{H}_2$ gives an isomorphism from the $U(\mathfrak{g})$-module $N_{\overline{0}}$ to the $U(\mathfrak{g})$-module $N_{\overline{1}}$ and converse.

Assume that $\mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L$ is not simple and N is a proper subsupermodule of $\mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L$. Then from the above we have $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} N = L \oplus L$ or $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} N = \mathrm{T}_{\lambda}^{\lambda'} L \oplus \mathrm{T}_{\lambda}^{\lambda'} L$. In the case $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} N = L \oplus L$ we obtain that N is annihilated by $\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2$ as $L \in \mathbf{Irr}_{\lambda}^{\mathfrak{g}}$. This is however not possible as $\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2$ does not annihilate $L(\mathcal{V}(\lambda))$ by Lemma 6 and hence $\mathbf{c} - (\lambda_1 - \lambda_2 + 1)^2 \notin \mathcal{I}_{\lambda}$. In the case $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} N = \mathrm{T}_{\lambda}^{\lambda'} L \oplus \mathrm{T}_{\lambda}^{\lambda'} L$ we obtain a similar contradiction using the element $\mathbf{c} - (\lambda_1 - \lambda_2 - 1)^2$. This proves that $\mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} -$ gives a well-defined and injective map from $\mathbf{Irr}_{\lambda}^{\mathfrak{g}}$ to Irr_{λ}.

The rest is similar to the proof of Proposition 16(i). For any $N \in \mathrm{Irr}_{\lambda}$ we fix a simple $\mathfrak{g}$-submodule L of $\mathrm{Res}\, N$ and consider $\mathcal{L}(L, N)$. Changing, if necessary, the parity of N and using Corollary 10 we get $\mathcal{L}(L, N) \cong \mathcal{L}_{\lambda}$ and, finally, $\mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong N$. This completes the proof. □

3.6. Typical regular integral supermodules. This case splits into two subcases with different formulations of the main result. In the first subcase we have the same result as in the previous subsection, but a rather different argument.

PROPOSITION 20. *Assume that λ is typical, regular, integral and that $\lambda_1 - \lambda_2 \neq -1$.*

(i) *The correspondence*

$$\begin{aligned} \mathbf{Irr}_{\lambda}^{\mathfrak{g}} &\leftrightarrow \mathrm{Irr}_{\lambda} \\ L &\mapsto \mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L \end{aligned}$$

is a bijection.

(ii) *For every simple $L \in \mathbf{Irr}_{\lambda}^{\mathfrak{g}}$ the module $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L$ is semi-simple and we have*

$$\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} \mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong L \oplus \mathrm{T}_{\lambda}^{\lambda'} L \oplus L \oplus \mathrm{T}_{\lambda}^{\lambda'} L,$$

where $\mathrm{T}_{\lambda}^{\lambda'} L$ is a simple module. Moreover, we also have the isomorphism $\mathrm{Res}\, \mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong L \oplus \mathrm{T}_{\lambda}^{\lambda'} L$.

PROOF. Similarly to the proof of Proposition 19(ii) we have the decomposition (6), where the first two direct summands are isomorphic to $U(\mathfrak{g})/\mathbf{I}_{\lambda}$ and result into the direct summand $L \oplus L$ of $\mathrm{Res}\, \mathcal{L}_{\lambda} \otimes_{U(\mathfrak{g})} L$.

Let us look at the summand $\mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda'))$. Under our assumptions on λ we have $\lambda_1 - \lambda_2 \in \{-2, -3, -4, \dots\}$. Applying the left exact bifunctor $\mathcal{L}(-,-)$ from the short exact sequence (2) to the short exact sequence

$$0 \to L^{\mathfrak{g}}(\lambda') \to M^{\mathfrak{g}}(-\lambda' - \alpha) \to L^{\mathfrak{g}}(-\lambda' - \alpha) \to 0\,, \tag{7}$$

we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
\mathcal{L}(L^{\mathfrak{g}}(-\lambda-\alpha), L^{\mathfrak{g}}(\lambda')) \arrow[r, hook] \arrow[d, hook] & \mathcal{L}(L^{\mathfrak{g}}(-\lambda-\alpha), M^{\mathfrak{g}}(-\lambda'-\alpha)) \arrow[r] \arrow[d, hook] & \mathcal{L}(L^{\mathfrak{g}}(-\lambda-\alpha), L^{\mathfrak{g}}(-\lambda'-\alpha)) \arrow[d, hook] \\
\mathcal{L}(M^{\mathfrak{g}}(-\lambda-\alpha), L^{\mathfrak{g}}(\lambda')) \arrow[r, hook] \arrow[d] & \mathcal{L}(M^{\mathfrak{g}}(-\lambda-\alpha), M^{\mathfrak{g}}(-\lambda'-\alpha)) \arrow[r, two heads] \arrow[d] & \mathcal{L}(M^{\mathfrak{g}}(-\lambda-\alpha), L^{\mathfrak{g}}(-\lambda'-\alpha)) \arrow[d] \\
\mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \arrow[r, hook] & \mathcal{L}(L^{\mathfrak{g}}(\lambda), M^{\mathfrak{g}}(-\lambda'-\alpha)) \arrow[r] & \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(-\lambda'-\alpha))\,.
\end{tikzcd}$$

As both $L^{\mathfrak{g}}(-\lambda - \alpha)$ and $L^{\mathfrak{g}}(-\lambda' - \alpha)$ are finite dimensional while F acts injectively on all other modules involved, we obtain

$$\mathcal{L}(L^{\mathfrak{g}}(-\lambda - \alpha), L^{\mathfrak{g}}(\lambda')) = \mathcal{L}(L^{\mathfrak{g}}(-\lambda - \alpha), M^{\mathfrak{g}}(\lambda')) = 0\,.$$

By dual arguments we also have $\mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(-\lambda' - \alpha)) = 0$. Hence, using arguments similar to the proof of Lemma 12 we obtain the following commutative diagram of $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules:

$$\begin{tikzcd}
\mathcal{L}(M^{\mathfrak{g}}(-\lambda-\alpha), L^{\mathfrak{g}}(\lambda')) \arrow[r, hook] \arrow[d, hook] & \mathcal{L}(M^{\mathfrak{g}}(-\lambda-\alpha), M^{\mathfrak{g}}(-\lambda'-\alpha)) \arrow[d, hook] \\
\mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \arrow[r, "\sim"] \arrow[d] & \mathcal{L}(L^{\mathfrak{g}}(\lambda), M^{\mathfrak{g}}(-\lambda'-\alpha)) \\
B &
\end{tikzcd},$$

where B is finite dimensional. This implies that there exists a short exact sequence of $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules

$$0 \to \mathcal{L}(M^{\mathfrak{g}}(-\lambda - \alpha), M^{\mathfrak{g}}(-\lambda' - \alpha)) \to \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \to B' \to 0\,,$$

where B' is finite dimensional. Tensoring the latter sequence with L and using the right exactness of the tensor product gives us the following exact sequence:

$$\begin{aligned} &\mathcal{L}(M^{\mathfrak{g}}(-\lambda - \alpha), M^{\mathfrak{g}}(-\lambda' - \alpha)) \otimes_{U(\mathfrak{g})} L \\ &\qquad\qquad \to \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \otimes_{U(\mathfrak{g})} L \to B' \otimes_{U(\mathfrak{g})} L \to 0\,. \end{aligned}$$

By an argument similar to the proof of Lemma 12 one shows that $B' \otimes_{U(\mathfrak{g})} L = 0$ as B' is finite dimensional while L is infnite-dimensional and simple. This gives us a surjection

$$\mathcal{L}(M^{\mathfrak{g}}(-\lambda - \alpha), M^{\mathfrak{g}}(-\lambda' - \alpha)) \otimes_{U(\mathfrak{g})} L \twoheadrightarrow \mathcal{L}(L^{\mathfrak{g}}(\lambda), L^{\mathfrak{g}}(\lambda')) \otimes_{U(\mathfrak{g})} L. \tag{8}$$

However, now both $M^{\mathfrak{g}}(-\lambda - \alpha)$ and $M^{\mathfrak{g}}(-\lambda' - \alpha)$ are projective in $\mathbf{O}$ and hence from [BG, 3.3] we have that the functor

$$\mathcal{L}(M^{\mathfrak{g}}(-\lambda - \alpha), M^{\mathfrak{g}}(-\lambda' - \alpha)) \otimes_{U(\mathfrak{g})} -$$

is a projective functor, isomorphic to $\mathrm{T}_{\lambda}^{\lambda'}$. It follows that the supermodule $\mathcal{L}(M^{\mathfrak{g}}(-\lambda - \alpha), M^{\mathfrak{g}}(-\lambda' - \alpha)) \otimes_{U(\mathfrak{g})} L$ is simple and hence the surjection (8) is, in fact, an isomorphism. This complets the proof of (ii).

The proof of the claim (i) is now completed similarly to the proof of Proposition 16(i) and Proposition 19(i). □

This second subcase requires a different formulation:

PROPOSITION 21. *Assume that λ is typical and $\lambda_1 - \lambda_2 = -1$.*

(i) *The correspondence*

$$\begin{aligned} \mathbf{Irr}^{\mathfrak{g}}_{\lambda'} &\leftrightarrow \mathrm{Irr}_{\lambda} \\ L &\mapsto \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L \end{aligned}$$

is a bijection.

(ii) *For every simple $L \in \mathbf{Irr}^{\mathfrak{g}}_{\lambda'}$ the module $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}} \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L$ is semi-simple and we have*

$$\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}} \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong L \oplus \mathrm{T}^{\lambda}_{\lambda'} L \oplus L \oplus \mathrm{T}^{\lambda}_{\lambda'} L\,,$$

where $\mathrm{T}^{\lambda}_{\lambda'} L$ is a simple module. Moreover, we also have the isomorphism $\mathrm{Res}\, \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong L \oplus \mathrm{T}^{\lambda}_{\lambda'} L$.

PROOF. Under our assumptions we have $L^{\mathfrak{g}}(\lambda) \cong M^{\mathfrak{g}}(\lambda)$ and hence a simplified version of the proof of Proposition 20(ii) gives the direct sum decomposition from (ii). Under our assumption the weight λ lies on the wall and hence the functor $\mathrm{T}^{\lambda}_{\lambda'}$ is a translation to the wall and thus sends simple $\mathfrak{g}$-modules to simple $\mathfrak{g}$-modules (see [BeGi, Proposition 3.1]). This proves the claim (ii).

The proof of the claim (i) is similar to the proof of Proposition 16(i) and Proposition 19(i). □

REMARK 22. The fact that translation to the wall sends simple $\mathfrak{g}$-modules to simple $\mathfrak{g}$-modules can be proved in a more elementary way than [BeGi, Proposition 3.1] (where a much more general result is established). The module $\mathrm{T}^{\lambda}_{\lambda'} L$ is nonzero and has finite length and hence there is a simple submodule L' of it. By adjunction of translations to the wall and out of the wall we get

$$0 \neq \mathrm{Hom}_{\mathfrak{g}}(L', \mathrm{T}^{\lambda}_{\lambda'} L) = \mathrm{Hom}_{\mathfrak{g}}(\mathrm{T}^{\lambda'}_{\lambda} L', L)\,,$$

which yields that L is a simple quotient of $\mathrm{T}^{\lambda'}_{\lambda} L'$. At the same time $\mathrm{T}^{\lambda}_{\lambda'} \mathrm{T}^{\lambda'}_{\lambda} \cong \mathrm{Id} \oplus \mathrm{Id}$ by the classification of projective functors ([BG, 3.3]). From this and the exactness of $\mathrm{T}^{\lambda}_{\lambda'}$ it follows that $\mathrm{T}^{\lambda}_{\lambda'} L \cong L'$.

3.7. Typical singular supermodules. Here we deal with the case when $\lambda = (t, t)$, $t \in \boldsymbol{k}$, $t \neq 0$. This turns out to be the most complicated case, in which we are able to get the least amount of information about the corresponding simple $\mathfrak{q}$-supermodules. We let $\mathrm{T}_{\lambda'} : \mathfrak{C}_{\lambda'} \to \mathfrak{C}_{\lambda'}$ be the translation functor through the wall, which is isomorphic to the indecomposable projective functor on $\mathbf{O}$, which sends the dominant Verma module $M^{\mathfrak{g}}(-\lambda'-\alpha)$ to the indecomposable projective cover of $L^{\mathfrak{g}}(\lambda')$ (see [BG, 3.3]).

PROPOSITION 23. *Assume that* $\lambda = (t, t)$, $t \in \boldsymbol{k}$, $t \neq 0$.

(i) *The correspondence*

$$\begin{aligned} \mathbf{Irr}^{\mathfrak{g}}_{\lambda'} &\leftrightarrow \mathrm{Irr}_{\lambda} \\ L &\mapsto \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L \end{aligned}$$

is a bijection.

(ii) *For every simple* $L \in \mathbf{Irr}^{\mathfrak{g}}_{\lambda'}$ *we have* $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}}\, \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong Y \oplus Y$, *where* Y *is an indecomposable* $\mathfrak{g}$*-module with simple top isomorphic to* L, *simple socle isomorphic to* L *and such that the homology of the sequence* $L \hookrightarrow Y \twoheadrightarrow L$ *is finite dimensional* (*and is a direct sum of several*, *possibly zero*, *copies of the trivial* $\mathfrak{g}$*-module*). *We also have* $\mathrm{Res}\, \mathcal{M}_{\lambda} \otimes_{U(\mathfrak{g})} L \cong Y$.

PROOF. As usual, we start with the claim (ii). First we claim that the $\mathfrak{g}$-modules $M(\mathcal{V}(\lambda))_{\overline{0}}$ and $M(\mathcal{V}(\lambda))_{\overline{1}}$ are indecomposable. They are isomorphic via the action of $\overline{H}_1+\overline{H}_2$ as λ is typical. If $X := M(\mathcal{V}(\lambda))_{\overline{0}}$ would be decomposable, then $X \cong M^{\mathfrak{g}}(\lambda) \oplus M^{\mathfrak{g}}(\lambda')$ by Lemma 5, which would yield that E annihilates all elements of weight λ' in $M(\mathcal{V}(\lambda))$. As $\overline{E}$ must annihilate at least two such elements (since we have four linearly independent elements of weight λ' and only two linearly independent elements of weight λ in $M(\mathcal{V}(\lambda))$), we would have a nonzero highest weight vector of weight λ' in $M(\mathcal{V}(\lambda))$, which would contradict the fact that the supermodule $M(\mathcal{V}(\lambda))$ is simple (see Lemma 6). This shows that X is indecomposable and hence projective in $\mathbf{O}$ by Lemma 5 and [Ma, Section 5].

Applying $\mathcal{L}(-, X)$ to the short exact sequence (7) and using the same arguments as in the proof of Lemma 12 we obtain an exact sequence of $U(\mathfrak{q})$-$U(\mathfrak{g})$-bimodules

$$0 \to \mathcal{L}(M^{\mathfrak{g}}(-\lambda'-\alpha), X) \to \mathcal{L}(L^{\mathfrak{g}}(\lambda'), X) \to B \to 0\,,$$

where B is finite dimensional.

Let now $L \in \mathbf{Irr}^{\mathfrak{g}}_{\lambda'}$. Similarly to the proof of (4) one shows that $B \otimes_{U(\mathfrak{g})} L = 0$ and hence, tensoring the above sequence with L, we get an exact sequence

$$0 \to \mathrm{Ker} \to \mathcal{L}(M^{\mathfrak{g}}(-\lambda'-\alpha), X) \otimes_{U(\mathfrak{g})} L \to \mathcal{L}(L^{\mathfrak{g}}(\lambda'), X) \otimes_{U(\mathfrak{g})} L \to 0\,.$$

Again by the same arguments as in the proof of Lemma 12 we get that Ker is finite dimensional.

As both modules $M^{\mathfrak{g}}(-\lambda'-\alpha)$ and X are projective in $\mathbf{O}$, the functor $\mathcal{L}(M^{\mathfrak{g}}(-\lambda'-\alpha), X) \otimes_{U(\mathfrak{g})} -$ is a projective functor by [BG, 3.3], more precisely, the translation functor $\mathrm{T}_{\lambda'}$ through the wall. As $\mathrm{T}_{\lambda'}$ is self-adjoint and annihilates finite dimensional modules, the image of $\mathrm{T}_{\lambda'}$ does not contain any nontrivial finite dimensional submodules, which yields $\mathrm{Ker} = 0$ and thus $\mathcal{L}(L^{\mathfrak{g}}(\lambda'), X) \otimes_{U(\mathfrak{g})} L \cong \mathrm{T}_{\lambda'}L =: Y$.

As L is simple, the standard properties of $\mathrm{T}_{\lambda'}$ (see e.g. [GJ, 3.6]) say that $\mathrm{T}_{\lambda'}L$ has a simple socle isomorphic to L, a simple top isomorphic to L, and that $\mathrm{T}_{\lambda'}$ kills the homology of the complex $L \hookrightarrow Y \twoheadrightarrow L$, which means that this homology is finite dimensional. The claim (ii) follows.

The proof of the claim (i) is similar to the proof of Proposition 16(i) and Proposition 19(i). $\square$

As we see, Proposition 23 does not describe the structure of the module $\mathrm{Res}\, N$ for $N \in \mathrm{Irr}_\lambda$ for typical singular λ completely (in the sense that the homology of the sequence in Proposition 23(ii) does heavily depend on the choice of the module L). The information about $\mathrm{Res}\, N$, which is obtained in Proposition 23, is known as the *rough structure* of the module $\mathrm{Res}\, N$ (see [KM, MS] for details).

3.8. Parity change. In this subsection we prove the last part of Theorem 1.

PROPOSITION 24. *For every strongly typical or atypical simple* $\mathfrak{q}$*-supermodule* N *we have* $N \not\cong \Pi N$. *For every typical simple* $\mathfrak{q}$*-supermodule* N, *which is not strongly typical*, *we have* $N \cong \Pi N$.

PROOF. The claim is trivial for the trivial supermodule N and the corresponding ΠN. Let us assume first that N is a nontrivial atypical supermodule and consider the action of the element $\overline{H}_1 + \overline{H}_2$ on N. As we have seen in Subsection 3.4, this action defines a $\mathfrak{g}$-homomorphism from $N_{\overline{0}}$ to $N_{\overline{1}}$, and a $\mathfrak{g}$-homomorphism from $N_{\overline{0}}$ to $N_{\overline{1}}$. One of these homomorphisms is zero while the other one is an isomorphism. Changing the parity swaps these two maps and proves the claim in the case of atypical supermodules.

For strongly typical supermodules we can distinguish N and ΠN via the action of the anticenter of $U(\mathfrak{q})$. By [Go, Section 10], the algebra $U(\mathfrak{q})$ contains a unique up to scalar element $T_{\mathfrak{q}} \in U(\mathfrak{q})_{\overline{0}}$, which commutes with all elements of $U(\mathfrak{q})_{\overline{0}}$ and anticommutes with all elements of $U(\mathfrak{q})_{\overline{1}}$. In particular, this elements acts as a scalar, say τ, on the $U(\mathfrak{q})_{\overline{0}}$-module $N_{\overline{0}}$ and thus as the scalar $-\tau$ on the $U(\mathfrak{q})_{\overline{0}}$-module $N_{\overline{1}}$. If N is strongly typical then $\tau \neq 0$ by [Go, Theorem 10.3], which yields that $T_{\mathfrak{q}}$ acts with the eigenvalue $-\tau \neq \tau$ on $(\Pi N)_{\overline{0}}$, implying $N \not\cong \Pi N$.

Let now N be typical but not strongly typical with annihilator $\mathcal{I}_\lambda$. Then from Subsection 3.4 we have that $\mathcal{V}(\lambda) \cong \Pi\mathcal{V}(\lambda)$, which yields

$$\mathcal{L}(L, \mathcal{V}(\lambda)) \cong \mathcal{L}(L, \Pi\mathcal{V}(\lambda)) \cong \Pi\mathcal{L}(L, \mathcal{V}(\lambda))$$

for any $\mathfrak{g}$-module L. From Lemma 15 and the proofs of Propositions 19, 20, 21 and 23 we have that N has the form either $\mathcal{L}(L, \mathcal{V}(\lambda)) \otimes_{U(\mathfrak{g})} L$ or $\Pi\mathcal{L}(L, \mathcal{V}(\lambda)) \otimes_{U(\mathfrak{g})} L$ for some simple $\mathfrak{g}$-module L. The claim of the proposition follows. $\square$

Now we are ready to prove our first main theorem, namely Theorem 1.

PROOF OF THEOREM 1. The claim Theorem 1(i) is Proposition 14. The claim Theorem 1(ii) follows from Lemma 15 and Propositions 16, 19, 20, 21 and 23. The claim Theorem 1(iii) is Proposition 24. $\square$

3.9. Weight supermodules. Weight supermodules form a very special and important class of supermodules. It is easy to see that all bijections between simple $\mathfrak{q}$-supermodules and $\mathfrak{g}$-modules, described in Lemma 15 and Propositions 16, 19, 20, 21 and 23, restrict to the corresponding subclasses of weight (super)modules. For a classification of simple weight

$\mathfrak{g}$-module we refer the reader to [Ma, Chapter 3]. In this subsection we present an alternative approach to the classification of simple weight $\mathfrak{q}$-supermodules using the coherent families approach from [Mat] (see [Ma, Section 3.5] for the corresponding arguments in the case of $\mathfrak{g}$-modules). For some other Lie superalgebras an analogous approach can be found in [Gr].

For $z \in \boldsymbol{k}$ denote by ${}^{z}U'$ the $U(\mathfrak{q})$-$U(\mathfrak{q})$ bimodule U', where the right action of $U(\mathfrak{q})$ is given by the usual multiplication, while the left action of $U(\mathfrak{q})$ is given by multiplication twisted by θ_z.

PROPOSITION 25. *Every simple weight* $\mathfrak{q}$*-supermodule is isomorphic to one of the following supermodules*:

(a) *Simple finite dimensional supermodule.*

(b) *Simple infinite dimensional highest weight supermodule.*

(c) *Simple infinite dimensional lowest weight supermodule.*

(d) *A simple supermodule of the form* $L^z := {}^{z}U' \otimes_{U(\mathfrak{q})} L$ *for some* $z \in \boldsymbol{k}$, *where* L *is a simple infinite dimensional highest weight supermodule.*

The supermodules described in Proposition 25(d) are called *dense* supermodules, see [Ma, Chapter 3].

PROOF. Let N be a simple weight $\mathfrak{q}$-supermodule. Assume first that N contains a nonzero vector v of weight λ such that $E(v) = 0$. If we have $\overline{E}(v) = 0$, then v is a highest weight vector. Using the universal property of Verma supermodules we get an epimorphism from either $M(\mathcal{V}(\lambda))$ or $M(\Pi\mathcal{V}(\lambda))$ to N and hence $N \cong L(\mathcal{V}(\lambda))$ or $N \cong L(\Pi\mathcal{V}(\lambda))$. If $\overline{E}(v) = w \neq 0$, we have $\overline{E}(w) = 0$ as $\overline{E}^2 = 0$ and $E(w) = E\overline{E}(w) = \overline{E}E(w) = 0$. Hence w is a highest weight vector and as above we get that N is a highest weight supermodule (this supermodule might be finite dimensional).

Similarly if N contains a nonzero weight vector v such that $F(v) = 0$, we obtain that N is a lowest weight supermodule.

Assume, finally, that both E and F act injectively on N. As $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{g}} N$ is a $\mathfrak{g}$-module of finite length, from [Ma, Chapter 3] we get that all weight spaces of N are finite dimensional and hence both E and F act, in fact, bijectively on N. Thus we can lift the $U(\mathfrak{q})$-action on N to a U'-action and twist the later by θ_z for any $z \in \boldsymbol{k}$. Denote the obtained U'-supermodule (and also its restriction to $U(\mathfrak{q})$) by N^z. By polynomiality of θ_z we get that for some $z \in \boldsymbol{k}$ the supermodule N^z contains a non-zero weight element v, annihilated by E. By the same arguments as in the first paragraph of this proof, we get that N^z contains a highest weight vector, say w. Let X be the $U(\mathfrak{q})$-subsupermodule of N^z, generated by w. As the action of F on N^z is injective, the action of F on X is injective as well. So, X cannot be a finite dimensional supermodule. Let Y be a simple subsupermodule in the socle of X. Then Y is a simple infinite dimensional highest weight supermodule (in particular, F acts injectively on Y as well).

By construction, the supermodule ${}^{-z}U' \otimes_{U(\mathfrak{q})} Y$ is a nonzero subsupermodule of N and hence is isomorphic to N. The claim of the proposition follows. $\square$

To obtain an irredundant complete list of pairwise nonisomorphic simple weight $\mathfrak{q}$-supermodules, one could use the following lemma:

LEMMA 26. *Let L be a simple infinite dimensional highest weight $\mathfrak{q}$-supermodule.*

(i) *If $z, z' \in \boldsymbol{k}$, then the supermodules L^z and $L^{z'}$ are isomorphic if and only if $z - z' \in \mathbf{Z}$.*

(ii) *There exists at most one coset $t + \mathbf{Z} \in \boldsymbol{k}/\mathbf{Z}$ such that the supermodule L^z is not simple if and only if $z \in \mathbf{Z}$ or $z \in t + \mathbf{Z}$.*

PROOF. If $z, z' \in \boldsymbol{k}$ and $z - z' \notin \mathbf{Z}$, then the supermodules L^z and $L^{z'}$ have differrent weights and hence are not isomorphic. If $z - z' \in \mathbf{Z}$, then to prove that L^z and $L^{z'}$ are isomorphic it is enough to check that $L^0 \cong L^1$. In the latter case it is easy to check that the map $v \mapsto F(v)$ from L^0 to L^1 is an isomorphism of $\mathfrak{q}$-supermodules. This proves the claim (i).

To prove the claim (ii) we first observe that if X is a proper subsupermodule of L^z, then F cannot act bijectively on X. Indeed, if F would act bijectively on X, then, by comparing the characters, we would have that $L \cap {}^{-z}U' \otimes_{U(\mathfrak{q})} X$ would be a proper subsupermodule of L, which is not possible as L is simple.

But if F does not act bijectively, then it acts only injectively on X. In this case there should exist a weight element $v \in X$, which does not belong to the image of F and is annihilated by E (see [Ma, Chapter 3]). Similarly to the proof of Proposition 25 one then obtains that either v or $\overline{E}(v)$ is a highest weight vector of X. However, the eigenvalue of $\mathbf{c}$ is not affected by θ_z and is given by a quadratic polynomial. Now the claim (ii) follows from the claim (i). □

3.10. The superalgebra $\mathfrak{pq}_2$. The superalgebra $\mathfrak{pq}_2$ is defined as the quotient of the superalgebra $\mathfrak{q}$ modulo the ideal, generated by the central element $H_1 + H_2$. Hence simple $\mathfrak{pq}_2$-supermodules are naturally identified with simple atypical $\mathfrak{q}$-supermodules, and thus a classification of all simple $\mathfrak{pq}_2$-supermodules follows directly from Subsections 3.3 and 3.4.

3.11. The superalgebra $\mathfrak{sq}_2$. The superalgebra $\mathfrak{sq} := \mathfrak{sq}_2$ is defines as a subsuperalgebra of $\mathfrak{q}$, generated by $\mathfrak{g}$ and the elements $\overline{E}$, $\overline{F}$, $\overline{H}_1 - \overline{H}_2$. As $\mathfrak{sq}$ is a subsuperalgebra of $\mathfrak{q}$, we have the natural restriction functor

$$\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{sq}} : \mathfrak{q}\text{-sMod} \to \mathfrak{sq}_2\text{-sMod}\,.$$

In this subsection we classify all simple $\mathfrak{sq}$-supermodules. The classification is divided into two cases, first we classify simple typical $\mathfrak{sq}$-supermodules and then we classify simple atypical $\mathfrak{sq}$-supermodules.

PROPOSITION 27. *The functor* $\mathrm{Res}^{\mathfrak{q}}_{\mathfrak{sq}}$ *induces a bijection between the set of isomorphism classes* (*up to parity change*) *of simple typical* $\mathfrak{q}$*-supermodules and the set of isomorphism classes* (*up to parity change*) *of simple typical* $\mathfrak{sq}$*-supermodules.*

To prove Proposition 27 we would need the following lemma:

LEMMA 28. (i) *Every simple* $\mathfrak{sq}$*-supermodule is a subsupermodule of the supermodule, induced from a simple* $\mathfrak{g}$*-module.*

(ii) *Every simple* $\mathfrak{sq}$*-supermodule is of finite length, when considered as a* $\mathfrak{g}$*-module.*

(iii) *Every simple* $\mathfrak{sq}$*-supermodule is a subsupermodule of the restriction of some simple* $\mathfrak{q}$*-supermodule.*

PROOF. The claim (i) is proved analogously to Proposition 13. The claim (ii) is proved analogously to Proposition 14. By the PBW theorem the algebra $U(\mathfrak{q})$ is free of rank two both as a left and as a right $U(\mathfrak{sq})$-supermodule. Hence the induction functor $\mathrm{Ind}_{\mathfrak{sq}}^{\mathfrak{q}} : \mathfrak{sq}\text{-sMod} \to \mathfrak{q}\text{-sMod}$ is exact and for any simple $\mathfrak{sq}$-supermodule L we have

$$\mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} \circ \mathrm{Ind}_{\mathfrak{sq}}^{\mathfrak{q}} L \cong L \oplus L .$$

In particular, it follows that $\mathrm{Ind}_{\mathfrak{sq}}^{\mathfrak{q}} L$ is of finite length as a $\mathfrak{g}$-module, and hence also as a $\mathfrak{sq}$-supermodule.

Now let L be a simple $\mathfrak{sq}$-supermodule and N be a simple quotient of $\mathrm{Ind}_{\mathfrak{sq}}^{\mathfrak{q}} L$. Using the adjunction we have

$$0 \neq \mathrm{Hom}_{\mathfrak{q}}(\mathrm{Ind}_{\mathfrak{sq}}^{\mathfrak{q}} L, N) = \mathrm{Hom}_{\mathfrak{sq}}(L, \mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} N) .$$

The claim (iii) follows. □

PROOF OF PROPOSITION 27. Because of Lemma 28(iii), to prove the claim of Proposition 27 it is enough to show that the restriction of every typical simple $\mathfrak{q}$-supermodule N to $\mathfrak{sq}$ is a simple $\mathfrak{sq}$-supermodule.

Note that, since we consider now typical supermodules, we have that the element $H_1 + H_2 \in \mathfrak{sq}$ still induces an isomorphism between the $\mathfrak{g}$-submodules $X_{\overline{0}}$ and $X_{\overline{1}}$ for any $\mathfrak{sq}$-supermodule X.

Let us first observe that the restriction of every typical simple highest weight $\mathfrak{q}$-supermodule $L(\mathcal{V}(\lambda))$ (or $\Pi L(\mathcal{V}(\lambda))$) to $\mathfrak{sq}$ is a simple $\mathfrak{sq}$-supermodule. From the previous paragraph we have that the highest weight space remains a simple supermodule over the Cartan subsuperalgebra after restriction. Hence if the supermodule $\mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} L(\mathcal{V}(\lambda))$ would be not simple, it would have to have a non-trivial new primitive element, that is $v \neq 0$ such that $E(v) = \overline{E}(v) = 0$. However, the action of E and $\overline{E}$ remain unchanged by the restriction and hence this is not possible.

From [Mu] we thus obtain that typical primitive ideals of $U(\mathfrak{sq})$ are annihilators of the typical supermodules $\mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} L(\mathcal{V}(\lambda))$. In fact, the previous paragraph (and Lemma 28(iii)) now implies the classification of all typical simple finite dimensional $\mathfrak{sq}$-supermodules (they are just restrictions of the corresponding typical simple finite dimensional $\mathfrak{q}$-supermodules).

Let now N be a typical simple infinite dimensional $\mathfrak{q}$-supermodule. The restriction $X := \mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} N$ has finite length as a $\mathfrak{g}$-module, and hence also as an $\mathfrak{sq}$-supermodule (since $\mathfrak{g}$ is a subalgebra of $\mathfrak{sq}$).

If we assume that $0 \neq Y \subsetneq X$ is a proper $\mathfrak{sq}$-subsupermodule, then it is also a $\mathfrak{g}$-submodule. From our explicit description of the $\mathfrak{g}$-module structure on N (see Propositions 19, 20, 21 and 23) we obtain that in this case the annihilator of Y contains the element

$\mathbf{c} - t$ for some $t \in \boldsymbol{k}$. However, from the first part of the proof we know that the element $\mathbf{c} - t$ does not annihilate the corresponding highest weight $\mathfrak{sq}$-supermodule and thus cannot belong to the corresponding primitive ideal. The obtained contradiction shows that $0 \neq Y \subsetneq X$ is not possible and completes the proof. □

To classify atypical simple $\mathfrak{sq}$-supermodules we consider the Lie algebra $\mathfrak{a} := \mathfrak{sl}_2$ (the subalgebra of $\mathfrak{g}$) as a Lie subsuperalgebra of $\mathfrak{sq}$. In this case we have the natural restriction functor

$$\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{sq}} : \mathfrak{sq}\text{-sMod} \to \mathfrak{a}\text{-Mod}\,.$$

PROPOSITION 29. *The functor* $\mathrm{Res}_{\mathfrak{a}}^{\mathfrak{sq}}$ *induces a bijection between the set of isomorphism classes of simple purely even* $\mathfrak{a}$-(*super*)*modules and the set of isomorphism classes* (*up to parity change*) *of simple atypical* $\mathfrak{sq}$-*supermodules.*

PROOF. Let N denote a simple atypical $\mathfrak{q}$-supermodule. Then from Lemma 18 it follows that either $N_{\overline{0}}$ or $N_{\overline{1}}$ is an $\mathfrak{sq}$-subsupermodule of $\mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} N$. Denote this subsupermodule by X and then we have $U(\mathfrak{sq})_{\overline{1}}X = 0$, which means that X is just an $\mathfrak{a}$-module, trivially extended to an $\mathfrak{sq}$-supermodule. It follows also that $\mathrm{Res}_{\mathfrak{sq}}^{\mathfrak{q}} N/X \cong \Pi X$. This defines a map from the set of isomorphism classes (up to parity change) of simple atypical $\mathfrak{sq}$-supermodules to the set of isomorphism classes of simple $\mathfrak{a}$-modules (considered as simple purely even $\mathfrak{a}$-supermodules).

From Lemma 15 and Proposition 16 we have that for any simple $\mathfrak{a}$-module L there exists a (unique) $\mathfrak{q}$-supermodule N, whose restriction to $\mathfrak{a}$ is isomorphic to $L \oplus L$. Hence the above map is bijective. The claim of the proposition follows. □

REMARK 30. The claim of Proposition 24 obviously extends to simple $\mathfrak{sq}$-super modules.

3.12. The superalgebra $\mathfrak{psq}_2$. The superalgebra $\mathfrak{psq}_2$ is defined as the quotient of the superalgebra $\mathfrak{sq}_2$ modulo the ideal, generated by the central element $H_1 + H_2$. Hence simple $\mathfrak{psq}_2$-supermodules are naturally identified with simple atypical $\mathfrak{sq}_2$-supermodules and thus a classification of all simple $\mathfrak{psq}_2$-supermodules follows directly from Proposition 29.

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