

JACOBI FIELDS AND THE STABILITY OF MINIMAL FOLIATIONS OF ARBITRARY CODIMENSION

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Abstract. In this article, we investigate the stability of leaves of minimal foliations of arbitrary codimension. We also study relations between Jacobi fields and vector fields which preserve a foliation, and we apply these results to Killing fields.

1. Introduction. A leaf of a minimal foliation \mathcal{F} is stable if the second derivative of the volume functional with respect to any compactly supported normal variational field is nonnegative. In this article, by direct calculations, we show that any leaf of a minimal foliation with an integrable orthogonal distribution is stable. Next, we introduce Jacobi fields of foliations, and we investigate relations between these fields and vector fields preserving a foliation (Propositions 3.3 and 3.4). Using these relations, we show directly (not using the notion of calibration) that any Killing vector field preserves a minimal foliation \mathcal{F} having all leaves compact and an integrable orthogonal distribution (Corollary 3.5). We also show that a Killing field preserves two orthogonal complementary minimal foliations on a closed manifold (Corollary 3.8). Finally, we give some consequences of these results. Throughout the paper everything (manifolds, distribution, metrics, etc.) is assumed to be C^∞ -differentiable and oriented.

2. Stability results. Let M be an m -dimensional oriented, connected Riemannian manifold. On M , we consider a foliation \mathcal{F} , and let $n = \dim \mathcal{F}$. Let D denote the distribution corresponding to \mathcal{F} , i.e., $D = T\mathcal{F}$, and D^\perp the distribution which is the orthogonal complement of D , $l = \dim D^\perp = m - n$. We assume that they are orientable and transversally orientable. Let $\langle \cdot, \cdot \rangle$ represent a metric on M and ∇ denote the Levi-Civita connection of the metric. Let $\Gamma(D)$ and ∇^\top denote the set of all vector fields tangent to D and the induced connection in D , respectively. Similarly, we have $\Gamma(D^\perp)$ and ∇^\perp . Moreover, $L(\Gamma(D), \Gamma(D^\perp))$ denotes the set of all C^∞ -linear transformations with the induced inner product (see equation (1)).

Throughout this paper, we will use the following index convention $1 \leq i, j, \dots \leq n$, $n + 1 \leq \alpha, \beta, \dots \leq m$. Repeated indices denote summation over their range. Let us take a local orthonormal frame $\{e_1, \dots, e_m\}$ adapted to D, D^\perp , i.e., $\{e_i\}$ are tangent to D and $\{e_\alpha\}$

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are tangent to D^\perp . Moreover, $\{e_1, \dots, e_m\}, \{e_i\}$ and $\{e_\alpha\}$ are compatible with the orientation of M, D and D^\perp , respectively. Then, for $A, B \in L(\Gamma(D), \Gamma(D^\perp))$, we obtain

$$(1) \quad \langle A, B \rangle = \langle A(e_i), B(e_i) \rangle.$$

Finally, if v is a vector tangent to M , then we write $v = v^\top + v^\perp$, where v^\top belongs to D and v^\perp to D^\perp .

Define the shape operator $A^V \in L(\Gamma(D), \Gamma(D))$ of \mathcal{F} with respect to $V \in \Gamma(D^\perp)$ by

$$A^V(X) = -(\nabla_X V)^\top \quad \text{for } X \in \Gamma(D).$$

Then, using the notation $A^\alpha = A^{e_\alpha}$, we have that the mean curvature vector field H of \mathcal{F} is given by

$$H = \text{Tr}(A^\alpha)e_\alpha.$$

We say that \mathcal{F} is minimal if $H = 0$, i.e., if each leaf of \mathcal{F} is a minimal submanifold of M . For $V \in \Gamma(D^\perp)$, we define mappings $\alpha_V, \nabla^\perp V \in L(\Gamma(D), \Gamma(D^\perp))$ by

$$\nabla^\perp V(X) = \nabla_X^\perp V, \quad \alpha_V(X) = [V, X]^\perp \quad \text{for } X \in \Gamma(D),$$

and the field $R(V) = (R(e_i, V)e_i)^\perp$. Here R denotes the curvature tensor of M .

In the next part of this article, we will need the following lemma.

LEMMA 2.1 ([1]). *Let $p \in M$, and let $\{e_1, \dots, e_m\}$ be a local orthonormal frame field adapted to D, D^\perp , such that $(\nabla_X e_i)^\top(p) = 0$ and $(\nabla_X e_\alpha)^\perp(p) = 0$ for any vector field X on M . Then we have at the point p*

$$e_\alpha(A^{\beta^i}_j) = (A^\beta A^\alpha)^i_j - \langle R(e_j, e_\alpha)e_i, e_\beta \rangle + \langle (\nabla_{e_\alpha} e_\gamma)^\top, e_j \rangle \langle e_i, (\nabla_{e_\gamma} e_\beta)^\top \rangle - \langle \nabla_{e_j} (\nabla_{e_\alpha} e_\beta)^\top, e_i \rangle.$$

Under the notation of Lemma 2.1, we have

COROLLARY 2.2. *If \mathcal{F} is a minimal foliation and D^\perp is integrable, then we have at the point p*

$$-\text{Tr}(A^\alpha A^\beta) + \langle R(e_\alpha), e_\beta \rangle = \langle (\nabla_{e_\alpha} e_\gamma)^\top, (\nabla_{e_\beta} e_\gamma)^\top \rangle - \text{div}_L((\nabla_{e_\alpha} e_\beta)^\top);$$

where $\text{div}_L(X) = \langle \nabla_{e_i} X, e_i \rangle$, for $X \in \Gamma(D)$.

Now, for $V, W \in \Gamma(D^\perp)$, we introduce an auxiliary function by

$$f_{V,W} = \langle \nabla^\perp V, \nabla^\perp W \rangle + \langle R(V), W \rangle - \langle A^V, A^W \rangle.$$

Then, we have the following lemma.

LEMMA 2.3. *If \mathcal{F} is a minimal foliation and D^\perp is integrable, then*

$$f_{V,W} = \langle \alpha_V, \alpha_W \rangle - \text{div}_L((\nabla_V W)^\top).$$

PROOF. Take a basis as in Lemma 2.1. Then at the point p we have the equalities

$$\begin{aligned} \langle \nabla^\perp V, \nabla^\perp W \rangle &= \langle (\nabla_{e_i} V)^\perp, (\nabla_{e_i} W)^\perp \rangle \\ &= \langle e_i(V^\alpha)e_\alpha + V^\alpha(\nabla_{e_i} e_\alpha)^\perp, e_i(W^\beta)e_\beta + W^\beta(\nabla_{e_i} e_\beta)^\perp \rangle \end{aligned}$$

$$= e_i(V^\alpha)e_i(W^\alpha),$$

and

$$\begin{aligned} \langle A^V, A^W \rangle &= \langle (\nabla_{e_i} V)^\top, (\nabla_{e_i} W)^\top \rangle \\ &= V^\alpha W^\beta \langle A^\alpha(e_i), A^\beta(e_i) \rangle = V^\alpha W^\beta \operatorname{Tr}(A^\alpha A^\beta), \end{aligned}$$

and also

$$\langle R(V), W \rangle = V^\alpha W^\beta \langle R(e_\alpha), e_\beta \rangle.$$

Thus we have

$$(2) \quad f_{V,W}(p) = e_i(V^\alpha)e_i(W^\alpha) + V^\alpha W^\beta \langle R(e_\alpha), e_\beta \rangle - V^\alpha W^\beta \operatorname{Tr}(A^\alpha A^\beta)|_p.$$

On the other hand, from Corollary 2.2, we have at the point p

$$\begin{aligned} &-V^\alpha W^\beta \operatorname{Tr}(A^\alpha A^\beta) + V^\alpha W^\beta \langle R(e_\alpha), e_\beta \rangle \\ &= \langle (\nabla_V e_\gamma)^\top, (\nabla_W e_\gamma)^\top \rangle - V^\alpha W^\beta \operatorname{div}_L((\nabla_{e_\alpha} e_\beta)^\top) \\ &= \langle (\nabla_V e_\alpha)^\top, (\nabla_W e_\alpha)^\top \rangle - \operatorname{div}_L((\nabla_V W)^\top) \\ &\quad + W^\beta (\nabla_{e_\alpha} e_\beta)^\top(V^\alpha) + V^\alpha (\nabla_{e_\alpha} e_\beta)^\top(W^\beta) \\ &= \langle (\nabla_V e_\alpha)^\top, (\nabla_W e_\alpha)^\top \rangle - \operatorname{div}_L((\nabla_V W)^\top) \\ &\quad + (\nabla_W e_\alpha)^\top(V^\alpha) + (\nabla_V e_\alpha)^\top(W^\alpha). \end{aligned}$$

Using this and (2), we obtain

$$\begin{aligned} f_{V,W}(p) &= e_i(V^\alpha)e_i(W^\alpha) + \langle (\nabla_V e_\alpha)^\top, (\nabla_W e_\alpha)^\top \rangle \\ &\quad - \operatorname{div}_L((\nabla_V W)^\top) + (\nabla_W e_\alpha)^\top(V^\alpha) + (\nabla_V e_\alpha)^\top(W^\alpha)|_p. \end{aligned}$$

Moreover, we have at the point p

$$\begin{aligned} \langle \alpha_V, \alpha_W \rangle &= \langle (\nabla_V e_i - \nabla_{e_i} V)^\perp, (\nabla_W e_i - \nabla_{e_i} W)^\perp \rangle \\ &= \langle (\nabla_V e_\alpha, e_i) + \langle \nabla_{e_i} V, e_\alpha \rangle \rangle \langle (\nabla_W e_\alpha, e_i) + \langle \nabla_{e_i} W, e_\alpha \rangle \rangle \\ &= \langle (\nabla_V e_\alpha, e_i) + e_i(V^\alpha) \rangle \langle (\nabla_W e_\alpha, e_i) + e_i(W^\alpha) \rangle. \end{aligned}$$

Thus

$$f_{V,W}(p) = \langle \alpha_V, \alpha_W \rangle(p) - \operatorname{div}_L((\nabla_V W)^\top)(p).$$

Since p is arbitrary, we complete the proof. □

Now, let L be a leaf of the foliation \mathcal{F} with the induced metric and the Levi-Civita connection $\tilde{\nabla}$. Similarly as before, we introduce the connection in $\Gamma((TL)^\perp)$, the second fundamental form $\tilde{A}^{\tilde{V}}$ of a leaf L and $\tilde{\nabla}^\perp \tilde{V}$ for an arbitrary $\tilde{V} \in \Gamma((TL)^\perp)$ (see [8]). Let L be a minimal submanifold of M , we say that L is *stable* if the inequality

$$\int_L f_{\tilde{V}} \geq 0$$

holds, where

$$f_{\tilde{V}} = \langle \tilde{\nabla}^\perp \tilde{V}, \tilde{\nabla}^\perp \tilde{V} \rangle + \langle R(\tilde{V}), \tilde{V} \rangle - \langle \tilde{A}^{\tilde{V}}, \tilde{A}^{\tilde{V}} \rangle$$

and \tilde{V} is an arbitrary vector field of $\Gamma((T(L))^\perp)$ having compact support on L (see, for example, [3]).

THEOREM 2.4. *If \mathcal{F} is a minimal foliation of a manifold M without boundary and the orthogonal distribution D^\perp is integrable, then any leaf L of \mathcal{F} is stable.*

PROOF. Let \tilde{V} be an arbitrary vector field from $\Gamma((T(L))^\perp)$ having compact support on L . Since, for each point $q \in L$, there exist a certain neighbourhood \tilde{U} of q in L and $V \in \Gamma(D^\perp)$ such that $V|_{\tilde{U}} = \tilde{V}|_{\tilde{U}}$, $W_{\tilde{V}}$ defined by

$$(3) \quad W_{\tilde{V}}|_{\tilde{U}} = (\nabla_V V)^\top|_{\tilde{U}},$$

is a well-defined vector field of $\Gamma(TL)$. Similarly, we can define $\alpha_{\tilde{V}} \in L(\Gamma(TL), \Gamma((TL)^\perp))$ such that

$$(4) \quad \alpha_{\tilde{V}}|_{\tilde{U}} = \alpha_V|_{\tilde{U}}.$$

Now, let p be a fixed point of L . Note that

$$f_{\tilde{V}}(p) = f_{V,V}(p),$$

where V as above. From Lemma 2.3, we have

$$f_{\tilde{V}}(p) = |\alpha_V|^2(p) - \operatorname{div}_L((\nabla_V V)^\top)(p).$$

Using (3) and (4), we obtain

$$f_{\tilde{V}}(p) = |\alpha_{\tilde{V}}|^2(p) - \operatorname{div}_L(W_{\tilde{V}})(p).$$

Since the point p is arbitrary, we get

$$\int_L f_{\tilde{V}} = \int_L |\alpha_{\tilde{V}}|^2 \geq 0.$$

This ends the proof. □

Note that, the above theorem can be proved using the notion of calibration [4]. In our case, the volume form of leaves, which is a smooth n -form on M , gives a calibration of \mathcal{F} (see [2]).

3. Jacobi and Killing fields. For the mapping $A : \Gamma(D^\perp) \rightarrow L(\Gamma(D), \Gamma(D))$ defined by

$$A(V) = A^V, \quad V \in \Gamma(D^\perp),$$

we can construct A^t , the transpose of A , i.e., if $B \in L(\Gamma(D), \Gamma(D))$ then

$$\langle A^t(B), V \rangle(p) = \langle A^V, B \rangle(p), \quad p \in M.$$

We then set

$$\hat{A} = A^t \circ A.$$

Furthermore, if $V \in \Gamma(D^\perp)$, we construct a new cross-section $\nabla^{\perp^2} V$ in D^\perp by setting

$$(5) \quad \nabla^{\perp^2} V = \nabla_{e_i}^\perp \nabla_{e_i}^\perp V - \nabla_{\nabla_{e_i}^\top e_i}^\perp V,$$

i.e., the trace of the connection of the mapping $\nabla^\perp V$.

Finally, we define $J : \Gamma(D^\perp) \rightarrow \Gamma(D^\perp)$ by

$$J(V) = -\nabla^{\perp^2} V + R(V) - \hat{A}(V).$$

DEFINITION 3.1. We say that a normal section $V \in \Gamma(D^\perp)$ is a Jacobi field of \mathcal{F} if $J(V) = 0$ on M .

Similarly, we can introduce Jacobi fields of a leaf L of the foliation \mathcal{F} (see [8]). Then V is a Jacobi field of \mathcal{F} if, for any leaf L of the foliation \mathcal{F} , $V|_L$ is a Jacobi field of L . Moreover, we will denote by α_V^t the transpose of α_V , i.e.,

$$\langle \alpha_V^t(W), X \rangle(p) = \langle W, \alpha_V(X) \rangle(p), \quad p \in M$$

for any $X \in \Gamma(D)$, $W \in \Gamma(D^\perp)$.

LEMMA 3.2. Let \mathcal{F} be a minimal foliation of a manifold M and assume that the orthogonal distribution is integrable. Then we have the formula

$$\langle J(V), W \rangle = \langle \alpha_V, \alpha_W \rangle + \text{div}_L(\alpha_V^t(W))$$

for $V, W \in \Gamma(D^\perp)$.

PROOF. Let p be a fixed point of L and $(\{e_i\}, \{e_\alpha\})$ be a local adapted frame field, such that $\nabla_X^\top e_i(p) = 0$ and $\nabla_X^\perp e_\alpha(p) = 0$ for each vector field X on M . Using Lemma 2.3 and the fact that $\langle \hat{A}(V), W \rangle = \langle A^V, A^W \rangle$, we obtain

$$\langle J(V), W \rangle = -\langle \nabla^{\perp^2} V, W \rangle - \langle \nabla^\perp V, \nabla^\perp W \rangle - \text{div}_L((\nabla_V W)^\top) + \langle \alpha_V, \alpha_W \rangle.$$

Then using (5), we have the following equalities at the point p :

$$\begin{aligned} \langle J(V), W \rangle &= -\langle \nabla_{e_i}^\perp \nabla_{e_i}^\perp V, W \rangle - \langle \nabla^\perp V, \nabla^\perp W \rangle - \text{div}_L((\nabla_V W)^\top) + \langle \alpha_V, \alpha_W \rangle \\ &= -e_i \langle \nabla_{e_i}^\perp V, W \rangle - \text{div}_L((\nabla_V W)^\top) + \langle \alpha_V, \alpha_W \rangle \\ &= -e_i \langle (\nabla_V e_i)^\perp, W \rangle + e_i \langle \alpha_V(e_i), W \rangle \\ &\quad - \text{div}_L((\nabla_V W)^\top) + \langle \alpha_V, \alpha_W \rangle \\ &= e_i \langle (\nabla_V W)^\top, e_i \rangle + e_i \langle \alpha_V(e_i), W \rangle - \text{div}_L((\nabla_V W)^\top) + \langle \alpha_V, \alpha_W \rangle \\ &= e_i \langle \alpha_V(e_i), W \rangle + \langle \alpha_V, \alpha_W \rangle \\ &= e_i \langle \alpha_V^t(W), e_i \rangle + \langle \alpha_V, \alpha_W \rangle \\ &= \langle \nabla_{e_i}(\alpha_V^t(W)), e_i \rangle + \langle \alpha_V, \alpha_W \rangle \\ &= \text{div}_L(\alpha_V^t(W)) + \langle \alpha_V, \alpha_W \rangle. \end{aligned}$$

Since the point p is arbitrary, we complete the proof. □

PROPOSITION 3.3. Let \mathcal{F} be a minimal foliation of a manifold M with the integrable orthogonal distribution. If a vector field X on M is foliation preserving, i.e., maps leaves to leaves. Then X^\perp is a Jacobi field.

PROOF. Since X is foliation preserving, we have

$$[X, F] \in \Gamma(D) \quad \text{for } F \in \Gamma(D).$$

Consequently, $\alpha_{X^\perp} = 0$. From Lemma 3.2, for $V = X^\perp$ and an arbitrary $W \in \Gamma(D^\perp)$, we have

$$\langle J(V), W \rangle = 0.$$

Thus X^\perp is a Jacobi field. □

Conversely, under an additional assumption, we have the following proposition.

PROPOSITION 3.4. *Let \mathcal{F} be a minimal foliation of a manifold M such that all leaves are closed and the orthogonal distribution is integrable. If $X \in \Gamma(TM)$ is a vector field such that X^\perp is a Jacobi field, then X is foliation preserving.*

PROOF. It suffices to show that $\alpha_V = 0$ for $V = X^\perp$. Since V is a Jacobi field, from Lemma 3.2, for any leaf L of \mathcal{F} , we have

$$0 = \int_L \langle J(V), V \rangle = \int_L (|\alpha_V|^2 + \text{div}_L(\alpha_V^t(V))) = \int_L |\alpha_V|^2.$$

Consequently, $\alpha_V = 0$ on M . □

COROLLARY 3.5. *Let \mathcal{F} be as in Proposition 3.4. If X is a Killing vector field on M , then X is foliation preserving.*

PROOF. Since the normal component of the Killing vector field is a Jacobi field for each leaf L (see [8]), X^\perp is a Jacobi field for each L and hence for \mathcal{F} . □

Using the notion of calibration, the above corollary was proved by Oshikiri [6].

REMARK 3.6. Corollary 3.5 can not be extended to the case when the orthogonal distribution is not integrable: The Hopf fibration of the unit sphere $S^3 \rightarrow S^2$ gives a counter example.

PROPOSITION 3.7. *Let \mathcal{F} and \mathcal{F}^\perp be minimal orthogonal foliations on a closed manifold M . If $X \in \Gamma(TM)$ is a vector field such that X^\perp is a Jacobi field, then X preserves \mathcal{F} .*

PROOF. Denote by H^\perp the mean curvature vector field of \mathcal{F}^\perp , then, from Lemma 3.2 for $V = W = X^\perp$, we obtain

$$\begin{aligned} 0 &= \int_M \langle J(V), V \rangle = \int_M (|\alpha_V|^2 + \text{div}_L(\alpha_V^t(V))) \\ &= \int_M (|\alpha_V|^2 + \text{div}_M(\alpha_V^t(V)) + \langle \alpha_V^t, H^\perp \rangle) \\ &= \int_M |\alpha_V|^2. \end{aligned}$$

Thus $\alpha_V = 0$ and X preserves \mathcal{F} . □

COROLLARY 3.8. *Let $\mathcal{F}, \mathcal{F}^\perp$ be as in Proposition 3.7. If X is a Killing vector field on M , then X is foliation preserving.*

PROPOSITION 3.9. *Let \mathcal{F} be a foliation with all leaves compact of a manifold M and $X \in \Gamma(TM)$ a vector field preserving \mathcal{F} . If $X^\perp(p) = 0$ and $p \in L \in \mathcal{F}$, then $X^\perp = 0$ on L .*

PROOF. Denote $V = X^\perp$, then $\alpha_V = 0$ on M and $V(p) = 0$. Let $q, q \neq p$, be an arbitrary point of L . Since L is complete, there exists a geodesic $c : (-\varepsilon, 1 + \varepsilon) \rightarrow L$ connecting p and q with $c(0) = p$ and $c(1) = q$ and a covering $\{U^I\}_{I=0}^N$ of c , with an orthonormal frame $\{e_\alpha^I\}$. Let $u^I = c^{-1}(U^I)$ for $I = 0, \dots, N$, and let $C^I \in \Gamma(D|_{U^I})$ be vector fields such that $C^I(c(t)) = \dot{c}(t)$ for $t \in u^I$. From the assumption, for each I , we have

$$\langle \alpha_V(C^I), e_\alpha^I(c(t)) \rangle = 0.$$

Since $V = V_I^\alpha e_\alpha^I$ (summation over α) on U^I , we have

$$\frac{d}{dt}(V_I^\alpha \circ c)(t) + (V_I^\beta \circ c)(t)(A^I)_\beta^\alpha(t) = 0,$$

for a matrix $A^I(t)$. Thus $V_I^\alpha \circ c$ is a solution of the set of linear differential equations. Without loss of generality, we may assume that $p \in U^0$ and $q \in U^N$. Consequently, $V_0^\alpha \circ c \equiv 0$. Inductively $V_I^\alpha \circ c \equiv 0$ for any I , and thus $V(q) = 0$. □

COROLLARY 3.10. *Let M be a manifold and \mathcal{F} a minimal foliation having all leaves closed and the integrable orthogonal distribution. If a Killing vector field X on M is tangent to a leaf L at some point, then X is tangent to L everywhere on L .*

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