ON RIEMANN SPACES WHOSE HOMOGENEOUS HOLONOMY GROUPS ARE INTEGRABLE

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1. With respect to the Riemann space V_n which has a one or two parametric homogeneous holonomy group, the next two theorems are already known.

THEOREM. If the homogeneous holonomy group of V_n is one parametric, then V_n is a direct product of a two dimensional Riemann space V_2 and an $(n-2)$ dimensional euclidean space E_{n-2}, that is $V_n = V_2 \times E_{n-2}$ (Liber's theorem) [1].

Using similar notations, another theorem can be stated as follows:

THEOREM. If the homogeneous holonomy group of V_n is two parametric, then $V_n = V_2 \times V_2 \times E_{n-4}$ (Kurita's theorem) [2].

In this paper we shall investigate the structure of Riemann spaces whose homogeneous holonomy groups are integrable and r-parametric in general. Since the one parametric linear homogeneous continuous groups can be supposed as special ones of integrable groups, and two parametric groups are always integrable, the result of this paper contains the above theorems as special cases.

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2. Let V_n be an n dimensional Riemann space. Following Cartan's lemma [3] we can choose frames at each point of V_n so that the connexion of the space is analytically the same as those of a space whose fundamental group is the holonomy group of V_n. We shall denote the homogeneous holonomy group of V_n by H_r and assume that H_r is r parametric and integrable. In the following, we call "homogeneous holonomy group" simply "holonomy group".

As H_r is integrable, there exists an array of invariant subgroups

$$H_r \supset H_{r-1} \supset H_{r-2} \supset \cdots \supset H_1 \supset I \quad (I:\ \text{identity});$$

each suffix shows the number of parameters. We can choose the generators $X_1 f, X_2 f, \cdots, X_r f$ of H_r so that the generators of any H_λ are $X_1 f, \cdots, X_\lambda f$ $(\lambda = 1, 2, \ldots, r)$. As $X_\lambda f$ has the form

$$X_\lambda f = a_{(\lambda)}{}^i_j x^j \frac{\partial f}{\partial x^i} \qquad (i, j = 1, 2, \ldots, n),$$

we shall denote each matrix $\| a_{(\lambda)}{}^i_j \|$ $(\lambda = 1, \ldots, r)$ by $A_1, A_2, \cdots, A_r$ respectively. Then there exists at least an array of linear spaces which are invariant under H_r [4]

(1) $$E_1 \subset E_2 \subset \dots \subset E_{n-1};$$

each $E_a (a = 1, 2, \dots, n-1)$ being a dimensional real or imaginary spaces. Among them, we shall denote the vector E_1 by x and in the first place, suppose that x is an imaginary vector. Since x is invariant under H_r, the equations

(2) $$A_1 x = \rho_1 x, \quad A_2 x = \rho_2 x, \dots, \quad A_r x = \rho_r x$$

hold good, where ρ_λ is one of the characteristic roots of the matrix A_λ, that is, one of the roots of the characteristic equation,

$$|A_\lambda - \rho_\lambda E| = 0 \quad (E\text{: unit matrix}).$$

If x has the form $(m + ni)x'$ (x': real vector), the real vector x' itself is invariant for H_r. This case will be considered later. Hence we suppose that the real and imaginary parts of x are linearly independent.

From (2), it follows immediately

(3) $$A_1 \bar{x} = \bar{\rho}_1 \bar{x}, \quad A_2 \bar{x} = \bar{\rho}_2 \bar{x}, \dots, A_r \bar{x} = \bar{\rho}_r \bar{x},$$

where $\bar{\rho}_\lambda$ is the conjugate complex number of ρ_λ and $\bar{x}$ is the vector conjugate to x (in the sense of complex numbers). The $\bar{\rho}_1, \bar{\rho}_2, \dots, \bar{\rho}_r$ being also the characteristic roots of $A_1, A_2, \cdots, A_r$ respectively. Equations (3) show that $\bar{x}$ is invariant for H_r as x. Hence the two dimensional vector space spanned by x and $\bar{x}$ is invariant undar H_r, furthermore this space being spanned by $x + \bar{x}$ and $i(x - \bar{x})$, must be real. That is, if the E_1 in the array of invariant vector spaces (1) is imaginary, there exists at least a two dimensional vector space invariant under H_r. The real and imaginary parts of x being linearly independent, this space does not degenerate.

In the next place, suppose that the E_1 in (1) is a real vector space. This means that V_n admits at least a parallel vector field, hence

$$V_n = E_1 \times V_{n-1},$$

where V_{n-1} is an $(n-1)$ dimensional Riemann space whose holonomy group is necessarily r parametric and integrable. Applying the same considerations for this V_{n-1} instead of V_n, if the first E_1 in the array of invariant vector spaces for the holonomy group of V_{n-1}:$E_1 \subset E_2 \subset \dots \subset E_{n-2}$, is real again, then V_n takes the form

$$V_n = E_2 \times V_{n-2}.$$

We may proceed in like manner for V_{n-2} and so on, and if in any step of this process there appear only real E_1's, our V_n should be flat, contrary to our assumption that H_r is r parametric. Hence there must appear in a certain step of the process an imaginary E_1, and then H_r leaves invariant at least a real two dimensional vector space.

Accordingly, in any case, V_n admits at least a two dimensional parallel vector space and by the Thomas' decomposition theorem, V_n has the form

$$V_n = V_2 \times V_{n-2}.$$

If V_2 is not flat, its holonomy group is one parametric, the holonomy group of V_{n-2} being $(r-1)$ parametric. If V_2 is flat, V_{n-2} has an r parametric and integrable group as its holonomy group. In both cases, we may proceed in like manner for V_{n-2} as for V_n and so on, finally having

$$V_n = V_2 \times V_2 \times \cdots \times V_2 \times V_p. \tag{4}$$

In this decomposition, when the number of V_2's which are not flat is just equal to r, V_p $(p = n - 2r)$ must be flat, otherwise H_r should have parameters more than r, contrary to the assumption. If, after the last step of the process, the numbers of V_2's which are not flat is less than r, the parameters of H_r are less than r, which is also absurd.

Conversely, if V_n is a direct product of r two dimensional Riemann spaces and an euclidean space, its holonomy group is the direct product of the holonomy groups of each two dimensional Riemann spaces. We can easily observe that it is abelian and hence integrable.

Accordingly we have the

THEOREM. *If the homogeneous holonomy group of Riemann space V_n is r parametric and integrable, V_n should be a direct product of r two dimensional Riemann spaces and an euclidean space. The converse is also true.*

And we also have the

COROLLARY. *If the homogeneous holonomy group of V_n is integrable, it must be at most $[n/2]$ parametric, where $[n/2]$ denotes the integer part of $n/2$.*

When the holonomy group of V_n is one parametric, there exists also an array of invariant vector spaces (1), hence the Theorem is applicable. Then $V_n = V_2 \times E_{n-2}$. This is the part of Liber's theorem concerning Riemann spaces. If the group is two parametric, it is always integrable, hence $V_n = V_2 \times V_2 \times E_{n-4}$. This is the theorem which Kurita has proved in a different manner.

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